Topological entropy zero and asymptotic pairs

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Abstract

Let T be a continuous map on a compact metric space (X,d). A pair of distinct points $x,y \in X$ is *asymptotic* if $\lim_{n\to\infty} d(T^n x, T^n y) = 0$. We prove the following four conditions to be equivalent: 1. $h_{\mathrm{top}}(T) = 0$; 2. (X,T) has a (topological) extension (Y,S) which has no asymptotic pairs; 3. (X,T) has a topological extension (Y',S') via a factor map that collapses all asymptotic pairs; 4. (X,T) has a symbolic extension (i.e., with (Y',S') being a subshift) via a map that collapses asymptotic pairs. The maximal factors (of a given system (X,T)) corresponding to the above properties do not need to coincide.

Notational conventions: We will consider dynamical systems given by the iterates of a single transformation T of the space X into itself. If we regard only the forward iterates T^n with $n \geq 0$, (T^0 is, by convention, the identity map), then we denote the system by $(X,T,\mathbb{N}_0)$, while the notation $(X,T,\mathbb{Z})$ indicates that T is necessarily invertible and we observe its both forward and backward iterates T^n with $n \in \mathbb{Z}$. In statements valid for both types of systems we will denote the system by $(X,T,\mathbb{S})$ understanding that $\mathbb{S} \in \{\mathbb{N}_0, \mathbb{Z}\}$ represents both options. When speaking about factors and extensions we will always require that the factor and the extension have the same $\mathbb{S}$.

1 Introduction

In ergodic theory of probability measure preserving transformations, one of the central problems is the classification into systems of entropy zero (deterministic systems) and those with positive entropy. We can consider five conditions which are well known to be equivalent for a measurable system $(X,\mathfrak{A},\mu,T,\mathbb{S})$ (as can be easily deduced from the definition of the Kolmogorov-Sinai entropy, the Krieger Generator Theorem and the characterization of the Pinsker sigma-algebra in processes with finitely many states):

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1. The Kolmogorov-Sinai entropy of $(X, \mathfrak{A}, \mu, T, \mathbb{S})$ is zero;
2. The sigma-algebra $\mathfrak{A}$ equals the full Pinsker sigma-algebra of the system, defined as the join $\bigvee_{\mathcal{P}} \Pi_{\mathcal{P}}$, over all finite partitions $\mathcal{P}$ of X, of the Pinsker sigma-algebras $\Pi_{\mathcal{P}}$ in the processes generated by $\mathcal{P}$ (recall that $\Pi_{\mathcal{P}} = \bigcap_{n\geq 0} \mathcal{P}_n^\infty$), where $\mathcal{P}_n^\infty = \bigvee_{i\geq n} T^{-i}\mathcal{P}$).
3. The system occurs as factor of another system (extension), such that the (lifted) sigma-algebra $\mathfrak{A}$ is contained in the full Pinsker sigma-algebra of the extension.
4. The system occurs as factor of a process over a finite partition $\mathcal{P}$, and the (lifted) sigma-algebra $\mathfrak{A}$ is contained in the Pinsker sigma-algebra $\Pi_{\mathcal{P}}$ of the extending process.
5. Every subinvariant sub-sigma-algebra $\mathfrak{F}$ of $\mathfrak{A}$ (i.e., such that $T^{-1}(\mathfrak{F}) \subset \mathfrak{F}$) is invariant (i.e., $T^{-1}(\mathfrak{F}) = \mathfrak{F}$), in other words, the transformation in every factor of the forward action $(X, \mathfrak{A}, \mu, T, \mathbb{N}_0)$ is invertible.

