Rank as a function of measure

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Abstract

We establish certain topological and algebraic properties of rank understood as a function on the set of invariant measures on a topological dynamical system. To be exact, we show that rank is of Young class LU (i.e., it is the limit of an increasing sequence of upper semicontinuous functions) and that the rank of a convex combination of mutually singular measures equals the sum of their ranks.

1 Introduction

For a topological dynamical system (X, T), where X is a compact metric space and $T : X \to X$ is continuous, we consider the set $\mathcal{M}_T(X)$ of all T-invariant Borel probability measures on X. It is well known that this is a nonempty, compact metric (in the weak-star topology), convex set, whose extreme points are precisely the ergodic measures (the collection of which we denote by $\mathsf{ex}\,\mathcal{M}_T(X)$). Moreover, $\mathcal{M}_T(X)$ is in fact a Choquet simplex, that is, every invariant measure μ admits a unique representation as an integral average of the ergodic measures (the ergodic decomposition). Thus, the system (X, T) gives rise to what we call an *assignment*, a function Ψ whose domain is a simplex K, and "values" are ergodic measure-preserving systems identified up to isomorphism. Every such assignment is determined by its restriction to the set $\mathsf{ex}\,K$ of extreme points of K; the restriction assumes only ergodic "values" and the entire assignment can be reconstructed from $\Psi|_{\mathsf{ex}\,K}$ according to the ergodic decomposition.

Trying to understand the interplay between topological and measurable dynamics, one encounters the following natural problem: characterize these abstract assignments that can be realized in topological dynamical systems.

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At the moment the complete solution of this problem seems beyond our reach. There should exist some "continuity" or at least "measurability" obstructions, but we have at our disposal no good topological or measurable structure in the collection of classes of measure-preserving systems modulo isomorphism. Nonetheless, we can produce a number of necessary consitions by studying the behavior of some isomorphism invariants with values in more friendly spaces. For instance, it is fairly intuitive, that if we consider an isomorphism invariant in form of a real number r (for example the Kolmogorov-Sinai entropy or rank), then $r(\Psi)$ should be measurable on the simplex K. Indeed, in any topological system the entropy function $h : \mathcal{M}_T(X) \to [0, \infty]$ is not only measurable; it is a nondecreasing limit of upper semicontinuous functions (i.e., of Young class LU), see [DS].[1] Since (except on some domains, e.g. discrete or countable) not every nonnegative function is of class LU, the entropy obstruction is nonvoid; it implies that not all possible assignments are admissible in topological systems.[2]

Following the same lines of investigation, in this paper we will seek for an obstruction related to another real-valued (in fact integer-valued) isomorphism invariant, namely the rank (as defined by Ornstein, Rudolph and Weiss in [ORW]). Notice that rank distinguishes systems of zero Kolmogorov-Sinai entropy, hence any obstruction that we find is complementary to the entropy obstruction.

Although its definition does not require ergodicity, rank has been studied mainly for ergodic systems. So, we will begin by examining how does rank react to convex combinations of mutually singular measures (and more general integral averages). We discover that instead of being affine (which is impossible for integer-valued functions) is obeys certain "additive rule". Next, we will show that just like the entropy function, the rank function is also of Young class LU. This restricts nontrivially the variety of assignments which assume "values" with entropy zero at uncountably many extreme points.

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[1] The entropy function is also affine. In [DS] it is shown that there are no other entropy obstructions: every affine LU function $h : \mathsf{ex}\, K \to [0, \infty]$ defined on any metrizable Choquet simplex can be modeled as the entropy function in a topological system.

[2] Interestingly, it has been proved (independently in [KO] for homeomorphisms an in [D1] for continuous maps) that if the simplex K has at most countably many extreme points then every assignment Ψ on K assuming ergodic but not periodic "values" on the extreme points (and extended to the rest of the simplex by averaging) can be realized in a topological (even minimal) system. This is in no collision with the entropy obstruction; on a countable set every function is of class LU. Notice that this fact generalizes the celebrated Jewett–Krieger Theorem, which can be viewed as a special case concerning the one-point simplex.

2 Preliminaries

Let $(X, \mathcal{B}, \mu)$ be a probability space. All subsets of X considered below are assumed to be measurable, and all partitions are (measurable and) finite.

Definition 2.1. We say that a sequence of partitions $\{\mathcal{P}_m\}_{m\geq 1}$ *sequentially generates* if, given any set A and $\varepsilon > 0$, for every large enough m there exists a set A_m being a union of elements of $\mathcal{P}_m$, such that $\mu(A_m \triangle A) < \varepsilon$.[3]

Given a partition $\mathcal{P}$ of X and a subset $Y \subset X$, the symbol $\mathcal{P}|_Y$ denotes the partition $\{P \cap Y : P \in \mathcal{P}\}$ of Y.

