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Classification of types of convergence of monotone nets of real functions

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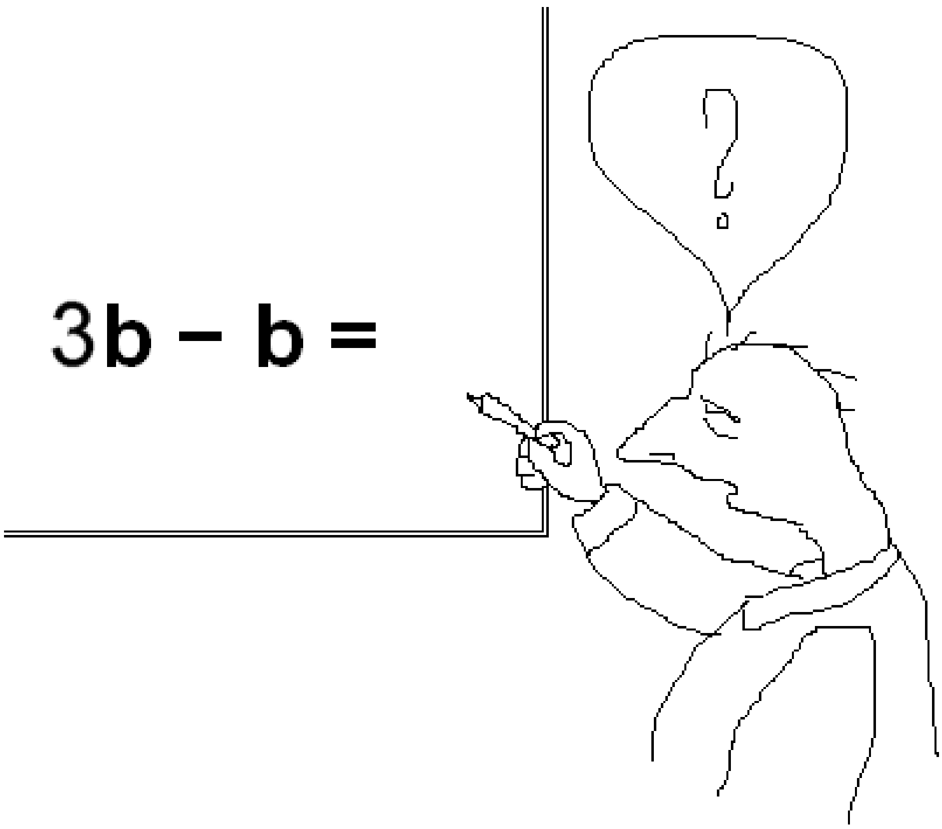
Did you know that William Shakespeare

was a mathematician

involved in difference equations?

Here is how WE know about that:


3b − b =
?

Actually, he never solved this equation.....

Monotone nets of nonnegative functions

$f_\kappa : X \to [0, \infty)$, $\{f_\kappa\}$ is a net (κ ranges over a directed family K).

$\forall \iota > \kappa \;\; f_\iota \geq f_\kappa$ (a nondecreasing net - we will say *increasing*)

$f_\kappa \nearrow f : X \to [0, \infty]$ (pointwise convergence)

We will assume that f is finite everywhere.

This convergence is either uniform, i.e.,

for every ϵ there is κ such that $f_\kappa \geq f - \epsilon$,

or not. Can we distinguish between non-uniform convergences?

The non-uniformity is measured by

$D_X = \lim_{\kappa \in K} \downarrow \sup_{x \in X} (f(x) - f_\kappa(x))$ (the *global defect of uniformity*)

Motivation - entropy theory in topological dynamics. $D_X = \mathbf{h}^*$.

