BEYOND HYPERBOLIC DYNAMICS

C I R M, LUMINY, June 2011

Boyle–Downarowicz Lectures

3. and 4. Defect Theory for monotone nets of functions

Motivation

• Symbolic extensions depend on entropy but not the classical notions.

• Require studying so-called **entropy structure** and its **superenvelopes**:

Main Theorem: Consider a topological dynamical system (X, T). A function E on invariant measures "reflects" the entropy in a symbolic extension if and only if it is an affine superenvelope of the entropy structure of (X, T).

Mike has explained how symbolic extensions of zero-dimensional systems are related to *superenvelopes* of the *entropy structure*.
In dimension zero both notions are relatively easy:

• **Entropy structure** (*entropy sequence*) is the sequence of functions $\mathcal{H} = \{h_k\}$ defined on $\mathcal{M}_T(X)$ as entropy functions with respect to refining clopen partitions.

h_k are upper semicontinuous, affine, and increase to the entropy function h which is finite (otherwise there are no symbolic extensions). The differences $h_{k+1} - h_k$ are also upper semicontinuous.

• **Superenvelope** of $\mathcal{H}$ is any function $E \geq h$ such that $E - h_k$ is upper semicontinuous for all k.

Without assuming dimension zero, still we can build symbolic extensions, as follows:

Theorem 0: Every topological dynamical system (X, T) has a *principal* zero-dimensional extension (X', T').

Principal means it preserves entropy for all invariant measures:

$$h(\mu, T) = h(\mu', T') \text{ whenever } \mu' \text{ maps to } \mu.$$

It is clear that any symbolic extension of (X', T') is a symbolic extension of (X, T). Also the converse is true, in a sense:

Fact 0: If (Y, S) is a symbolic extension of (X, T), then (Y, S) has a symbolic principal extension (Y', S') which is a (symbolic) extension of (X', T').

In particular, the family of extension entropy functions h_{ext}^{φ} for symbolic extensions (and thus the symbolic extension entropy function h_{sex} and the topological symbolic extension entropy $\mathbf{h}_{\text{sex}}$) are the same for (X, T) as for (X', T'), and they are characterized as the superenvelopes of the entropy structure of (X', T').

But the principal zero-dimensional extension is usually rather hard to describe. The clopen partitions of X' do not translate to any reasonable objects in X. We would like to characterize the symbolic extension entropies directly in terms of (X, T), where we have no clopen partitions, while entropy functions with respect to partitions are not upper semicontinuous, etc.

We would like to define *entropy structure* and *superenvelopes* directly on invariant measures of the system.
Amazingly, at least is some types of systems (smooth systems), these notions are related to other, much more familiar notions (Lyapunov exponents, degree of smoothness).

GOAL OF THE LECTURE

• Introducing superenvelopes for abstract nets of functions;

• Introducing uniform equivalence relation, which preserves superenvelopes;

• Introducing entropy notions which will replace the entropy sequence in general systems (among them **Newhouse local entropy**);

• Prove uniform equivalence of these notions and entropy sequence in a principal zero-dimensional extension;

• Apply Newhouse entropy structure for explicit computation of a superenvelope (hence symbolic extension netropy) for smooth interval maps.

Monotone nets of nonnegative functions

$f_\kappa : \mathfrak{X} \to [0, \infty)$, $\{f_\kappa\}$ is a net (κ ranges over a directed family $\mathcal{K}$).

$\forall \iota > \kappa \ \ f_\iota \geq f_\kappa$ (a nondecreasing net - we will say *increasing*)

$f_\kappa \nearrow f : \mathfrak{X} \to [0, \infty]$ (pointwise convergence)

We will assume that f is finite everywhere.

We have a "dual" net of tails $\{\theta_\kappa\}$, where $\theta_\kappa = f - f_\kappa$.
The functions θ_k are nonnegative and decrease pointwise to zero.
The net $\{\theta_\kappa\}$ and the limit function f determine the net $\{f_\kappa\}$.

This convergence is either uniform, i.e.,

for every ϵ there is κ such that $\theta_\kappa \leq \epsilon$,

or not. Can we distinguish between non-uniform convergences?

