BASIC FACTS CONCERNING ACTIONS OF AMENABLE GROUPS ON COMPACT SPACES

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based on the seminal paper by

D. Ornstein and B. Weiss

Entropy and isomorphism theorems for actions of amenable groups

J. d'Anal. Math., 48 (1987), 1–141.

and a joined work with

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PRELIMINARIES ON AMENABLE GRUOPS

A group G is **amenable** if there exists a finitely additive left-invariant probability measure on G. Abelian groups, nilpotent groups, solvable groups, groups with polynomial or subexponential growth are amenable. A group that contains the free subgroup with two generators is not amenable.

Here we will use this equivalent definition:

DEFINITION 1. A countable, infinite, discrete group G is called **amenable** if it has a **Følner sequence** i.e., a sequence $(F_n)_{n\geq 1}$ of finite sets $F_n \subset G$ $(n \geq 1)$ satisfying, for every $g \in G$, the condition

$$\frac{|F_n \cap gF_n|}{|F_n|} \underset{n\to\infty}{\longrightarrow} 1.$$

- $gF = \{gf : f \in F\}$
- $|\cdot|$ denotes the cardinality of a set

A related very important notion is this:

DEFINITION 2: Let E be a finite subset of G and choose $\delta \in (0,1)$. We will say that a finite set F is (E,δ)**-invariant** if

$$\frac{|F \triangle EF|}{|F|} \leq \delta,$$

- $EF = \{ef : e \in E, f \in F\}$
- $\triangle$ denotes the symmetric difference

Some trivial observations

- If E contains the unity of G then (E, δ)-invariance is just the condition

$$|EF| \leq (1+\delta)|F|.$$

- If a set F is (E, δ)-invariant, so is Fg, for every $g \in G$.

- It is clear, that if (F_n) is a Følner sequence then for every finite set $E \subset G$ and every $\delta > 0$, F_n is *eventually* (i.e., for sufficiently large n) (E, δ)-invariant.

- If (F_n) is a Følner sequence and E is a finite set then both (EF_n) and $(E \cup F_n)$ are Følner sequences as well. (In this manner we can easily produce a Følner sequence containing the unity.)

DEFINITION 3: Fix an arbitrary (usually infinite) set $H \subset G$. For every finite set F we will denote

$$D_F(H) = \inf_{g \in G} \frac{|H \cap Fg|}{|F|}$$

(notice that the multiplication by g is now on the right) and we define

$$D(H) = \sup\{D_F(H) : F \subset G, |F| < \infty\}.$$

$D(H)$ will be called the **lower Banach density** of H.

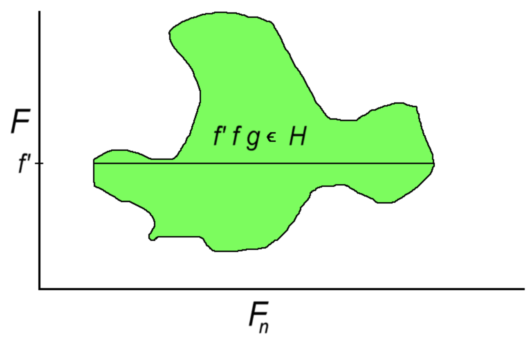
LEMMA 1: If (F_n) is a Følner sequence then for every set $H \in G$ we have

$$D(H) = \lim_{n \to \infty} D_{F_n}(H).$$

Proof. Fix some $\delta > 0$ and let F be a finite set such that $D_F(H) \geq D(H) - \delta$. Let n be so large that F_n is (F, δ)-invariant. Given $g \in G$, we have

$$\frac{|H \cap Ffg|}{|F|} \geq D_F(H),$$

for every $f \in F_n$. This implies that there are at least $D_F(H)|F||F_n|$ pairs (f', f) with $f' \in F, f \in F_n$ such that $f'fg \in H$. This in turn implies that there exists at least one $f' \in F$ for which there are not less than $D_F(H)|F_n|$ corresponding fs in F_n (see figure),



i.e., $|H \cap f'F_n g| \geq D_F(H)|F_n|$.