There have been several attempts to create analogs of determinism (and of the Pinsker factor as well) in topological dynamics. Let us mention some: Furstenberg [F] calls a topological dynamical system $(X, T, \mathbb{S})$ “deterministic” when its topological entropy $h_{\text{top}}(T)$ is zero. This is an obvious analogy to the first above measure-theoretic understanding of determinism. Kaminski, Siemaszko and Szymanski [K-S-S] consider another class: they call a system “topologically deterministic” whenever the transformation in every topological factor of the forward action $(X, T, \mathbb{N}_0)$ is invertible (a homeomorphism). This idea clearly follows the analogy to the criterion 5 above. The authors show that their definition leads to a proper subclass of the class of systems with topological entropy zero. Blanchard, Host and Maass ([B-H-M]) assign the term “deterministic” to equicontinuous systems (which is obviously an even smaller class).

In this paper we will define three more topological notions of determinism, analogous to the remaining three of the listed characterizations of entropy zero in ergodic theory (items 2–4 in the above list). All of them are expressed in terms of asymptotic pairs and factors/extensions. Then we will show that, just like in ergodic theory, these three notions coincide with topological entropy zero. This fact can be considered yet another analogy between measure-theoretic and topological dynamics. At the same time, it provides three new criteria for topological entropy zero, shedding some new light on this popularly used property. Notice that asymptotic pairs is a much simpler concept than “exponential growth of (n, ϵ)-sepatared sets” or than entropy pairs, used so far to distinguish between zero and positive topological entropy in general topological dynamical systems.

The analogy does not reach further, to the notion of Pinsker factor. Our approach allows to define four types of topological analogs of the Pinsker factor; they are all topological factors (of a given topological system (X, T)) maximal with respect to some property. At the end of the paper we mention them and the fact that these factors in general do not coincide.

2 Pinsker-like systems

Before we introduce our classes of systems, we recall an elementary definition from topological dynamics.

Definition 2.1. *A pair of distinct points $\langle x, y\rangle$ in a topological dynamical system $(X, T, \mathbb{S})$ is (forward) asymptotic if*

$$\lim_{n\to\infty} d(T^n x, T^n y) = 0.$$

Suppose we want to mimic the notion of the Pinsker sigma-algebra from ergodic theory. For a process generated by a finite partition $\mathcal{P}$, this sigma-algebra equals $\Pi_{\mathcal{P}} = \bigcap_{n\geq 1} \mathcal{P}_n^\infty$. If a function f on X is $\Pi_{\mathcal{P}}$-measurable then its value $f(x)$ at $x = (x(n))_{n\in\mathbb{S}}$ is (almost surely) determined by the unilateral sequence $x[n, \infty)$ starting at any positive n.

In topological dynamics an analog of a process over a finite partition is a symbolic system (subshift) over a finite alphabet Λ. The following definition attempts to copy the measure-theoretic concept of measurability of a factor of a process generated by a finite partition $\mathcal{P}$, with respect to the Pinsker sigma-algebra $\Pi_{\mathcal{P}}$: the image of each point x via the topological factor map should be determined by the unilateral sequence $x[n, \infty)$ starting at any positive n.

Definition 2.2. *Let $(X, T, \mathbb{S})$ be a subshift. A topological factor $(Y, S, \mathbb{S})$ of $(X, T, \mathbb{S})$ (with a factoring map $\pi : X \to Y$) is Pinsker-like if*

$$\forall_{x,x'\in X}\ \forall_{n\in\mathbb{N}}\quad x[n, \infty) = x'[n, \infty) \implies \pi(x) = \pi(x').$$

In other words, π collapses asymptotic pairs.

The last phrasing of this condition can be applied not only to subshifts: factors that collapse asymptotic pairs can be considered in any topological dynamical system. They will also be called *Pinsker-like factors*. One has to realize, that there is an (*a priori*) essential difference between Pinsker-like factors of subshifts and of arbitrary systems. A pair $\langle x, x'\rangle$ in a subshift is asymptotic whenever it is "ϵ-asymptotic", i.e, when $\limsup d(T^n x, T^n x') < \epsilon$ for a sufficiently small epsilon. In general systems asymptoticity cannot be weakened this way. The requirement that a factor map collapses all asymptotic pairs is stronger for subshifts (than for general systems) because it means that all "ϵ-asymptotic" pairs are already collapsed. We will distinguish two seemingly new classes of topological systems, as defined below, by analogy to the characterizations 4 and 3 of measure-theoretic determinism listed in the introduction:

Definition 2.3. *We will call a topological dynamical system $(X, T, \mathbb{S})$ (strongly) Pinsker-like (PL) if there exists a subshift, such that $(X, T, \mathbb{S})$ is its Pinsker-like factor. A system $(X, T, \mathbb{S})$ is weakly Pinsker-like (WPL) if it occurs as a*

Pinsker-like factor of any topological dynamical system (not necessarily a subshift).

Both classes PL and WPL (of strongly and weakly Pinsker-like systems, respectively) are closed under taking factors, which follows from the completely trivial observation below:

Lemma 2.4. *Let π be a topological factor map (between two topological dynamical systems) which is a composition of several factor maps, at least one of which is Pinsker-like. Then π is Pinsker-like.*

Proof. Just observe that any factor map sends an asymptotic pair either to an asymptotic pair or collapses it. □

3 NAP-systems

We will now introduce yet another class of systems, by analogy to the second characterization of measure-theoretic determinism (item 2 in the introduction). A measure-theoretic system is deterministic if it is its own Pinsker factor (via the identity map). In our analogy, this would mean that a topological system should be its own Pinsker-like factor, via identity, i.e., that identity collapses all asymptotic pairs. This is possible only in systems which simply do not have (distinct) asymptotic pairs. This leads to the following class:

Definition 3.1. *A topological dynamical system $(X, T, \mathbb{S})$ is called NAP (no asymptotic pairs) if it has no asymptotic pairs.*

The class of NAP systems is not closed under taking factors. There is a quite complicated example in [B-H-S]. From the results of the next section it will follow that any nonperiodic subshift of entropy zero has a NAP extension, while nonperiodic subshifts are never NAP (this elemntary fact goes back to 1969, [B-W]). Below we give a very simple explicit example:

Example 3.2. *There exists NAP-system $(X, T, \mathbb{Z})$, which has a nontrivial factor $(Y, S, \mathbb{Z})$ in which all distinct pairs are asymptotic.*

Proof. We begin with describing the factor system. We let $(Y, S, \mathbb{Z})$ be the one-point compactification of the integers with the map $n \mapsto n+1$ (and $\infty \mapsto \infty$). It is obvious that all distinct pairs in this system are asymptotic. The extension $(X, T, \mathbb{Z})$ is a subsystem of the product space $Y \times \mathbb{T}$, where $\mathbb{T}$ is the circle treated as the additive group $[0, 1)$ with addition modulo 1. On this space we introduce the following action: we fix an irrational number $\alpha \in (0, 1)$ and we define T by the formula

$$T(\langle n, t\rangle) = \langle n+1, t+\alpha+\tfrac{1}{n}\rangle \quad \text{(for } n = 0 \text{ we simply skip } \tfrac{1}{n}\text{)}.$$

Note that on the invariant circle $\{\infty\} \times \mathbb{T}$ we have the irrational rotation by α. We restrict the system to this invariant circle and the two-sided orbit of the point $x_0 = \langle 0, 0 \rangle$. It is easy to see that we obtain a closed invariant set X on which T is a homeomorphism, extending $(Y, S, \mathbb{Z})$. It remains to show that there are no asymptotic pairs in X.

If a pair $\langle x, x' \rangle$ consists of two points from the invariant circle, then the distance between $T^n(x)$ and $T^n(x')$ is constant (i.e., does not depend on n). If x belongs to the circle and x' is on the single orbit outside the circle then the projection of $T^n(x')$ onto the circle rotates by the varying angle $\alpha + \frac{1}{n}$, while x rotates by the constant angle α. The differences $\frac{1}{n}$ decrease to zero, but form a divergent series, so it is easy to see, that this pair of points behaves proximally ([A]), but not asymptotically. Finally consider a pair $\langle x, x' \rangle$ where both points are outside the invariant circle. Then $x = T^m(x_0)$, $x' = T^{m+k}(x_0)$, for some $m \in \mathbb{Z}$ and a positive integer k. The projections of the points $T^n(x) = T^{m+n}(x_0)$ and $T^n(x') = T^{m+n+k}(x_0)$ onto the circle differ by

