BEYOND HYPERBOLIC DYNAMICS

C I R M, LUMINY, June 2011

Boyle–Downarowicz Lectures

6. Symbolic extensions of smooth interval maps

The main result of this lecture is a joint work with **Alejandro Maass**.
Latest progress is due to **David Burguet** and has been presented in his talk.

In order to fully appreciate the theory of entropy structure we need an example of an important class of systems for which this theory allows us to compute (or at least estimate) the symbolic extension entropy, while explicit construction of symbolic extensions, in the same generality, seems inaccessible. A spectacular such example is the class of smooth transformations of compact Riemannian manifolds, especially those of dimension 1 and 2, for which we have a beautiful direct dependence between $\mathbf{h}_{\text{sex}}$ and the degree of smoothness. In this case, we can not only estimate the symbolic extension entropy function in terms of more familiar parameters, but even indicate a concrete function that is realized as h_{ext}^{φ} in a symbolic extension.

State of art for smooth systems

1. C^∞ implies asymptotic h-expansiveness (Buzzi, 1997) which is equivalent to the existence of principal symbolic extensions (Boyle, Fiebig, Fiebig, 2002).

2. Successful application of entropy structures to smooth systems in dimension ≥ 2 (D. & Newhouse, 2005):
∘ examples of C^1 maps with no symbolic extensions,
∘ examples with $\mathbf{h}_{\text{sex}}(T) \geq \mathbf{h}_{\text{top}}(T) + \frac{R(f)}{r-1}$,
∘ conjecture $\mathbf{h}_{\text{sex}}(T) \leq \mathbf{h}_{\text{top}}(T) + \frac{\dim \cdot R(f)}{r-1}$.

3. Proof of the conjecture in dimension 1 (D. & Maass, 2009; the subject of today's lecture)

4. C^r examples on the interval with $\mathbf{h}_{\text{sex}}(f) \geq \mathbf{h}_{\text{top}}(f) + \frac{R(f)}{r-1}$ (Burguet, 2008).

5. Proof of the conjecture in dimension 2 (with adjustments of the constant) (Burguet; the subject of his lecture)

Formulation of the result

Let $f : [0, 1] \to [0, 1]$ be a C^1 transformation of the interval. For $\mu \in \mathcal{M}_f([0, 1])$ we define

$$\chi(\mu) = \int \log |f'| \, d\mu$$

(for ergodic measure this is the *Lyapunov exponent* of μ). This function is upper semicontinuous (not continuous). Let $\chi^+ = \max\{0, \chi\}$ (still upper semicontinuous) and we let $\overline{\chi^+}$ denote the integral average of χ^+ over the ergodic decomposition (this is an upper semicontinuous and affine function on $\mathcal{M}_f([0, 1])$).

We also set $R(f) = \sup_\mu \overline{\chi^+}$. It is not hard to see (using the Ergodic Theorem), that

$$R(f) = \lim_n \sup_x \log^+ |(f^n)'(x)|.$$

It is important that the function χ and the constant $R(f)$ are invariants of C^1 conjugacy.

We define the *degree of smoothness* r of f inductively:

• for $r \leq 1$, f is of class C^r if it is r-Hölder, i.e., there exists a constant c such that $|f(x) - f(y)| \leq c|x - y|^r$.

• For $r > 1$ we require that f is differentiable, and f' is of class C^{r-1}. Attention! our C^n is slightly weaker that standard.
Exception: We understand C^1 in the standard sense.

Theorem 14: Let f be a C^r transformation of the interval $[0,1]$, where $r > 1$. Then, the function

$$u = \frac{\overline{\chi^+}}{r-1}$$

is an affine repair function of the entropy structure. Hence the function $h + \frac{\overline{\chi^+}}{r-1}$ is an affine superenvelope, and thus it is the extension entropy function in some symbolic extension.

Corollary: We have

$$h_{\text{sex}} \leq h + \frac{\overline{\chi^+}}{r-1} \quad \text{and} \quad \mathbf{h}_{\text{sex}}(f) \leq \mathbf{h}_{\text{top}}(f) + \frac{R(f)}{r-1}.$$

(This is exactly the conjecture in dimension 1.)