Since $f' \in F$ and F_n is (F, δ)-invariant (and hence so is $F_n g$), we have

$$|H \cap f' F_n g| \leq |H \cap F F_n g| \leq |H \cap F_n g| + \delta |F_n|,$$

which yields

$$|H \cap F_n g| \geq (D_F(H) - \delta)|F_n|.$$

We have proved that $D_{F_n}(H) \geq D_F(H) - \delta \geq D(H) - 2\delta$, which ends the proof. □

DEFINITION 4: Let $\{A_\alpha\}$ be a (possibly infinite) family of finite sets. We will say that this family is **ε-disjoint** if there exist pairwise disjoint sets $A'_\alpha \subset A_\alpha$ such that, for every α,

$$|A'_\alpha| \geq (1-\varepsilon)|A_\alpha|.$$

The following lemma plays the key role in many dynamical constructions (entropy, topological entropy, symbolic extensions, etc.)

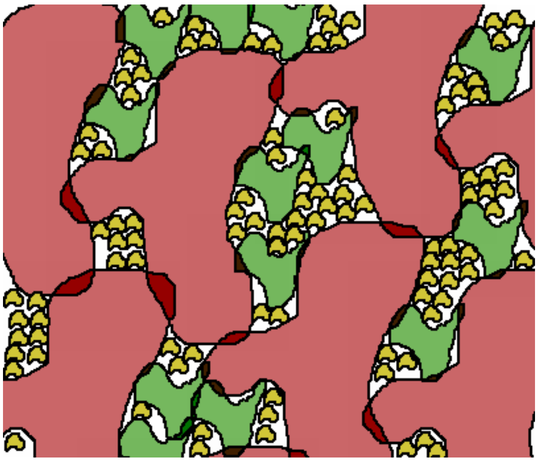
LEMMA 2: Let (F_n) be a Følner sequence. Then for every $\varepsilon \in (0, \frac{1}{2})$ and $n_0 \in \mathbb{N}$ there exist $k \geq 1$ and some numbers $n_k \geq n_{k-1} \geq \cdots \geq n_1 = n_0+1$, and sets $C_k, C_{k-1}, \ldots, C_1$ contained in G such that the family

$$\{F_{n_i}c : 1 \leq i \leq k,\ c \in C_i\}$$

is ε-disjoint, and its union

$$H = \bigcup_{i=1}^{k} F_{n_i}C_i$$

has lower Banach density larger than $1 - \varepsilon$.



Proof. Too long!

SUBADDITIVITY

Let us consider a a non-negative function f defined on finite subsets of G.

We say that f is **monotone** if $F \subset F' \implies f(F) \leq f(F')$.

We say that f is **left-invariant** if $f(F) = f(Fg)$ for any $g \in G$.

We say that f is **subadditive** if $f(F \cup F') \leq f(F) + f(F')$.

EXAMPLES:

• Given a subset $H \subset G$, the function $f(F) = \sup_g |H \cap Fg|$ is non-negative, monotone, left-invariant and subadditive. This function is used to define **upper Banach density**.

• In a classical dynamical system (X, T) the functions

$$f(F) = \mathbf{H}(\mathcal{U}^F)$$

or

$$f(F) = H_\mu(\mathcal{P}^F)$$

are non-negative, monotone, left-invariant and subadditive.

THEOREM 1: Let f be a non-negative, monotone, left-invariant, subadditive function on finite subsets of G. Then the limit

$$\lim_{n\to\infty} \frac{f(F_n)}{|F_n|}$$

exists for every Følner sequence (F_n) and does not depend on that sequence.