$$k\alpha + \tfrac{1}{m+n} + \tfrac{1}{m+n+1} + \cdots + \tfrac{1}{m+n+k-1}.$$

The finite sum of the harmonic series visible in the above formula decreases to zero as n grows, hence the distance between such pair converges to $k\alpha \pmod 1$. Because α is irrational, for any k this limit is positive. So such a pair is not asymptotic either. □

Clearly, the transformation in every NAP-system is injective, so it is a homeomorphism on the surjective part of the system. But there are invertible systems that are not NAP (for example nonperiodic bilateral subshifts). It is easy to see that every system topologically deterministic in the sense of Kaminski, Siemaszko and Szymanski is NAP, but not vice-versa.

Since the class of NAP-systems is not closed under taking factors (which makes it a poor analog of the measure-theoretic class of deterministic systems), it is reasonable to enlarge this class by admitting all factors of NAP-systems. Such enlarged class (denoted FNAP) is going to be our last topological analog of determinism, corresponding to the property 2 in the introduction.

4 Equality between the classes

Leaving aside the class of topologically deterministic systems in the sense of Kaminski, Siemaszko and Szymanski (which is essentially smaller than the class of all systems with topological entropy zero), we have distinguished four classes of systems, each closed under taking factors, each being a topological analog of measure-theoretic determinism: systems with topological entropy zero (TEZ), Pinsker-like systems (PL), weakly Pinsker-like systems (WPL), and finally systems with no asymptotic pairs and their factors (FNAP). Our main result follows:

Theorem 4.1.

$$TEZ = PL = WPL = FNAP.$$

The inclusion FNAP $\subset$ WPL is obvious: a factor of a NAP system is its factor via a map that collapses all asymptotic pairs (because there are none). The inclusion PL $\subset$ WPL is trivial. The inclusion WPL $\subset$ TEZ has been proved by Blanchard, Host and Ruette [B-H-R]. It remains to show the inclusions TEZ $\subset$ PL and TEZ $\subset$ FNAP. This is done in the two lemmas below, which we formulate in the "universal language", i.e., without using our abbreviations:

Lemma 4.2. *Every topological dynamical system $(X, T, \mathbb{S})$ with topological entropy zero is a factor of a subshift with topological entropy zero, via a factor map that collapses all asymptotic pairs.*

Lemma 4.3. *Every topological dynamical system $(X, T, \mathbb{S})$ with topological entropy zero is a factor of a zero-dimensional system with no asymptotic pairs.*

Since nonperiodic symbolic systems always possess asymptotic pairs, except the periodic case, the NAP extension must not be symbolic. This is why there is no trivial inclusion between the classes FNAP and PL. The equality is obtained via the class TEZ.

Before the proofs of the above lemmas let us say a few words about zero-dimensional dynamical systems. By an easy exercise, every such system $(X, T, \mathbb{S})$ can be represented in the following symbolic-array form: every element of X is an array $x = [x_{k,n}]_{k\in\mathbb{N},n\in\mathbb{S}'}$, where $\mathbb{S}' \in \{\mathbb{N}_0, \mathbb{Z}\}$ (the index set $\mathbb{S}'$ need not match the acting set $\mathbb{S}$; in case $\mathbb{S}' = \mathbb{Z}$ we say that the arrays are *bilateral*), and there are finite sets Λ_k (alphabets) such that $x_{k,n} \in \Lambda_k$ for all k and n. The transformation T on X is the horizontal shift $S([x_{k,n}]) = [x_{k,n+1}]$.