The non-uniformity is measured by

$D_{\mathfrak{X}} = \lim_{\kappa \in K} \downarrow \sup_{x \in \mathfrak{X}} \theta_\kappa(x)$ (the *global defect of uniformity*)

If $\mathfrak{X}$ is a metric space, we can localize this parameter:

$$D_x = \inf_{\epsilon>0} \lim_{\kappa\in K} \downarrow \sup_{y\in B(x,\epsilon)} \theta_\kappa(y) = \lim_{\kappa\in K} \downarrow \inf_{\epsilon>0} \sup_{y\in B(x,\epsilon)} \theta_\kappa(y).$$

For a function $f : \mathfrak{X} \to \mathbb{R}$, the function

$$\widetilde{f}(x) := \limsup_{y\to x} f(y) = \inf_{\epsilon>0} \sup_{y\in B(x,\epsilon)} f(y)$$

is called the *upper-semicontinuous envelope* of f. Thus,

$$D_x = \lim_{\kappa\in K} \downarrow \widetilde{\theta_\kappa}(x).$$

In this way we have discovered that nonuniformity of the convergence has to do with upper semicontinuity.

Upper semicontinuous functions

A function $f : \mathfrak{X} \to \mathbb{R}$ is *upper semicontinuous* if one of the following equivalent conditions holds

1. $\forall x \in \mathfrak{X} \ \limsup_{y \to x} f(y) \leq f(x)$ (f is upper semicontinuous *at* x);

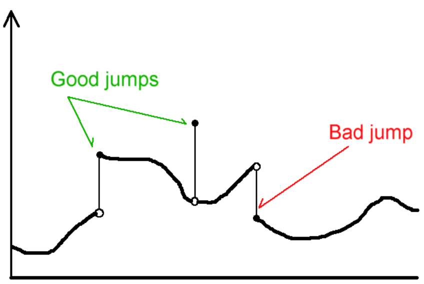
2. $\forall a \in \mathbb{R} \quad f^{-1}((-\infty, a))$ is open (or $f^{-1}([a, \infty))$ is closed);

3. the area above the graph is open
(or the area below and on the graph is closed);

4. f is a pointwise infimum of a family of continuous functions.

5. f is a pointwise limit of a decreasing net of continuous functions.

6. $f = \widetilde{f}$



The upper semicontinuous envelope has already been introduced:

$$\widetilde{f}(x) = \limsup_{y \to x} f(y).$$

We also have:

$$\widetilde{f} = \inf\{g : g \geq f, g \text{ is continuous}\}.$$

For a function $f : \mathfrak{X} \to \mathbb{R}$ we define

$$\dddot{f} = \widetilde{f} - f$$

and we call it the *defect of upper semicontinuity function* (or just *defect*).

Both the upper semicontinuous envelope and defect operations are subadditive:

$$\widetilde{f+g} \leq \widetilde{f} + \widetilde{g} \quad \text{and} \quad \overset{\cdots\cdots\cdots}{f+g} \leq \dddot{f} + \dddot{g}.$$

On a compact domain every upper semicontinuous function is bounded above. Moreover, we have the following *exchange of suprema and infima* statement

Fact 1: If $\{g_\kappa\}$ is a decreasing net (to a limit g) of nonnegative upper semicontinuous functions on a compact metric domain $\mathfrak{X}$, then

$$\sup_{x\in\mathfrak{X}} g(x) = \sup_{x\in\mathfrak{X}} \lim_{\kappa} \downarrow g_\kappa(x) = \lim_{\kappa} \downarrow \sup_{x\in\mathfrak{X}} g_\kappa(x).$$

Proof: Obviously, $L \leq R$. The sets $g_\kappa \geq L + \epsilon$ are closed (hence compact) and decrease. Their intersection is empty because on the intersection $g \leq L + \epsilon$. So only finitely many of them are nonempty. $\square$

One of the consequences:

Fact 2: *A net of upper semicontinuous functions* $\{g_\kappa\}$ *decreasing to a continuous limit* g *on a compact domain converges uniformly.*

Proof: The functions $g_\kappa - g$ are upper semicontinuous and decrease to zero. Now apply the above. $\square$

We go back to the increasing net $\{f_\kappa\}$ (or equivalently, to the decreasing net $\{\theta_\kappa\}$). Recall that we have

$$D_x = \lim_{\kappa\in K} \downarrow \widetilde{\theta}_\kappa(x).$$

We easily see that $D_x \leq D_{\mathfrak{X}}$.