If X is a metric space, we can localize this parameter:

$$D_x = \inf_{\epsilon>0} \lim_{\kappa\in K} \downarrow \sup_{y\in B(x,\epsilon)} (f(y) - f_\kappa(y)) =$$
$$\lim_{\kappa\in K} \downarrow \inf_{\epsilon>0} \sup_{y\in B(x,\epsilon)} (f(y) - f_\kappa(y)).$$

For a function $f : X \to \mathbb{R}$, the function $\widetilde{f}(x) := \inf_{\epsilon>0} \sup_{y\in B(x,\epsilon)} f(y)$ is called the *upper-semicontinuous envelope* of f. Thus,

$$D_x = \lim_{\kappa\in K} \downarrow (\widetilde{f - f_\kappa})(x).$$

Upper semicontinuous functions

A function $f : X \to \mathbb{R}$ is *upper semicontinuous* if one of the following equivalent conditions holds

1. $\forall x \in X \quad \limsup_{y \to x} f(y) \leq f(x) \quad$ (f is u.s.c. *at* x);

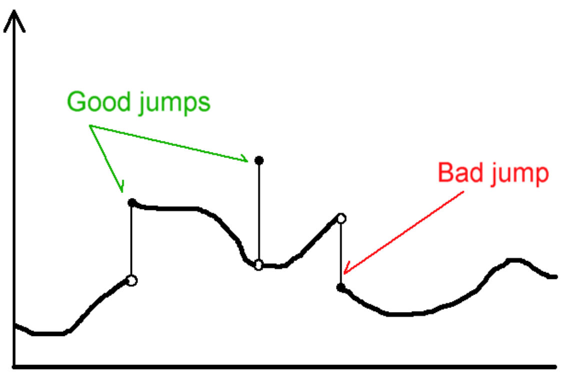
2. $\forall a \in \mathbb{R} \quad f^{-1}((-\infty, a))$ is open (or $f^{-1}([a, \infty))$ is closed);

3. the area above the graph is open
(or the area below and on the graph is closed);

4. f is a pointwise infimum of a family of continuous functions.

5. f is a pointwise limit of a decreasing net of continuous functions.

6. $f = \widetilde{f}$



The upper semicontinuous envelope has already been introduced: $\widetilde{f}(x) = \limsup_{y \to x} f(y)$.

We also have: $\widetilde{f} = \inf\{g : g \geq f, g \text{ is continuous}\}$.

For a function $f : X \to \mathbb{R}$ we also define

$\dddot{f} = \widetilde{f} - f$

and we call it the *defect of upper-semicontinuity function* (or just *defect*).

On a compact domain every u.s.c. function is bounded above. Moreover, we have the following *exchange of suprema and infima* statement

Fact 1: If g_κ is a decreasing net of nonnegative u.s.c. functions on a compact metric domain X, then

$$\sup_{x\in X} \lim_{\kappa} \downarrow g(x) = \lim_{\kappa} \downarrow \sup_{x\in X} g(x).$$

We will also need this:

Fact 2: *A net of u.s.c. functions decreasing to a continuous limit on a compact domain converges uniformly.*

Proof: The sets $f_\kappa - f \geq \epsilon$ are closed (hence compact) and decrease, their intersection is empty, so only finitely many of them are nonempty. $\square$

We go back to the increasing net f_κ. Recall, that we have

$$D_x = \lim_{\kappa \in K} \downarrow \widetilde{(f - f_\kappa)}(x).$$

We easily see that $D_x \leq D_X$.

If $D_x = 0$ for all x then we say that the convergence is *locally uniform.* In general, this does not imply uniform convergence. However,