An elementary example of a zero-dimensional system is an *odometer*. The simplest way to define an odometer is via inverse limits: let $\mathbf{p} = (p_k)_{k\in\mathbb{N}}$ be a sequence of integers such that p_{k+1} is a multiple of p_k. Let $\mathbb{Z}_k = \mathbb{Z}/p_k\mathbb{Z}$ with the transformation $S_k(z) = z+1 \bmod p_k$. For each $k \geq 1$ the system $(\mathbb{Z}_k, S_k)$ is a topological factor of $(\mathbb{Z}_{k+1}, S_{k+1})$ via the congruence mod p_k. The *odometer* to base (p_k) is defined as the inverse limit of the sequence of systems $(\mathbb{Z}_k, S_k)$. Formally, it is the set

$$G_{\mathbf{p}} = \{(z_k)_{k\in\mathbb{N}} : \forall_k\, z_k \in \mathbb{Z}_k, z_k = z_{k+1} \bmod p_k\},$$

with the transformation $T((z_k)_k) = (z_k + 1 \bmod p_k)_k$. In the symbolic-array form every element of the odometer is a bilateral array over two symbols: "empty space" and "division marker" (pictured as a short vertical line on the left side of the empty space), such that in row number k the division markers appear periodically: one every p_k positions, and the markers in row $k+1$ are allowed only at the coordinates n where the division markers occur in the row k.

The next tool are *principal zero-dimensional extensions*. The following is true: every topological dynamical system has a zero-dimensional (bilateral) principal extension. We will not need the full generality of the meaning of the word ,,principal". In case $(X, T, \mathbb{S})$ has topological entropy zero, an extension $(Y, S, \mathbb{S})$ of $(X, T, \mathbb{S})$ is *principal* whenever it also has topological entropy zero. The existence of such an extension is guaranteed in several ways: one of them uses Mean Dimension Theory by E. Lindenstrauss and B. Weiss ([L-W], [L]) and is described in detail in [B-D]. An alternative method is newly proved in [D-H]. Passing to the product with an odometer, we can always assume that the elements of the extension are bilateral *binary marked* arrays. "Binary marked" means that the arrays are essentially filled with zeros and ones, with additional "division markers" occurring by the same rules as in the odometer. The rectangular $p_k \times k$ block occurring in rows 1 through k between the markers in the last row of such a marked array will be called a k-rectangle. An example of a binary marked array and a k-rectangle is shown on the Figure 1 below.

$$\begin{array}{l}
\ldots|\,0\,1|1\,1|\mathbf{0\,0}|\mathbf{0\,1}|\mathbf{0\,0}|\mathbf{0\,1}|\mathbf{1\,0}|\mathbf{1\,1}|1\,0|1\,1|0\,1|1\ldots \\
\ldots\; 1\,1\;0\,1|\mathbf{1\,0\;1\,1\;1\,0}|\mathbf{0\,1\;1\,1\;1\,0}|1\,1\;0\,0\;1\,0|1\ldots \\
\ldots\; 0\,1\;0\,0|\mathbf{1\,1\;0\,1\;1\,0\;1\,0\;1\,1\;0\,0}|1\,1\;0\,0\;0\,1\;1\ldots \\
\ldots\; 1\,1\;1\,1\;0\;1\;1\;1\;0\;1\;1\;1\;1\;1\;1\;1|1\;1\;0\,1\;1\,1\;0\ldots \\
\qquad\qquad\qquad\vdots\;\vdots\;\vdots
\end{array}$$

Figure 1: *A binary marked array (the base* (p_k) *starts with* $p_1 = 2,\ p_2 = 6,\ p_3 = 12, \ldots$*. The boldface numbers form a* 3*-rectangle.*