If $D_x = 0$ for all x then we say that the convergence is *locally uniform.* In general, this does not imply uniform convergence. However,

Theorem 1: *If $\mathfrak{X}$ is compact, then $D_{\mathfrak{X}} = \sup_{x\in\mathfrak{X}} D_x$.*

Proof:

$$D_{\mathfrak{X}} = \lim_{\kappa\in K} \downarrow \sup_{x\in\mathfrak{X}} \theta_\kappa(x) \leq$$

$$\lim_{\kappa\in K} \downarrow \sup_{x\in\mathfrak{X}} \widetilde{\theta}_\kappa(x) = \sup_{x\in\mathfrak{X}} \lim_{\kappa\in K} \downarrow \widetilde{\theta}_\kappa(x) =$$

$$\sup_{x\in\mathfrak{X}} D_x. \qquad \square$$

As we know, if all functions θ_κ are upper semicontinuous then they decrease to zero uniformly (on a compact domain). So, in this case, rather surprisingly we can predict the "type of convergence" by examining the properties of the individual functions. We can weaken the upper semicontinuity condition as follows:

Definition 1: A decreasing net $\{g_\kappa\}$ is *asymptotically upper semicontinuous* if the defects $\ddot{g}_\kappa$ decrease to zero pointwise.

Fact 3: A decreasing to zero net $\{\theta_\kappa\}$ converges locally uniformly (uniformly, on compact domain) if and only if it is asymptotically upper semicontinuous.

Proof:

$$D_x = \lim_{\kappa \in K} \downarrow \widetilde{\theta}_\kappa(x) = \lim_{\kappa \in K} \downarrow \widetilde{\theta}_\kappa(x) - \lim_{\kappa \in K} \downarrow \theta_\kappa(x) = \lim_{\kappa \in K} \downarrow \dddot{\theta}_\kappa(x).$$

□

If a net $\{g_\kappa\}$ decreases to a nonzero function (especially, discontinuous) then the above equivalence fails, nonetheless we will consider asymptotic upper semicontinuity a **very desirable property**.

Repair functions

For a single function f we have $\dddot{f} + f = \widetilde{f}$ is upper semicontinuous, so by adding the function $\dddot{f}$ we have *repaired* the function f.
(The defect function fills in all the "bad jumps".)

Can we, similarly, "repair" a decreasing net $\{g_\kappa\}$ by adding to it just ONE nonnegatIVE function?

Definition 2: A nonnegative function u *repairs* (or is a *repair function* of) a decreasing net $\{g_\kappa\}$ if the net $\{u + g_\kappa\}$ is asymptotically upper semicontinuous.

By repairing a net we obviously cannot make it uniformly convergent, but by being asymptotically upper semicontinuous the repaired net is in a sense "closer" to this ideal situation.

Because the infimum of any family of upper semicontinuous functions is upper semicontinuous, it is easy to see that if the family of repair functions is nonempty then its pointwise infimum $u_{\min}$ is a repair function. We will call it *the smallest repair function* of the net.

It may happen, however, that there are no repair functions. In such case we agree to define the smallest repair function to be the constant infinity function.

Finding the smallest possible repair function $u_{\min}$ of the net of tails $\{\theta_\kappa\}$ is the most important issue in this talk.

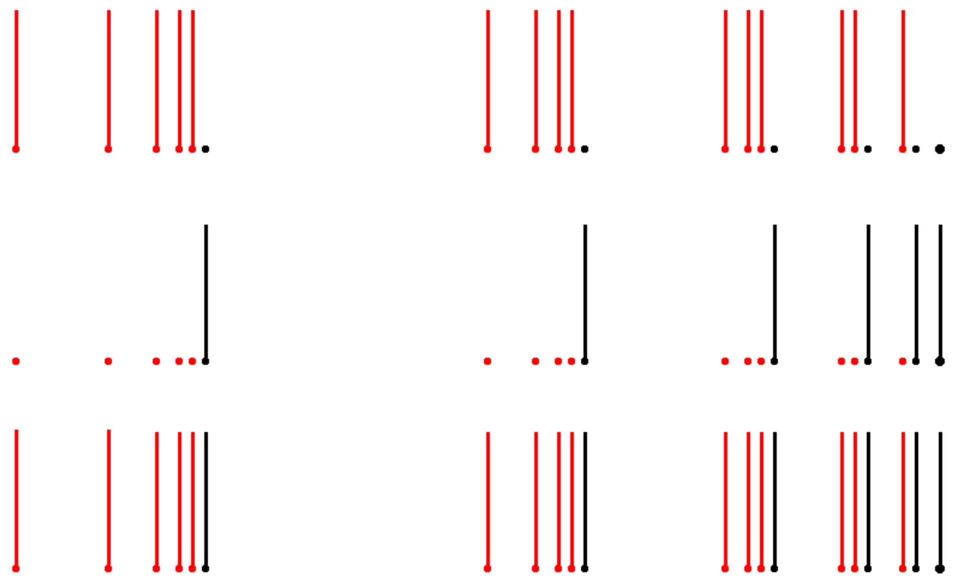
A reasonable choice of a candidate to be a repair function is the "limit defect", i.e., the function

$$u_1(x) = \lim_{\kappa \in K} \downarrow \overset{\dots}{\theta}_\kappa(x),$$

(which happens to coincide with the local defect of uniformity D_x).