Using the Margulis-Ruelle inequality $\mathbf{h}_{\text{top}}(f) \leq R(f)$ we can also write

$$\mathbf{h}_{\text{sex}}(f) \leq R(f) + \frac{R(f)}{r-1} = \frac{r}{r-1} R(f).$$

Proof of Theorem 14; reformulation

Let $\theta_{\mathcal{V}}$ denote the tail of the Newhouse entropy structure $\mathcal{H}^{\mathrm{New}}$, i.e., $\theta_{\mathcal{V}}(\mu) = h(\mu) - h(T|\nu, \mathcal{V})$, where $\mathcal{V}$ is an open cover. We need to show that for each invariant measure μ, for $\mathcal{V}$ fine enough,

$$\overset{\cdots\cdots\cdots}{u + \theta_{\mathcal{V}}}(\mu) < \gamma,$$

i.e., that $\widetilde{u + \theta_{\mathcal{V}}}(\mu) - u(\mu) - \theta_{\mathcal{V}}(\mu) < \gamma$. Since $\theta_{\mathcal{V}}(\mu)$ is small for fine $\mathcal{V}$, this is the same as showing

$$\widetilde{u + \theta_{\mathcal{V}}}(\mu) - u(\mu) < \gamma.$$

By the definition of the upper semicontinuous envelope, we must show that if ν is sufficiently close to μ then

$$u(\nu) + \theta_{\mathcal{V}}(\nu) - u(\mu) < \gamma,$$

which we rewrite as

$$h(T|\nu, \mathcal{V}) \leq \frac{\overline{\chi^+}(\mu) - \overline{\chi^+}(\nu)}{r - 1} + \gamma.$$

We will show the above in case ν is ergodic. (Dropping ergodicity requires some general technical work.) Assuming ergodicity of ν, by easy convexity arguments it suffices to prove the simpler inequality:

$$h(T|\nu, \mathcal{V}) \leq \frac{\chi(\mu) - \chi(\nu)}{r - 1} + \gamma$$

(known as the **Antarctic Theorem**).

Proof of the Antarctic Theorem, Counting Lemma

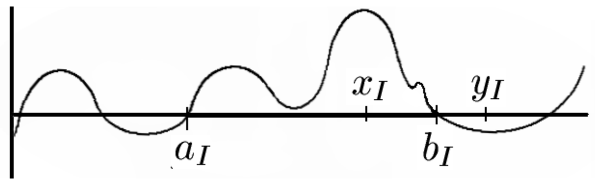
Lemma 2: Let $g : [0,1] \to \mathbb{R}$ be a C^r function, where $r > 0$. Then there exists a constant $c > 0$ such that for every $0 < s < 1$ the number of components of the set $\{x : g(x) \neq 0\}$ on which $|g|$ reaches or exceeds the value s is at most $c \cdot s^{-\frac{1}{r}}$.

Proof. If g has a constant sign then there is only one component and the lemma holds with $c = 1$. Otherwise we proceed inductively, as follows: For $0 < r \leq 1$, g is Hölder, i.e., there exists a constant $c_1 > 0$ such that $|g(x) - g(y)| \leq c_1 |x - y|^r$. If $|g(x)| \geq s$ and y is a zero point for g then

$$|x - y| \geq {c_1}^{-\frac{1}{r}} \cdot s^{\frac{1}{r}}.$$

The component containing x is at least that long and the number of such components is at most $c \cdot s^{-\frac{1}{r}}$, where $c = {c_1}^{\frac{1}{r}}$.

Now take $r > 1$ and suppose that the lemma holds for $r - 1$. Let g be of class C^r. We count the components $I = (a_I, b_I)$ of $\{x : g(x) \neq 0\}$ where $|g|$ exceeds s. Unless $a_I = 0$ or $b_I = 1$, I contains a critical point. Let x_I denote the largest critical point $x \in I$ satisfying $|g(x)| \geq s$. Unless I is the last or last but one component, there is a critical point larger than or equal to b_I. Let y_I be the smallest such critical point. So, except for at most three components, I determines an interval (x_I, y_I).