Definition 2.2. For partitions $\mathcal{P}$ and $\mathcal{Q}$ we will write $\mathcal{P} \succ_\varepsilon \mathcal{Q}$ if there exists a set Y_ε of measure at least $1-\varepsilon$ such that $\mathcal{P}|_{Y_\varepsilon} \succ \mathcal{Q}|_{Y_\varepsilon}$ (i.e., $\mathcal{P}$ refines $\mathcal{Q}$ relatively on Y_ε).

A fairly straightforward proof of the following statement is left to the reader as an exercise:

Lemma 2.3. *A sequence of finite partitions $\{\mathcal{P}_m\}_{m\geq 1}$ sequentially generates if and only if, given any finite partition $\mathcal{Q}$ and $\varepsilon > 0$, we have $\mathcal{P}_m \succ_\varepsilon \mathcal{Q}$ for all large enough m.*

Throughout the rest of this section X denotes a separable metric space. Most of the definitions and statements hold for more general topological spaces, but we will not need a wider generality. Exceptionally, the space X from this part of preliminaries will be later interpreted as either $\mathcal{M}_T(X)$ or $\mathsf{ex}\,\mathcal{M}_T(X)$ rather than the phase space X in a dynamical system.

Definition 2.4. A function $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ is called *upper semicontinuous* (u.s.c.) if for all $t \in \mathbb{R}$ the sets $\{x \in X : f(x) < t\}$ are open.[4]

For example, the characteristic function of a closed set is upper semicontinuous. It is an easy exercise to verify that a function f is upper semicontinuous if and only if it is the pointwise limit of a non-increasing sequence of continuous functions from X to $\mathbb{R} \cup \{-\infty, \infty\}$.

[3]Note that if a sequence of partitions $\{\mathcal{P}_m\}_{m\geq 1}$ sequentially generates, then it also *generates*, i.e., $\bigvee_m \mathcal{P}_m = \mathcal{B}$ up to measure μ. Clearly, the reverse implication is not true.

[4]One is accustomed to *real-valued* u.s.c. functions, and such are always bounded from above. In our setup a u.s.c. function is either bounded from above or it assumes infinity as a value (necessarily on a closed set).

Definition 2.5. A function $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ is of Young class LU (an LU function for short) if it is the pointwise limit of a nondecreasing sequence f_n of upper semicontinuous functions.

The reader will easily verify the class LU is closed under finite sums, finite infima and countable suprema.

Lemma 2.6. *A function $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ is of Young class LU if and only if, for any $t \in \mathbb{R}$, the set $\{x : f(x) > t\}$ is of type F_σ.*

Proof. If f is of class LU, then $f = \lim_{n\to\infty} \uparrow f_n$, where the f_n are u.s.c. Therefore

$$\{x : f(x) \le t\} = \bigcap_k \{x : \forall_n \ f_n(x) < t + \tfrac{1}{k}\} = \bigcap_{n,k} \{x : f_n(x) < t + \tfrac{1}{k}\}$$

and thus it is of type G_δ, so its complement is of type F_σ.

To prove the other implication, let $f_t = t \cdot 1_{\{x : f(x) > t\}}$. By assumption, the set $\{x : f(x) > t\}$ is of type F_σ, so its characteristic function is of Young class LU (being a nondecreasing limit of characteristic functions of closed sets, which are u.s.c). Now it remains to note that $f = \sup f_q$ with q ranging over the rationals (and use the fact that the class LU is closed under countable suprema). ■

3 Rank of measure preserving systems

Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system on a standard probability space.[5] We will be studying the properties of an isomorphism invariant called rank. The notion has been known since 1982 (see [ORW]). In fact rank-one was known much earlier (since the 1970's) but the term "rank" was not yet used. The reader is referred also to the survey by Ferenczi ([F]) for more details and information concerning finite rank. The definition we provide below (Definition 3.2) is not identical, yet equivalent (in standard spaces) to the one found in the aforementioned sources.

Definition 3.1. Let $n_1, \dots, n_k \in \mathbb{N}$. Let $\mathcal{T}_1 = \{B_1, TB_1, \dots, T^{n_1-1}B_1\}$, $\mathcal{T}_2 = \{B_2, TB_2, \dots, T^{n_2-1}B_2\}, \dots, \mathcal{T}_k = \{B_k, TB_k, \dots, T^{n_k-1}B_k\}$ be k

[5] We adopt the definition of a standard probability space as one isomorphic to a compact metric space with a Borel probability measure extended to the completed sigma-algebra of Borel sets. The assumption that the space is standard is needed only to avoid problems with ergodic decomposition, which requires disintegration (over the sigma-algebra of invariant sets).