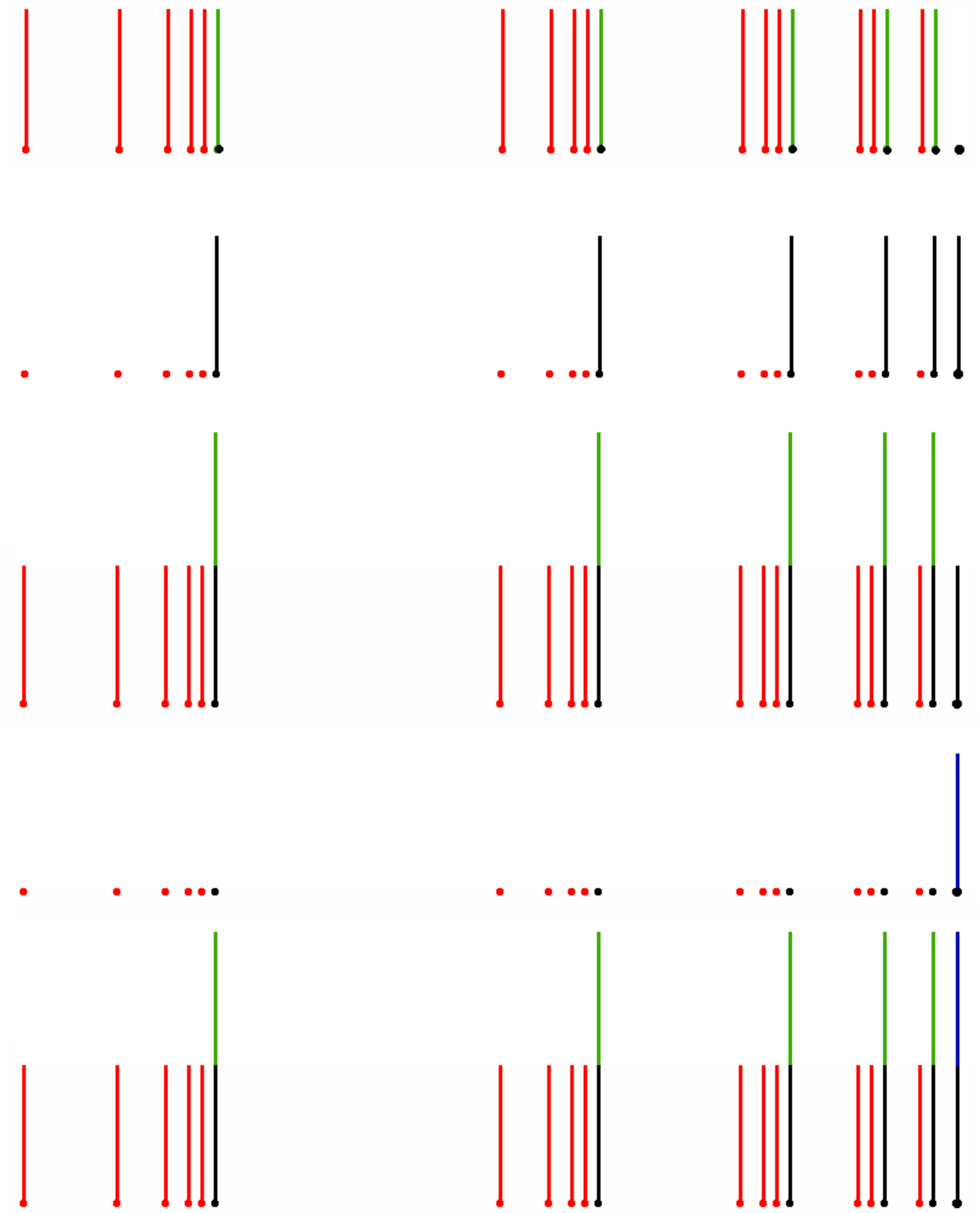
Well... sometimes it works, sometimes it does not!