Theorem 1: *If X is compact, then $D_X = \sup_{x \in X} D_x$.*

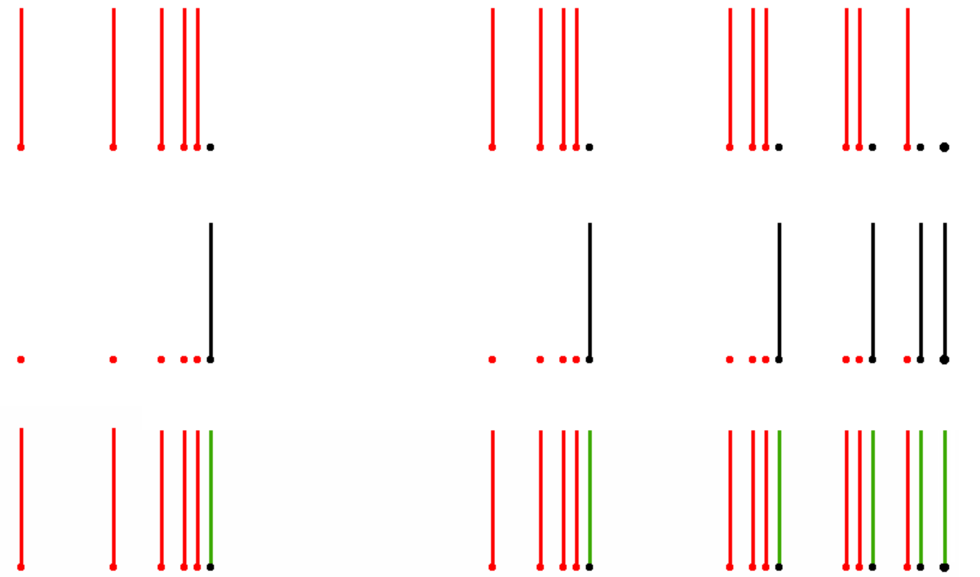
Proof: We have $D_X = \lim_{\kappa \in K} \downarrow \sup_{x \in X} (f - f_\kappa)(x) \leq$

$\lim_{\kappa \in K} \downarrow \sup_{x \in X} \widetilde{(f - f_\kappa)}(x) = \sup_{x \in X} \lim_{\kappa \in K} \downarrow \widetilde{(f - f_\kappa)}(x) = \sup_{x \in X} D_x$. $\square$

It suffices to study the net of *tails* $\{\theta_\kappa\}$, where $\theta_\kappa = f - f_\kappa$. This net decreases to zero: $\theta_\kappa \searrow 0$.

The global defect of uniformity D_X or the defect function D_x do not distinguish all possible *types* of convergence.

Example 1: The "pick-up sticks game". Top figure: Order all the red sticks anyhow by the naturals. Let θ_n be the function obtained by *removing* the first n sticks. Bottom figure: The same game with red and green sticks together. Middle picture: the defect function is the same in both games.



The first (top) example restricted to the set of points with positive defect (the black points) has no defect (all the functions are zero). The second (bottom) example restricted to the same set generates the defect *of the second order* at the rightmost point. It is clear that the two examples represent different "types" of nonuniformity.

Another approach:

We have

$$D_x = \lim_{\kappa} \downarrow \widetilde{\theta}_k(x) = \lim_{\kappa} \downarrow \widetilde{\theta}_k(x) - \lim_{\kappa} \downarrow \theta_k(x)$$
$$= \lim_{\kappa} (\widetilde{\theta}_k - \theta_\kappa)(x) = \lim_{\kappa} \dddot{\theta}_\kappa(x)$$

Thus, we can call D_x the *persistent defect of upper-semicontinuity* at x.

But this persistent defect is insufficient to capture the complexity of the convergence.