Proof of Lemma 4.2. The construction of symbolic extensions is in general a difficult task, discussed at length in [B-D]. The proof here is an exercise in constructing symbolic extensions in the easiest case of entropy zero systems. By Lemma 2.4, and the existence of zero-dimensional principal extensions, it suffices to build the desired extension in case $(X, T, \mathbb{S})$ is zero dimensional, in form of bilateral binary marked arrays. Let $\mathcal{R}_{k,n}$ denote the family of all rectangles of height k and length n, appearing in the first k rows of X. Because the system has entropy zero, the cardinalities of these families grow subexponentially with n, in particular, for each k there exists n_k such that $\log \#(\mathcal{R}_{k,n_k}) < n_k 2^{-k}$ (and the right hand side is an integer). By dropping some of the division markers we can easily arrange that $n_k = p_k$, the length of the k-rectangles. After this is done we let $\mathcal{R}_k$ denote the family of all k-rectangles, and we have $\#(\mathcal{R}_k) < 2^{p_k 2^{-k}}$. This implies, that there exists an injective function (code) π_k from all k-rectangles into the family of all binary blocks of length $p_k 2^{-k}$. We can now create the symbolic extension. Initially it will be not precisely symbolic, as its elements will consist of a pair: an element of the odometer (to base (p_k)) and a symbolic row. For each $x \in X$ we create its "preimage", y, as follows: we take the same element of the odometer as is represented by the markers in x. The symbolic row of y is filled inductively: above the left half of each 1-rectangle R of x we put in y the image of R via the code π_1 (this image has length exactly half of the length of R). After this step "half" of y is filled with zeros and ones, leaving the rest to be filled in the steps to come. In the following steps we apply an additional twist: the image $\pi_k(R)$ of each k-rectangle R in x is placed not above R, but instead, above the neighboring k-rectangle (to the right). The contents of $\pi_k(R)$ is written there into the consecutive free slots in that sector (starting from the left). This will use only half of the free slots available, leaving the rest to be used in the steps to come (see Figure 2).

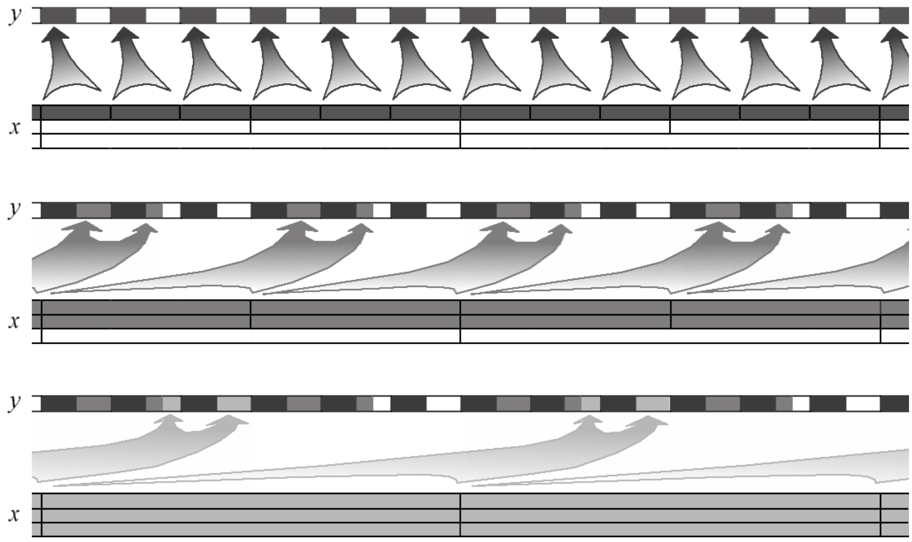

Figure 2: *Three first steps of the construction of the symbolic preimage y of x. The arrows show where the information is stored.*

It is easy to see, that after all steps are completed, depending on the positioning of the markers, the symbolic row of y is either completely filled, or there remain some unfilled slots. We fill these slots in every possible way, producing multiple preimages for x; this causes that $(Y, S, \mathbb{S})$ is not conjugate to $(X, T, \mathbb{S})$, only it is a topological extension. We skip the standard description of the factor map from Y to X, which relies on simply uncoding all the k-rectangles from the contents of appropriate places in the symbolic row of y, located with the help of the odometer part of y.