Example 1: The "pick-up sticks game".
Top figure: Order all the red sticks (anyhow) by the naturals. Let θ_k be the function obtained by *removing* the first k sticks. The middle picture shows the function u_1. The functions $u + \theta_k$ are all upper semicontinuous (bottom picture). It is so because θ_k restricted to the support of u_1 converge uniformly, in other words all the defect "comes from outside" the support of u_1.



Example 2: A similar "pick-up sticks game", now played with both red and green sticks (top picture).

Each green stick will be removed in a finite step, and then the game will locally look the same as in th preceding example. So, the limit defect function u_1 is the same as before (second picture).



This time, the sequence $\{u_1 + \theta_k\}$ is not asymptotically upper semi-continuous (third picture). On the support of u_1 the functions θ_k do not converge uniformly, so there is some "internal defect", represented by the limit defect function of the sequence $u_1 + \theta_k$ (last but one picture). We have detected *the defect of the second order*. If we add both limit defect functions, in this example we do repair the net (last picture).

The transfinite solution

Definition 3: The *transfinite sequence* (u_α) (α are the ordinals) associated with the decreasing to zero net $\{\theta_\kappa\}$ is defined as follows:

- $u_\alpha = 0$,

and, when u_β are already defined for all $\beta < \alpha$, we let

- $u_\alpha = v_\alpha + \lim\limits_\kappa \overset{\cdots\cdots}{v_\alpha + \theta_\kappa}$,

where $v_\alpha = \sup\limits_{\beta<\alpha} u_\beta$.

(If v_α is infinite at some point, we let $u_\alpha \equiv \infty$.)

Notice that $u_\alpha = v_\alpha + \lim\limits_\kappa \widetilde{v_\alpha + \theta_\kappa} - v_\alpha - \lim\limits_\kappa \theta_\kappa = \lim\limits_\kappa \downarrow \widetilde{v_\alpha + \theta_\kappa}$.

Interpretation: $u_1 = \lim\limits_\kappa \overset{\cdots}{\theta}_\kappa$ is the limit defect (of the first order). Then $\overset{\cdots\cdots}{u_1 + \theta_\kappa}$ is the the limit defect of the net "repaired" using u_1 (the defect of the second order). The function u_2 is the sum of these two defects, so it can be called *cumulative defect of second order.* Analogously, $\lim\limits_\kappa \overset{\cdots\cdots}{v_\alpha + \theta_\kappa}$ is the limit defect of the net "repaired" by adding the cumulative defect of all orders up to α (the defect of order α) and u_α is the *cumulative defect of order α.*

Fundamental facts:

1. $v_{\alpha+1} = u_\alpha$, so $u_{\alpha+1} = \lim\limits_{\kappa} \downarrow \widetilde{u_\alpha + \theta_\kappa}$.
2. The transfinite sequence u_α increases.
3. This sequence “stops” at some α_0 i.e., $u_\beta = u_{\alpha_0}$ for all $\beta > \alpha_0$.
4. If $\mathfrak{X}$ is compact then $\alpha_0 < \omega_1$ (α_0 is a countable ordinal).

Theorem 2: *u_{α_0} is the smallest repair function $u_{\min}$ for the net θ_κ.*

Proof: First, we will show that u_{α_0} repairs the net $\{\theta_\kappa\}$. Indeed,

$$\lim_{\kappa} \overset{\cdots\cdots}{u_{\alpha_0} + \theta_\kappa} = \lim_{\kappa} \overset{\cdots\cdots}{v_{\alpha_0+1} + \theta_\kappa} = u_{\alpha_0+1} - u_{\alpha_0} = 0.$$

Now we show that u_{α_0} is the smallest repair function. Suppose that $u \geq 0$ repairs the net $\{\theta_\kappa\}$. We have $u_\alpha \leq u$ for $\alpha = 0$. Suppose the same holds for all $\beta < \alpha$. Then $v_\alpha = \sup\limits_{\beta<\alpha} u_\beta \leq u$, hence

$$u_\alpha = \lim\nolimits_\kappa \widetilde{v_\alpha + \theta_\kappa} \leq \lim\nolimits_\kappa \widetilde{u + \theta_\kappa} = \lim\nolimits_\kappa \overset{\cdots\cdots}{u + \theta_\kappa} + u = u.$$

We have proved that $u_\alpha \leq u$ for all α including α_0. $\square$

Superenvelopes

Definition 4: For an increasing net $\{f_\kappa\}$ of nonegative functions, with a finite limit f, a function E is called a **superenvelope** if and only if $E = f + u$, where u is a repair function of the corresponding net of tails.