Asymptotic upper-semicontinuity

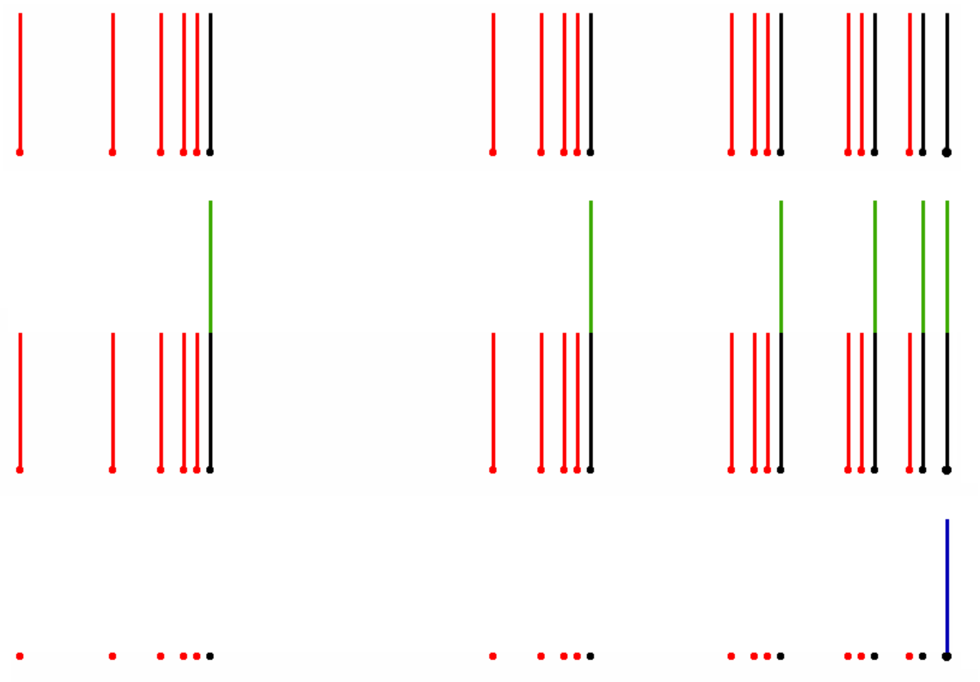
For a single function f we have $\overset{\cdots\cdots\cdots\cdots}{\dddot{f} + f} = \overset{\cdots\cdot}{\widetilde{f}} = 0$, so by adding the defect function $\dddot{f}$ we have *repaired* the function f (made it u.s.c.).

We will say that a function u *repairs* the net $\{\theta_\kappa\}$ if $\overset{\cdots\cdots\cdots\cdots}{u + \theta}_\kappa \underset{\kappa}{\longrightarrow} 0$. (i.e., the net $\{u + \theta_\kappa\}$ is *asymptotically upper-semicontinuous*).

Notice that $\theta_\kappa \searrow 0$ locally uniformly if and only if $\dddot{\theta}_\kappa \underset{\kappa}{\longrightarrow} 0$, so the net requires no reparation. By repairing a net we create a net which is in a sense closer to being (locally) uniformly convergent.

A good candidate to be a "repair function" seems to be the persistent defect, i.e., the function $u_1(x) = D_x$.

It is so in the first example, but not in the second example:



An attempt to repair the net $\{\theta_\kappa\}$ by adding the function

$u_1(x) = D_x.$

It is succesful in the first example, and fails in the second one. The bottom picture shows the persistent defect of the net $\{u_1 + \theta_\kappa\}$.

(We will call it *the defect of the second order*).

Finding the smallest possible repair function u is the most important issue in this talk. The task is equivalent to "solving" the "difference equation" for the unknown function $E \geq f$:

$$\text{(1)} \qquad \overset{\cdots\cdots\cdots}{E - f_\kappa} \underset{\kappa}{\longrightarrow} 0.$$

Any function E with this property is called a *superenvelope* of the net $\{f_\kappa\}$. We are looking for the *smallest* superenvelope. The smallest repair function for $\{\theta_\kappa\}$ then equals $u = E - f$, simply because

$$u + \theta_\kappa = (E - f) + (f - f_\kappa) = E - f_\kappa.$$

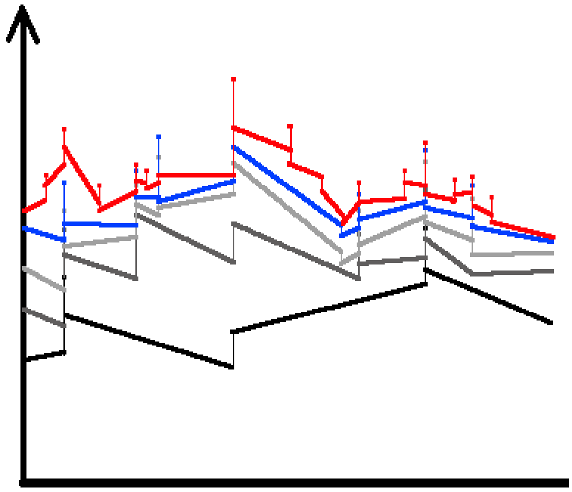
Note that a *finite* such function E (equivalently u) need not exist. If it doesn't, we say the the net is *unrepairable.*