Now consider two different arrays x and x' in X. If they have different positioning of the markers then they factor to two distinct elements of the odometer. Any pair of their preimages y and y' contains the same distinct pair of elements of the odometer. Since the odometer is distal (the map is an isometry in an appropriate metric), such elements are never asymptotic. Now suppose x and x' have the same structure of markers. They must still differ at at least one position (k, n). This implies that for every $k' \geq k$ the k'-rectangles covering this position in x and x' are different. Then they have different images by $\pi_{k'}$. It is thus clear that the symbolic preimages y of x and y' of x' (any choice of a pair of such preimages) will differ at infinitely many positions tending toward infinity. In other words, y and y' cannot be (forward) asymptotic. We have proved that asymptotic pairs $\langle y, y' \rangle$ must have a common image in X.

Now we calculate the topological entropy of the extension $(Y, S, \mathbb{S})$. Clearly the odometer has entropy zero, so we only need to compute the entropy of the symbolic row, and here it suffices to count the blocks occurring between neighboring pairs of k-markers. Take a block B in some $y \in Y$, lying between two consecutive k-markers. Its content is almost completely determined by the k-rectangle of the image x, positioned directly below B. This rectangle determines all but $p_k 2^{-k+1}$ entries in B. This implies that the number of all such blocks B is at most the number of all k-rectangles in X (which is not larger than $2^{p_k 2^{-k}}$) times $2^{p_k 2^{-k+1}}$. The logarithm of this product divided by the length p_k goes to zero with k. Thus $h_{\text{top}}(S) = 0$.

To complete the construction we must replace the odometer in the construction of Y by something symbolic. The standard method is to extend the odometer to a binary Toeplitz system of entropy zero (see e.g. [D] for a survey on Toeplitz systems). Such an extension induces an extension of Y to a symbolic system with two rows; the element of the odometer will be now replaced by a symbolic row containing an element of the Toeplitz system. By Lemma 2.4, after this modification of Y, the property that the factor map onto X collapses asymptotic pairs will be preserved. This ends the proof. □

Proof. (of Lemma 4.3) This time the task is, for an arbitary system $(X, T, \mathbb{S})$ with topological entropy zero, to find a zero-dimensional extension which is NAP (possesses no asymptotic pairs).

Here is how we construct the NAP extension. Let $(X, T, \mathbb{S})$ be the initial system (of entropy zero). By Lemma 4.2, there exists a bilateral subshift extension, also of entropy zero, say $(Y_1, S_1, \mathbb{S})$, via a map π_1 that collapses asymptotic pairs. Applying the same theorem again, $(Y_1, S_1, \mathbb{S})$ has a bilateral subshift extension $(Y_2, S_2, \mathbb{S})$ via a map π_2 that collapses asymptotic pairs. And so on. We obtain a sequence of bilateral subshifts $(Y_k, S_k, \mathbb{S})$ bound by factor maps π_k that collapse asymptotic pairs. The zero-dimensional extension is obtained as the corresponding inverse limit of subshifts. Suppose this inverse limit has an asymptotic pair $\langle y, y' \rangle$. For each $k \geq 1$ the image of this pair in Y_k (denoted $\langle y_k, y'_k \rangle$) is either an asymptotic pair or $y_k = y'_k$. In either case, such pair is collapsed by π_k (for every $k \geq 1$, in particular for each $k \geq 2$), i.e., $y_{k-1} = y'_{k-1}$ for all $k \geq 2$. We have shown that $y = y'$, so there are no (distinct) asymptotic pairs in the inverse limit. □

5 Hierarchy of maximal factors

With the above notions of topological determinism one can associate (at least) four types of “maximal factors” (of a given topological dynamical system): the maximal factor of entropy zero (so-called “topological Pinsker factor (TPF) – it is determined by the smallest closed forward invariant equivalence relation which contains all entropy pairs [B-L]), the maximal topologically deterministic factor in the sense of Kaminski, Siemaszko and Szczepanski (TDF; it corresponds to the intersection of all closed forward invariant equivalence relations such that every closed forward invariant equivalence relation containing it is also backward invariant), the maximal Pinsker-like factor (PLF, determined by the smallest closed forward invariant equivalence relation which contains all asymptotic pairs) and the maximal NAP-factor (NAPF, whose existence can be easily shown using inverse limits and Zorn's Lemma). Unlike in the case of the corresponding classes of systems, these four types of factors are all essentially different, so this is where the analogy to ergodic theory ends.