Equivalently, E is a superenvelope if

- $E \geq f$, and

•• $\overset{\dots\dots}{E - f_k} \to 0$ pointwise.

The smallest repair function u_{min} produces the **smallest superenvelope** $E_{\mathrm{min}} = u_{\mathrm{min}} + f$.

Suppose that the net $\{f_\kappa\}$ (equivalently $\{\theta_\kappa\}$) has the property of *upper semicontinuous differences*, i.e.,

$$\forall \iota > \kappa \qquad f_\iota - f_\kappa \quad (= \theta_\kappa - \theta_\iota) \text{ is upper-semicontinuous,}$$

then (by an easy exercise) the condition (••) takes on a simpler form:

$$\overset{\dots\dots}{E - f_\kappa} = 0 \quad \text{for every } \kappa \in K.$$

This is exactly how superenvelopes were defined by Mike in zero-dimensional dynamics (the *entropy sequence* did have upper semi-continuous differences).

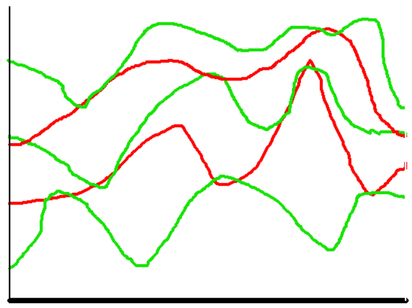
Uniform equivalence

We now introduce an equivalence relation which classifies monotone nets by the *type* of convergence.

Definition 5: Two increasing nets of functions $\{f_\kappa\}$ $(\kappa \in K)$ and $\{g_\iota\}$ $(\iota \in J)$ are *uniformly equivalent* if

$$\forall_{\epsilon>0,\ \kappa\in K,\ \iota\in J}\ \ \exists_{\kappa'\in K,\ \iota'\in J}\ \ f_{\kappa'} \geq g_\iota - \epsilon \ \text{ and } \ g_{\iota'} \geq f_\kappa - \epsilon$$

$$\text{(for decreasing nets: } f_{\kappa'} \leq g_\iota + \epsilon \ \text{ and } \ g_{\iota'} \leq f_\kappa + \epsilon).$$



For example, a monotone net converges uniformly to the limit function f if and only if it is uniformly equivalent to the constant net $\{f\}$ $(n \in \mathbb{N})$.

Theorem 3: Two uniformly equivalent nets have common:

1. limit function f,
2. global defect of uniformity $D_{\mathfrak{X}}$,

   and, if $\mathfrak{X}$ is a metric space, also

3. the defect function D_x,
4. all repair functions and superenvelopes (hence $u_{\min}$ and $E_{\min}$),
5. the entire transfinite sequence (u_α).

Proof of 4.: Let $\{\theta_\kappa\}$ and $\{\theta'_\iota\}$ be the nets of tails of two uniformly equivalent nets $\mathcal{H}$ and $\mathcal{H}'$, respectively. Let u be a repair function for $\{\theta_\kappa\}$. Fix a point $x \in \mathfrak{X}$ and an $\epsilon > 0$. Let κ be such that both

$$\theta_\kappa(x) < \epsilon \quad \text{and} \quad \overset{\cdots\cdots}{(u + \theta_\kappa)}(x) < \epsilon.$$

Let ι be such that $\theta'_\iota < \theta_\kappa + \epsilon$, i.e., $\theta'_\iota - \theta_\kappa < \epsilon$ (at all points). Then also $\widetilde{\theta'_\iota - \theta_\kappa} \leq \epsilon$.

$$\begin{aligned}\overset{\cdots\cdots}{(u + \theta'_\iota)}(x) \leq \overset{\cdots\cdots}{(u + \theta_\kappa)}(x) + \overset{\cdots\cdots}{(\theta'_\iota - \theta_\kappa)}(x) = \\ \overset{\cdots\cdots}{(u + \theta_\kappa)}(x) + (\widetilde{\theta'_\iota - \theta_\kappa})(x) - (\theta'_\iota - \theta_\kappa)(x) \\ \leq \overset{\cdots\cdots}{(u + \theta_\kappa)}(x) + (\widetilde{\theta'_\iota - \theta_\kappa})(x) + \theta_\kappa(x) \leq 3\epsilon. \quad \square\end{aligned}$$

Uniform convergence

Theorem 4: The following are equivalent

- $f_\kappa \to f \quad$ uniformly
- θ_κ is asymptotically upper semicontinuous,
- $u_{\min} = 0$,
- $E_{\min} = f$,
- $\alpha_0 = 0$,
- $\{f_\kappa\}$ is uniformly equivalent to $\{f\}$.