If the net $\{f_\kappa\}$ (equivalently $\{\theta_\kappa\}$) has the property of *upper-semicontinuous differences*, i.e.,

$$\forall \iota > \kappa \qquad f_\iota - f_\kappa \quad (= \theta_\kappa - \theta_\iota) \;\; \text{is upper-semicontinuous,}$$

then (by an easy exercise) the condition (1) takes on a simpler form:

$$\overset{\cdots\cdots\cdots}{E - f_\kappa} = 0 \quad \text{for every } \kappa \in K.$$



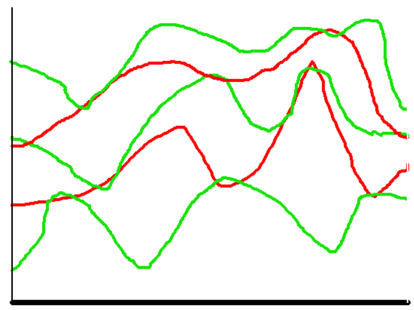
We will solve the equation (1) using transfinite induction.

Before we proceed, we introduce an equivalence relation which classifies monotone nets by the *type* of convergence.

Definition 1: Two increasing nets of functions $\{f_\kappa\}$ $(\kappa \in K)$ and $\{g_\iota\}$ $(\iota \in J)$ are *uniformly equivalent* if

$$\forall_{\epsilon>0,\ \kappa\in K,\ \iota\in J}\ \ \exists_{\kappa'\in K,\ \iota'\in J}\ \ f_{\kappa'} \geq g_\iota - \epsilon \ \text{ and } \ g_{\iota'} \geq f_\kappa - \epsilon$$

(for decreasing nets: $f_{\kappa'} \leq g_\iota + \epsilon$ and $g_{\iota'} \leq f_\kappa + \epsilon$).



For example, a monotone net converges uniformly to the limit function f if and only if it is uniformly equivalent to the constant net $\{f\}$ $(n \in \mathbb{N})$.

Two uniformly equivalent nets have common:

1. limit function f,
2. global defect of uniformity D_X,

   and, if X is a metric space, also

3. the defect function D_x,
4. all the superenvelopes E (hence also the smallest one).

Proof of 4.: Let E be a superenvelope of $\{f_\kappa\}$. The condition that $E \geq$ the limit function is satisfied for both nets. Fix a point $x \in X$ and a $\gamma > 0$. Let κ be such that both

$$f_\kappa(x) > f(x) - \gamma \quad \text{and} \quad \overset{\cdots\cdots\cdots}{(E - f_\kappa)}(x) < \gamma.$$

Let ι' be such that $g_{\iota'} > f_\kappa - \gamma$, i.e.,

$f_\kappa - g_{\iota'} < \gamma$ (at all points).

Then $\widetilde{f_\kappa - g_{\iota'}} \leq \gamma$. Since $g_{\iota'}(x) \leq f(x)$, we also have

$g_{\iota'}(x) - f_\kappa(x) < \gamma$.

Now we have

$$\overset{\cdots\cdots\cdots}{(E - g_{\iota'})}(x) \leq \overset{\cdots\cdots\cdots}{(E - f_\kappa)}(x) + \overset{\cdots\cdots\cdots}{(f_\kappa - g_{\iota'})}(x) =$$
$$\overset{\cdots\cdots\cdots}{(E - f_\kappa)}(x) + (\widetilde{f_\kappa - g_{\iota'}})(x) - (f_\kappa(x) - g_{\iota'}(x)) \leq 3\gamma.$$

We have proved that E is a superenvelope of $\{g_\iota\}$. $\square$

Subnets and sub-nets

If $\{f_\kappa\}$ $(\kappa \in K)$ is a net, and $J \subset K$ satisfies

$$\forall_{\kappa \in K} \quad \exists_{\iota \in J} \quad \kappa < \iota$$

then J is a directed family and $\{f_\iota\}$ $(\iota \in J)$ is called a *subnet* of $\{f_\kappa\}$.