Notice that these intervals are disjoint for different I.

There are two possible cases: either

a) $y_I - x_I > s^{\frac{1}{r}}$, or

b) $y_I - x_I \leq s^{\frac{1}{r}}$.

Clearly, the number of components I satisfying a) is smaller than $s^{-\frac{1}{r}}$. If a component satisfies b) then, by the mean value theorem, $|g'|$ attains on (x_I, b_I) a value at least $s/s^{\frac{1}{r}} = s^{\frac{r-1}{r}}$. This value is attained on a component of the set $\{x : g'(x) \neq 0\}$ contained in (x_I, y_I). Because g' is of class C^{r-1}, by the inductive assumption, the number of such intervals (x_I, y_I) (hence of components I satisfying b)) does not exceed $c \cdot (s^{\frac{r-1}{r}})^{-\frac{1}{r-1}} = c \cdot s^{-\frac{1}{r}}$. Jointly, the number of all components I is at most $3 + (c+1) \cdot s^{-\frac{1}{r}} \leq (c+4) \cdot s^{-\frac{1}{r}}$. $\square$

Letting $g = f'$ we obtain the following

Corollary: Let $f : [0,1] \to [0,1]$ be a C^r function, where $r > 1$. Then there exists a constant $c > 0$ such that for every $s > 0$ the number of branches of monotonicity of f on which $|f'|$ reaches or exceeds s is at most $c \cdot s^{-\frac{1}{r-1}}$.

Definition 11: Let f be as in the formulation of the above Corollary. Let $\mathcal{I} = (I_1, I_2, \ldots, I_n)$ be a finite sequence of branches of monotonicity of f, (i.e., any formal finite sequence whose elements belong to the countable set of branches, admitting repetitions). Denote

$$a_i = \min\{-1, \max\{\log |f'(x)| : x \in I_i\}\}.$$

Choose $S \leq -1$. We say that $\mathcal{I}$ *admits* the value S if

$$\frac{1}{n} \sum_{i=1}^{n} a_i \geq S.$$

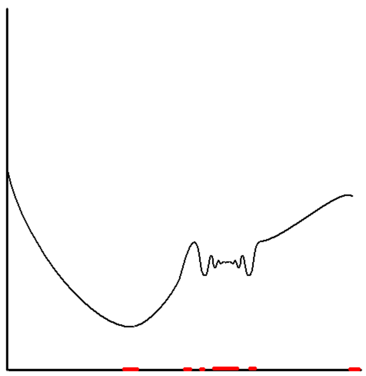
Clearly, if there exists a sequence of points $y_i \in I_i$ with $\log |f'(y_i)| \leq -1$ for each i and satisfying $\frac{1}{n} \sum_{i=1}^{n} \log |f'(y_i)| \geq S$, then $\mathcal{I}$ admits the value S.

Lemma 3: Let $f : [0,1] \to [0,1]$ be a C^r function, where $r > 1$. Fix $\gamma > 0$. Then there exists $S_\gamma \leq -1$ such that for every n and $S < S_\gamma$ the logarithm of the number of sequences $\mathcal{I}$ of length n which admit the value S is at most

$$n \frac{-S}{r-1}(1+\gamma).$$

We will skip the proof, which relies on the preceding lemma.

Let $C = \{x : f'(x) = 0\}$ be the critical set. Fix $\gamma > 0$. Fix some open neighborhood U of C on which $\log |f'| < S_\gamma$. Notice that U^c can be covered by finitely many open intervals on which f is monotone. Let $\mathcal{V}$ be the cover consisting of U and these intervals.