Subnets and sub-nets

If $\{f_\kappa\}$ $(\kappa \in K)$ is a net, and $J \subset K$ satisfies

$\forall_{\kappa \in K} \quad \exists_{\iota \in J} \quad \kappa < \iota$

then J is a directed family and $\{f_\iota\}$ $(\iota \in J)$ is called a *subnet* of $\{f_\kappa\}$.

If $J \subset K$ is just a directed family then we call $\{f_\iota\}$ $(\iota \in J)$ a *sub-net* of $\{f_\kappa\}$.

Unlike for sequences, a sub-net need not be a subnet.
A subnet of a convergent net converges to the same limit. In fact,

Theorem 5: (trivial) All subnets of an increasing net of functions are pairwise uniformly equivalent.

This fails for sub-nets.

Example 3: Fix some $f : \mathfrak{X} \to [0, \infty)$ and let $\{f_\kappa\}$ be the net of all functions $0 \le f_\kappa \le f$, $f_\kappa \ne f$ ordered by the usual inequality between functions. This net converges (increases) *uniformly* to f:

for every ϵ there is κ such that $f_\kappa \ge f - \epsilon$.

In thie example, any subnet converges uniformly to f. But there are plenty of sub-nets converging to other limits or converging to f but not uniformly, so they are not uniformly equivalent to the whole net.

The following theorem is a key tool to establish uniform equivalence between some nets of functions in some specific situations.

Theorem 6: *Let $\{f_\kappa\}$ be an increasing net of nonnegative functions on a compact metric space $\mathfrak{X}$, with upper semicontinuous differences. Let $\{f_\iota\}$ be a sub-net. Then $\{f_\iota\}$ is uniformly equivalent to $\{f_\kappa\}$ if and only if it has the same limit function f.*

Proof: One implication is obvious, since uniformly equivalent nets have the same limit function. It is also obvious that every element of the net $\{f_\iota\}$ is dominated by one from the net $\{f_\kappa\}$, namely by itself.

For the converse, we fix some $\kappa \in K$ and for each $\iota \in J$ choose an index $\kappa \vee \iota \in K$ such that

$\kappa \leq \kappa \vee \iota$ and $\iota \leq \kappa \vee \iota$.

So, for each ι, $f_\iota \leq f_{\kappa\vee\iota} \leq f$, which implies that $f_{\kappa\vee\iota} \underset{\iota}{\longrightarrow} f$. Thus

$$(\star) \qquad f_{\kappa\vee\iota} - f_\iota \quad \underset{\iota}{\longrightarrow} \quad 0.$$

These difference functions are nonnegative, and, by assumption, upper semicontinuous. We intend to use our Fact 2. Recall:

A net of u.s.c. functions decreasing to a continuous limit on a compact domain converges uniformly.

The convergence $(\star)$, however, need not be monotone...

To get a monotone net, with each ι we associate the function

$$g_\iota = \inf_{\iota' \leq \iota}(f_{\kappa \vee \iota'} - f_{\iota'}).$$

Now we have a net of nonnegative upper semicontinuous functions, decreasing to zero on a compact domain. This convergence is already uniform.

So, for every $\epsilon > 0$ there exists some $\iota \in J$ with $g_\iota < \epsilon$.

Since $f_{\kappa \vee \iota'} \geq f_\kappa$ and $f_{\iota'} \leq f_\iota$ for every $\iota' \leq \iota$, we have

$$\epsilon > g_\iota \geq f_\kappa - f_\iota.$$

(The right hand side need not be nonnegative, but it doesn't matter.)

We have proved that $f_\iota > f_\kappa - \epsilon$. $\square$

U.s.d.a.-nets on simplices

Choquet simplex.

Let $\mathbb{K}$ denote a convex set, and let ex$\mathbb{K}$ denote the set of extreme points of $\mathbb{K}$. For a probability distribution ξ on $\mathbb{K}$ the **barycenter** of ξ is a point $\mathrm{bar}(\xi) \in \mathbb{K}$ such that

$$\int f(x) d\xi = f(\mathrm{bar}(\xi))$$

for every continuous affine function f on $\mathbb{K}$.