If $J \subset K$ is just a directed family then we call $\{f_\iota\}$ $(\iota \in J)$ a *sub-net* of $\{f_\kappa\}$.

Unlike for sequences, a sub-net need not be a subnet.
A subnet of a convergent net converges to the same limit. In fact,

Theorem: All subnets of an increasing net of functions are pairwise uniformly equivalent.

This fails for sub-nets.

Example: Fix some $f : X \to [0, \infty)$ and let $\{f_\kappa\}$ be the net of all functions $0 \leq f_\kappa \leq f$ ordered by the usual inequality between functions. This net converges (increases) *uniformly* to f:

for every ϵ there is κ such that $f_\kappa \geq f - \epsilon$.

In thie example, any subnet converges uniformly to f. But there are plenty of sub-nets converging to other limits or converging to f but not uniformly, so they are not uniformly equivalent to the whole net.

The following theorem is a key tool to establish uniform equivalence between some nets of functions in some specific situations.

Theorem: *Let $\{f_\kappa\}$ be an increasing net of nonnegative functions on a compact metric space X, with upper-semicontinuous differences. Let $\{f_\iota\}$ be a sub-net. Then $\{f_\iota\}$ is uniformly equivalent to $\{f_\kappa\}$ if and only if it has the same limit function f.*

Proof: One implication is obvious, since uniformly equivalent nets have the same limit function. It is also obvious that every element of the net $\{f_\iota\}$ is dominated by one from the net $\{f_\kappa\}$, namely by itself.

For the converse, we fix some $\kappa \in K$ and for each $\iota \in J$ choose an index $\kappa(\iota) \in K$ such that

$\kappa \leq \kappa(\iota)$ and $\iota \leq \kappa(\iota)$.

So, for each ι, $f_\iota \leq f_{\kappa(\iota)} \leq f$, which implies that $f_{\kappa(\iota)} \underset{\iota}{\longrightarrow} f$. Thus

$$(2) \qquad f_{\kappa(\iota)} - f_\iota \quad \underset{\iota}{\longrightarrow} \quad 0.$$

Also, by assumption, these difference functions are u.s.c..
We intend to use the (already mentioned) fact:

Fact: *A net of u.s.c. functions decreasing to a continuous limit on a compact domain converges uniformly.*

The convergence (2), however, need not be monotone...

To get a monotone net, with each ι we associate the function

$$g_\iota = \inf_{\iota' \leq \iota}(f_{\kappa(\iota')} - f_{\iota'}).$$

Now we have a net of nonnegative u.s.c. functions a decreasing to zero on a compact domain. This convergence is already uniform.

So, for every $\epsilon > 0$ there exists some $\iota \in J$ with $g_\iota < \epsilon$.

Since $f_{\kappa(\iota')} \geq f_\kappa$ and $f_{\iota'} \leq f_\iota$ for every $\iota' \leq \iota$, we have

$$\epsilon > g_\iota \geq f_\kappa - f_\iota.$$

(The right hand side need not be nonnegative, but it doesn't matter.)

We have proved that $f_\iota > f_\kappa - \epsilon$. $\square$

The transfinite solution

Consider the net $\theta_\kappa \searrow 0$ defined on a metric space X.

Definition 2: The *transfainite sequence* u_α (α are the ordinals) associated with the net θ_k is defined as follows:

$u_\alpha = 0,$

and for an ordinal α such that u_α are already defined for all $\beta < \alpha$ let

$$v_\alpha = \sup_{\beta<\alpha} u_\beta.$$

If v_α is infinite at some point, we let $u_\alpha \equiv \infty$. Otherwise,

$$u_\alpha = v_\alpha + \lim_\kappa \overset{\cdots\cdots\cdots\cdots}{v_\alpha + \theta_\kappa}.$$

(Notice that $u_\alpha = v_\alpha + \lim_\kappa \widetilde{v_\alpha + \theta_\kappa} - v_\alpha - \lim_\kappa \theta_\kappa = \lim_\kappa \downarrow \widetilde{v_\alpha + \theta_\kappa}$.)