Theorem 5.1. *We have the following factorization*

$$TPF \;\mapsto\; MPLF \;\mapsto\; MNAPF \;\mapsto\; MDF\,.$$

These four types of factors are essentially different.

Proof. Let us first explain the factorizations: The maximal Pinsker-like factor has entropy zero, so it factors through the topological Pinsker factor. The maximal NAP factor is NAP, so the factor map leading to it must collapse all asymptotic pairs (the image of a not collapsed asymptotic pair is an asymptotic pair). So it is Pinsker-like, thus it factors through the MPLF. The MDF factor is deterministic in the sense of [K-S-S], so it is NAP, and hence it factors through the maximal NAP factor.

The first arrow is not by identity in any zero entropy system that possesses asymptotic pairs (for example in a nonperiodic subshift of entropy zero). The second arrow is not by identity in Example 5.2 below. The third arrow is not by identity in any NAP system which is not deterministic (like the one in Example 3.2). □

Example 5.2. *There exists a bilateral subshift* $(X,T,\mathbb{Z})$ *such that the maximal Pinsker-like factor* $(Y,S,\mathbb{Z})$ *is not NAP.*

Proof. Let $(X,T,\mathbb{Z})$ be the orbit-closure (in the $\mathbb{Z}$-action) of the following (bilateral) sequence over two symbols:

$$x = \dots 11111110000000111110000011100010111000111110000011111110000000\dots$$

In addition to the countable orbit of this sequence, the system contains also the points

$$a = \dots 000000111111\dots\;, \quad b = \dots 111111000000\dots$$

and their countable orbits, and the fixpoints

$$c = \dots 000000\dots\;, \quad d = \dots 111111\dots$$

The dynamics of this system is shown on the figure below.

It is elementary to see, that all points in the orbit of a are asymptotic to the fixpoint d, and all points in the orbit of b are asymptotic to c. The maximal factor collapsing asymptotic pairs must also collapse the pair $\langle c,d\rangle$, because the corresponding relation must be closed. So, all points a,b,c,d and their orbits are collapsed to one point. That is all. No other collapsing is necessary (we leave it to the reader). So obtained factor looks exactly the same as the factor $(Y,S,\mathbb{Z})$ in Example 3.2: it is a one-point compactification (by a fixpoint) of a single discrete orbit. As before, all pairs in this factor are asymptotic, so this maximal factor is not NAP. □

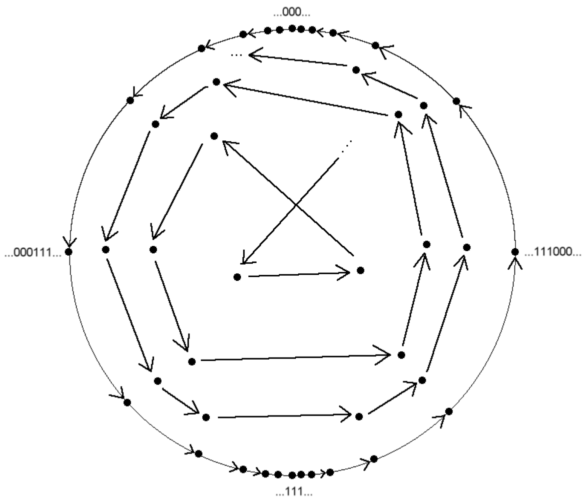

Figure 3: *The dynamics in the example. The backward orbit of the central point is not shown. It is more or less symmetric to the forward orbit.*

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