Proof. Take two Følner sequences (F_n) and (F'_n). It suffices to show that

$$\liminf_{n\to\infty} \frac{f(F'_n)}{|F'_n|} \geq \limsup_{n\to\infty} \frac{f(F_n)}{|F_n|}.$$

For a subsequence (n_k), we have

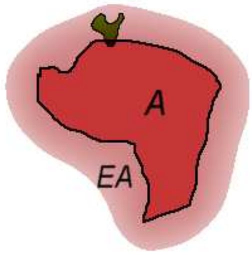
$$\liminf_{n\to\infty} \frac{f(F'_n)}{|F'_n|} = \lim_{k\to\infty} \frac{f(F'_{n_k})}{|F'_{n_k}|}.$$

Since (F'_{n_k}) is also a Følner sequence, it now suffices to prove that, for arbitrary Følner sequences the following holds:

$$\limsup_{n\to\infty} \frac{f(F'_n)}{|F'_n|} \geq \limsup_{n\to\infty} \frac{f(F_n)}{|F_n|}.$$

Fix an arbitrary n and find the ε-disjoint cover $H = \bigcup_{i=1}^{k} F'_{n_i} C_i$ as in Lemma 2, with $D(H) > 1-\varepsilon$ and all n_i larger than n.

There exists a finite set E such that if any set A intersects some $F'_{n_i}c$ then EA contains it ($E = \bigcup_{i=1}^{k} F'_{n_i} {F'}_{n_i}^{-1}$ is good).



Let n_0 be such that

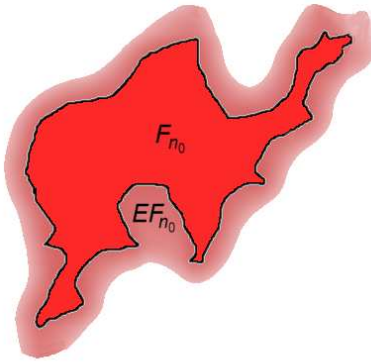
- F_{n_0} is (E, δ)-invariant
- $D_{F_{n_0}}(H) > 1 - \varepsilon$
- $\frac{f(F_{n_0})}{|F_{n_0}|} \geq \limsup_n \frac{f(F_n)}{|F_n|} - \varepsilon.$

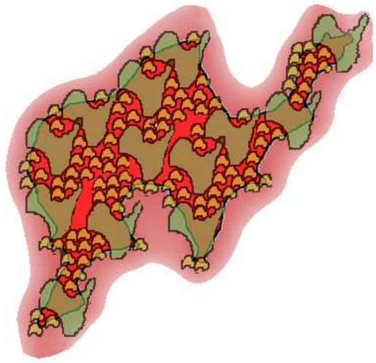
Then we have

$$\limsup_{n\to\infty}\frac{f(F_n)}{|F_n|}\lesssim\frac{f(F_{n_0})}{|F_{n_0}|}\leq\frac{f(EF_{n_0})}{|F_{n_0}|}\approx\frac{f(EF_{n_0})}{|EF_{n_0}|}\leq$$

$$\frac{\sum_{i=1}^{k}b_if(F'_{n_i})+b_0f(\{g\})}{|EF_{n_0}|}\approx$$

$$\frac{\sum_{i=1}^{k}b_if(F'_{n_i})}{\sum_{i=1}^{k}b_i|F'_{n_i}|}\in\operatorname{conv}\left\{\frac{f(F'_{n_i})}{|F'_{n_i}|}:i=1,\ldots,k\right\}\leq\sup_{m>n}\frac{f(F'_m)}{|F'_m|}$$





This implies the desired inequality

$$\limsup_{n\to\infty}\frac{f(F_n)}{|F_n|}\leq\limsup_{n\to\infty}\frac{f(F'_n)}{|F'_n|}.$$

□

BASICS ON ACTIONS OF AMENABLE GROUPS

THEOREM 2: Suppose a countable, discrete, amenable group G acts by homeomorphisms (denoted ϕ_g) on a compact metric space X. Then there exists a Borel probability measure μ on X *invariant under the action*, i.e. which satisfies $\phi_g(\mu) = \mu$ for all $g \in G$.