Interpretation: $u_1 = \lim_\kappa \overset{\cdots}{\theta}_\kappa$ is the persistent defect of the first order ($= D_x$). Then $\lim_\kappa \overset{\cdots\cdots\cdots\cdots}{v_\alpha + \theta_\kappa}$ is the *persistent defect of order* α; the defect of the net obtained (in the limit) by "repairing" θ_κ using u_β where $\beta < \alpha$. Thus u_α is the *cumulative defect of order* α.

Fundamental facts:
1. The transfinite sequence u_α increases. In particular, $v_{\alpha+1} = u_\alpha$.
2. This sequence "stops" at some α_0 i.e., $u_\beta = u_{\alpha_0}$ for all $\beta > \alpha_0$.
3. If X is compact then $\alpha_0 < \omega_1$ (α_0 is a countable ordinal).
4. The entire sequence is an invariant of uniform equivalence.

Theorem 2: *If u_{α_0} is finite then it is the smallest repair function for the net θ_κ.*

Proof: First, we will show that u_{α_0} repairs the net θ_κ. Indeed,

$$\lim_\kappa \overset{\cdots\cdots\cdots}{u_{\alpha_0} + \theta_\kappa} = \lim_\kappa \overset{\cdots\cdots\cdots\cdots}{v_{\alpha_0+1} + \theta_\kappa} = u_{\alpha_0+1} - u_{\alpha_0} = 0.$$

Now we show that u_{α_0} is the smallest repair function. Suppose that $u \geq 0$ repairs the net θ_k. We have $u_\alpha \leq u$ for $\alpha = 0$. Suppose the same holds for all $\beta < \alpha$. Then $v_\alpha = \sup\limits_{\beta<\alpha} u_\beta \leq u$, hence

$$u_\alpha = \lim_\kappa \widetilde{v_\alpha + \theta_\kappa} \leq \lim_\kappa \widetilde{u + \theta_\kappa} = \lim_\kappa \overset{\cdots\cdots}{u + \theta_\kappa} + u = u.$$

We have proved that $u_\alpha \leq u$ for all α including α_0. $\square$

Corollary: If $\theta_\kappa = f - f_\kappa$ for an increasing net f_κ then $E = u_{\alpha_0} + f$ is the smallest superenvelope of f_κ.

The order of accumulation of the defects

Definition 3: We call α_0 *the order of accumulation of the defects* of the net f_κ (or of θ_κ is one prefers). For $x \in X$ we define $\alpha(x)$ as the smallest ordinal α such that $u_\alpha(x) = u_{\alpha_0}(x)$, and call it *the order of accumulation of the defects at* x.

Clearly, $\alpha_0 = \sup_{x \in X} \alpha(x)$.

In the unrepairable case α_0 is the smallest ordinal such that $u_\alpha \equiv \infty$ and then $\alpha(x) = \alpha_0$ for every x.

The ordinal α_0 and the ordinal function $x \mapsto \alpha(x)$ are invariants of the uniform equivalence relation.

Topological order of accumulation in metric spaces.
Recall that a point $x \in X$ (metric space) has the *topological order of accumulation* 0 if it is an isolated point. The collection of all such points is an open set. Suppose we have determined all points of order of accumulation β for all $\beta < \alpha$ and their set X_α is open. A point has *topological order of accumulation* α if it isolated (relatively) in the complement of X_α. We denote by $\operatorname{ord}(x)$ the topological order of accumulation of x (whenever it is defined). If $\operatorname{ord}(x)$ is defined at every point then we set $\operatorname{ord}(X) = \sup_{x \in X} \operatorname{ord}(x)$ and call it the *topological order of accumulation of* X.