The map $\mathrm{bar} : \mathcal{M}(\mathbb{K}) \to \mathbb{K}$ is continuous and affine. The Choquet Theorem asserts that $\mathrm{bar} : \mathcal{M}(\mathrm{ex}\mathbb{K}) \to \mathbb{K}$ is a surjection (every point is a generalized average of the extreme points).

Definition 6: $\mathbb{K}$ is a simplex if and only if $\mathrm{bar} : \mathcal{M}(\mathrm{ex}\mathbb{K}) \to \mathbb{K}$ is a bijection. (Attention, since ex$\mathbb{K}$ need not be closed, $\mathcal{M}(\mathrm{ex}\mathbb{K})$ need not be compact and this bijection need not be a homeomorphism.)

Then the inverse of the barycenter map is the *extreme decomposition*. We are familiar with this: every invariant measure on a topological dynamical system has its *ergodic decomposition*. It is the same: $\mathcal{M}_T(X)$ is a simplex, ex$\mathcal{M}_T(X)$ are the ergodic measures.

Definition 7: A u.s.d.a.-net is an increasing net of nonnegative functions $\{f_\kappa\}$ starting with $\kappa = 0$, defined on a simplex $\mathbb{K}$, such that:

- $f_0 = 0$,
- $f_\iota - f_\kappa$ is upper semicontinuous for each $\iota \geq \kappa$

(the net $\{f_\kappa\}$ has upper semicontinuous differences),

- each f_κ is affine.

Note that, since we start with the zero function, not only the differences, but each f_κ is upper semicontinuous.

Theorem 7: If $\{f_\kappa\}$ is a u.s.d.a.-net then the minimal superenvelope equals the pointwise infimum of all **affine superenvelopes**.

$$E_{\min} = \inf\{E_A : E_A \text{ is an affine superenvelope}\}$$

(in particular E_A is concave).

Corollary: The above holds for any increasing net $\{f_\kappa\}$ of nonnegative functions defined on a Choquet simplex, which is uniformly equivalent to a u.s.d.a.-net.

This will be very useful in identifying the *sex entropy function.*

Theorem 8: If $\{f_\kappa\}$ is a u.s.d.a.-net then the pointwise supremum of the minimal superenvelope equals the infimum of pointwise suprema over all affine superenvelopes:

$$\sup_{x\in\mathbb{K}} E_{\min}(x) = \inf\{\sup_{x\in\mathbb{K}} E_A(x) : E_A \text{ is an affine superenvelope}\}.$$

This will be very useful in establishing the *sex entropy variational principle.* We skip the tedious proofs.

The case of a Bauer simplex

A **Bauer simplex** is a simplex $\mathbb{K}$ with $\mathrm{ex}\mathbb{K}$ closed.

The map $\mathrm{bar} : \mathcal{M}(\mathrm{ex}\mathbb{K}) \to \mathbb{K}$ is an affine homeomorphism.

Bauer simplices are precisely the simplices of the form $\mathcal{M}(\mathfrak{X})$, where $\mathfrak{X}$ is compact.

On a Bauer simplex we have a 1-1 correspondence (be restriction) between all **affine** continuous functions on $\mathbb{K}$ and all continuous functions on $\mathrm{ex}\mathbb{K}$ and the same for upper semicontinuous functions and Borel-measurable functions. This correspondence preserves the operations $\widetilde{\quad}$ and $\dddot{\quad}$.

Thus, there is a 1-1 correspondence between all u.s.d.a.-nets on $\mathbb{K}$ and u.s.d.-nets on $\mathrm{ex}\mathbb{K}$, and it preserves limits, repair functions, superenvelopes, transfinite sequence, and uniform equivalence.

Thus all examples of u.s.d.-nets (with some specific behavior) on compact spaces without convex structure lift to analogous examples of u.s.d.a.-nets on Bauer simplices.

But more funny behaviors of u.s.d.a.-sequences can be observed on simplices which are not Bauer. Mike showed examples of some of them:

• the minimal superenvelope need not attain its pointwise supremum on extreme points:

$$\sup_{x \in \mathbb{K}} E_{\min}(x) > \sup_{x \in \operatorname{ex}\mathbb{K}} E_{\min}(x),$$

• the minimal superenvelope need not be affine,
• the poinwise supremum of the minimal superenvelope need not coincide with the poitwise supremum of any affine superenvelope:

$$\sup_{x \in \mathbb{K}} E_{\min}(x) < \sup_{x \in \mathbb{K}} E_A(x), \quad \text{for any affine superenvelope } E_A.$$

None of these pathologies can occur on Bauer simplices.