- $\phi_g(\mu)$ is defined by the formula $\phi_g(\mu)(A) = \mu(\phi_g^{-1}(A))$.

Proof. Let ξ be any Borel probability measure on X and choose a Følner sequence (F_n). Set

$$M_n(\xi) = \frac{1}{|F_n|} \sum_{g \in F_n} \phi_g(\xi).$$

Clearly, this is a probability measure on X. By compactness (in the weak-star topology) of the collection of all probability measures, the sequence $M_n(\xi)$ has an accumulation point μ. Using the defining property of the Følner sequence, one easily verifies that μ is invariant. $\square$

Elementary facts

• The set of all invariant probability measures is convex and compact in the weak-star topology.

• The extreme points of this compact convex set are precisely the ergodic measures, i.e., measures giving to any invariant Borel set either the value 0 or 1.
(A set A is invariant if $\phi_g(A) = A$ for every $g \in G$.)

• An analog of the Birkhoff Ergodic Theorem holds:
If μ is an ergodic measure and φ in an absolutely integrable function then

$$\int \varphi \, dM_n(\delta_x) = \frac{1}{|F_n|} \sum_{g \in F_n} \varphi(\phi_g(x)) \underset{n \to \infty}{\longrightarrow} \int \varphi \, d\mu.$$

This holds only for Følner sequences (F_n) satisfying an additional *Shulman Condition* (I'll skip it). Every Følner sequence has a subsequence with this property.

ENTROPY AND TOPOLOGICAL ENTROPY

Let $\mathcal{U}$ and $\mathcal{P}$ be a finite open cover and a finite measurable partition of X, respectively. Set

$$\mathbf{H}(\mathcal{U}) = \log N(\mathcal{U}),$$

(where $N(\mathcal{U})$ is the minimal cardinality of a subcover of $\mathcal{U}$) and

$$H_\mu(\mathcal{P}) = -\sum_{A\in\mathcal{P}} \mu(A)\log(\mu(A)).$$

For a finite set $F \subset G$ denote

$$\mathcal{U}^F = \bigvee_{g\in F} \phi_g^{-1}(\mathcal{U}) \quad \text{and} \quad \mathcal{P}^F = \bigvee_{g\in F} \phi_g^{-1}(\mathcal{P}).$$

The functions $f(F) = \mathbf{H}(\mathcal{U}^F)$ and $g(F) = H_\mu(\mathcal{P}^F)$ are non-negative, monotone, left-invariant and subadditive. By Theorem 1, the limits

$$\mathbf{h}(\mathcal{U}) = \limsup_{n\to\infty} \frac{\mathbf{H}(\mathcal{U}^{F_n})}{|F_n|} \quad \text{and} \quad h_\mu(\mathcal{P}) = \limsup_{n\to\infty} \frac{H_\mu(\mathcal{P}^{F_n})}{|F_n|}$$

exist and do not depend on the choice of the Følner sequence (F_n). Finally, we define

$$\mathbf{h}(G\text{-action}) = \sup_{\mathcal{U}} \mathbf{h}(\mathcal{U}) \quad \text{and} \quad h_\mu(G\text{-action}) = \sup_{\mathcal{P}} h_\mu(\mathcal{P}).$$

KNOWN FACTS

• If a partition $\mathcal{P}_0$ generates (under the action, modulo μ) the entire Borel sigma-algebra, then

$$h_\mu(\mathcal{P}_0) = h_\mu(G - \text{action}).$$

• If the action is *expansive* then

$$\mathbf{h}(\mathcal{U}) = \mathbf{h}(G - \text{action})$$

for any cover $\mathcal{U}$ finer than the expansive constant.

• The Shannon–McMillan–Breiman Theorem holds.

• The Variational Principle holds.

• Many other important facts about entropy hold...

• Work in progress: The theory of **entropy structure** and **symbolic extensions** extends to the actions of amenable groups.

That's all, thank you!