Theorem 3: *Let f_κ be an increasing net of nonnegative functions on a compact metric space X. Then, $\alpha(x) \leq \operatorname{ord}(x)$ whenever $\operatorname{ord}(x)$ is defined. In particular, $\alpha_0 \leq \operatorname{ord}(X)$, if the last is defined.*

Proof: Suppose $\operatorname{ord}(x)$ is defined. We will show that x has an open neighborhood U_x where $u_{\operatorname{ord}(x)+1} \equiv u_{\operatorname{ord}(x)}$. This easily implies, that the transfinite sequence at x does not grow above $u_{\operatorname{ord}(x)}$.

If x is an isolated point, then $U_x = \{x\}$ is a neighborhood of x and, since $\widetilde{\theta}_\kappa(x) = \theta_\kappa(x)$ for every κ, we have $u_1(x) = 0$, i.e., $u_{\operatorname{ord}(x)+1} \equiv u_{\operatorname{ord}(x)}$ on U_x.

Suppose we have proved that every point x' with $\operatorname{ord}(x') < \alpha$ has a neighborhood on which $u_{\operatorname{ord}(x')+1} \equiv u_{\operatorname{ord}(x')}$. Then, as we know,

$u_{\alpha+1}(x') = u_\alpha(x')$.

Now take a point x with $\operatorname{ord}(x) = \alpha$. There is a neighborhood U_x of x which contains only x and points x' of topological order smaller than α, for which $u_{\alpha+1}(x') = u_\alpha(x')$. It remains to check that $u_{\alpha+1} = u_\alpha$ at the point x. We have

$$u_{\alpha+1}(x) = \lim_{\kappa} \downarrow \widetilde{(u_\alpha + \theta_\kappa)}(x) = \inf_{\kappa} \inf_{V} \sup_{y \in V} (u_\alpha + \theta_\kappa)(y) =$$
$$\inf_{V} \inf_{\kappa} \sup_{y \in V} (u_\alpha + \theta_\kappa)(y),$$

where V ranges over all neighborhoods of x such that $\overline{V} \subset U_x$.

Consider two cases:

(a) If for some V there is a subnet along which the last supremum is attained at x. Restricting to this V and this subnet, we can write

$$u_{\alpha+1}(x) \leq \inf_{\kappa} \, (u_\alpha + \theta_\kappa)(x) = u_\alpha(x).$$

(b) Otherwise, for every V and all sufficiently large κ, we can replace the supremum over V by the supremum over $\overline{V} \setminus \{x\}$. On this set $u_\alpha = \sup_{\beta<\alpha} u_\beta = v_\alpha$. Thus

$$u_{\alpha+1}(x) \le \inf_V \lim_\kappa \downarrow \sup_{y \in \overline{V} \setminus \{x\}} (v_\alpha + \theta_\kappa)(y)$$
$$\le \inf_V \lim_\kappa \downarrow \sup_{y \in \overline{V} \setminus \{x\}} (\widetilde{v_\alpha + \theta_\kappa})(y) \le \inf_V \lim_\kappa \downarrow \sup_{y \in \overline{V}} (\widetilde{v_\alpha + \theta_\kappa})(y).$$

Since $\overline{V}$ is compact, we can exchange $\lim\downarrow$ with sup in the last expression. This yields

$$u_{\alpha+1}(x) \le \inf_V \sup_{y \in \overline{V}} \lim_\kappa \downarrow (\widetilde{v_\alpha + \theta_\kappa})(y) = \inf_V \sup_{y \in \overline{V}} u_\alpha(y) = \widetilde{u}_\alpha(x) = u_\alpha(x),$$

because u_α is upper-semicuntinuous. □

Final remarks:

There exist examples of nets (sequences) on countable compact spaces of any countable order of accumulation $\alpha(X)$ such that

$$\alpha(x) = \mathrm{ord}(x)$$

at every point. These example are of the "pick-up sticks game" type.

The models where all sticks have the same length work for $\mathrm{ord}(X) \leq \omega_0$. They are completely analogous to the second example of the pick-up sticks game on the space with $\mathrm{ord}(X) = 2$.

Larger $\mathrm{ord}(X)$ requires varying lengths of the sticks. Already for $\omega_0 + 1$ the construction is too complicated to be explained in this talk.

That's all, thank you.