Lemma 4: Let T be a C^r transformation of the interval or of the circle X, where $r > 1$. Let U and $\mathcal{V}$ be as described above. Let ν be an ergodic measure and let

$$S(\nu) = \int_U \log |f'| \, d\nu.$$

Then

$$h(T|\nu, \mathcal{V}) \leq \frac{-S(\nu)}{r-1}(1+\gamma)$$

Sketch of the proof. It suffices to consider the case of $S(\nu)$ finite. The key decission is the choice of the set F (of large measure). We choose F to be the set of points on which the nth Cesaro means of the function $\mathbf{1}_U \log |f'|$ are close to $S(\nu)$ for n larger than some n_0. Let $x \in F$ and $n \geq n_0$. Consider a set

$$V^n = V_0 \cap T^{-1}(V_1) \cap \cdots \cap T^{-n+1}(V_{n-1})$$

containing x, with $V_i \in \mathcal{V}$ (as in the definition of local entropy). Consider the finite subsequence of times $0 \leq i_j \leq n-1$ when $V_{i_j} = U$. Let $n\zeta$ denote the length of this subsequence and assume $\zeta > 0$. For a fixed δ let E be an (n, δ)-separated set in $V^n \cap F$ and let $y \in E$. The sequence (i_j) contains only (usually not all) times i when $f^i(y) \in U$. Thus, since $y \in F$, we have

$$S(\nu) \leq \frac{1}{n}\Big(\sum_j \log|f'(T^{i_j}(y))| + A(y)\Big) + \epsilon,$$

where A is the similar sum over the times of visits to U not included in the sequence (i_j). Clearly $A \leq 0$, so it can be skipped. Dividing by ζ we obtain

$$\frac{S(\nu)-\epsilon}{\zeta} \leq \frac{1}{n\zeta}\sum_j \log|f'(T^{i_j}(y))|.$$

The right hand side above is smaller than S_γ. This implies that along the subsequence (i_j) the trajectory of y traverses a sequence $\mathcal{I}$ (of length $n\zeta$) of branches of monotonicity of f admitting the value $\frac{S(\nu)-\epsilon}{\zeta}$ smaller than S_γ. By Lemma 3, the logarithm of the number of such sequences $\mathcal{I}$ is dominated by

$$n\frac{-S(\nu)+\epsilon}{r-1}(1+\gamma)$$

At times i other than i_j the set V_i contains only one branch of monotonicity. This contributes to the cardinality of E a subexponential term. The proof is concluded by dividing by n, letting $n \to \infty$ (then ϵ tends to 0) and then letting the measure of F tend to 1. $\square$

Sketch of the proof of the Antarctic Theorem.

Fix an invariant measure μ with $\chi(\mu) > 0$ and some $\gamma > 0$. Clearly, then $\mu(C) = 0$. Since $\log|f'|$ is μ-integrable, The open neighborhood U of C (on which $\log|f'| < S_\gamma$) in the definition of $\mathcal{V}$ can be made so small that

$$\int_{\overline{U}} \log|f'(x)|\, d\mu > -\epsilon.$$

Then

$$\int_{\overline{U}^c} \log|f'(x)|\, d\mu < \chi(\mu) + \epsilon.$$

The above integral is an upper semicontinuous function of the measure ($\overline{U}^C$ is an open set on which $\log|f'|$ is finite and continuous and negative on the boundary). Thus the inequality holds for any invariant measure ν sufficiently close to ν. All the more

$$\int_{U^c} \log|f'(x)|\, d\nu < \chi(\mu) + \epsilon$$

(we have included the boundary to the set of integration, and the function is negative on that boundary). Then

$$-S(\nu) = \int_{U^c} \log|f'(x)|\, d\nu - \chi(\nu) \le \chi(\mu) - \chi(\nu) + \epsilon.$$

By Lemma 4, we have

$$h(T|\nu, \mathcal{V}) \le \frac{\chi(\mu) - \chi(\nu) + \epsilon}{r-1}(1+\gamma).$$

Since the numerator is bounded, we can replace the multiplicative error term γ by a small additive term as in the formulation of the Antarctic Theorem. $\square$

THANK YOU FOR YOUR ATTENTION