disjoint measurable towers, i.e. all the sets T^iB_l appearing in the towers are measurable and

$$T^iB_l \cap T^{i'}B_{l'} = \emptyset \text{ for } (l,i) \neq (l',i').$$

A *k-tower partition* $\mathcal{P}$ associated with $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k$ is the partition of X consisting of the sets T^iB_l ($l \in \{1,2,\dots,k\}, i \in \{0,1,\dots,n_l-1\}$) and $R := X \setminus \bigcup_{(l,i)} T^iB_l$. We will write

$$\mathcal{P} = \{T^iB_l, R\}_{l\in\{1,2,\dots,k\}, i\in\{0,1,\dots,n_l-1\}}$$

skipping the ranges of the indices whenever possible, and we will call the sets B_l the *bases of the towers*, the sets T^iB_l (including the bases) *the level sets*, and R will be referred to as the *the remainder.*

Definition 3.2. We are going to define *rank* of the system $(X, \mathcal{B}, \mu, T)$ which we will denote by $\mathsf{rank}(\mu)$. Let $k \in \mathbb{N}$. We say that $\mathsf{rank}(\mu) \leq k$ if there exists a generating sequence of k-tower partitions, such that the measures of the remainders tend to zero.[6] Otherwise we say that $\mathsf{rank}(\mu) > k$. $\mathsf{rank}(\mu) = k$ means that $\mathsf{rank}(\mu) \leq k$ and $\mathsf{rank}(\mu) > k-1$. Finally, $\mathsf{rank}(\mu) = \infty$ if $\mathsf{rank}(\mu) > k$ for any natural k.

Notice that the definition does not require μ to be ergodic. If μ is not ergodic then we have the following formula (called the *ergodic decomposition*

$$\mu = \int \nu_y \, d\xi_\mu(y),$$

where ξ_μ is the projection of μ onto the sigma-algebra Σ of invariant sets, and the measures ν_y are ergodic and supported by the atoms y of Σ. Distinct ergodic measures are mutually singular.

Remark 3.3. It is well known (for ergodic systems, but easily extends nonergodic systems as well) that finite rank implies zero Kolmogorov–Sinai entropy, which further implies that the transformation T is invertible (μ-almost everywhere). We will use this fact several times.

We will prove the following "additive rule":

Theorem 3.4. *The rank function satisfies*

$$\mathsf{rank}(\mu) = \sum_{\nu\in\mathsf{supp}(\xi_\mu)} \mathsf{rank}(\nu).$$

[6]The requirement on the remainders is meant to assure that every periodic orbit of positive measure eventually requires a separate tower. Without this condition a fixpoint of positive measure could be included in all remainders and the resulting "rank" would be lowered by 1. The condition is automatically fulfilled for nonatomic measures, and can be dropped in any ergodic systems except the trivial one-point system.

In other words, $\mathsf{rank}(\mu)$ is finite only for measures which are convex combinations of finitely many ergodic measures, and then it equals the sum of ranks of the ergodic components (of course only those with strictly positive coefficients). Other measures have infinite rank.

Proof. If μ is a finite combination of ergodic measures, this follows directly from Theorem 3.5 below. Every other measure can be represented as a convex combination of arbitrarily (still finitely) many mutually singular measures (each of rank at least 1 – there is no rank zero), so, by the same theorem its rank is infinite. Clearly, the sum on the right is, for such measures, also infinite, so the equality holds. ■

Theorem 3.5. *Let $(X, \mathcal{B}, \mu, T)$ be a dynamical system. Suppose that $\mu = p\mu_1 + q\mu_2$, where $p \in (0,1), q = 1-p$, μ_1 and μ_2 are mutually singular T-invariant measures on $\mathcal{B}$. Then:*

$$\mathsf{rank}(\mu) = \mathsf{rank}(\mu_1) + \mathsf{rank}(\mu_2).$$

We begin the proof with a simple lemma.

Lemma 3.6. *In a probability space consider a two-element partition $\mathcal{Q} = \{Q_1, Q_2\}$ and another finite partition $\mathcal{P} = \{P_1, \dots, P_l\}$ such that $\mathcal{P} \succ_\varepsilon \mathcal{Q}$. Let*

$$S_1 = \bigcup\{P \in \mathcal{P} : \mu(P \cap Q_1) > \mu(P \cap Q_2)\},$$
$$S_2 = \bigcup\{P \in \mathcal{P} : \mu(P \cap Q_1) \le \mu(P \cap Q_2)\},$$

and let $\mathcal{R} = \{S_1, S_2\}$. Then $\mathcal{R} \succ_\varepsilon \mathcal{Q}$.

Proof. In order to create a set on which (relatively) $\mathcal{P}$ refines $\mathcal{Q}$, we must discard from the space, for each $i = 1, 2, \dots, l$, either $P_i \cap Q_1$ or $P_i \cap Q_2$. Of course, we will discard least (in measure) if each time we discard the smaller part (if the parts are equal we discard, say, $P_i \cap Q_1$). Since $\mathcal{P} \succ_\varepsilon \mathcal{Q}$, in the above manner we will discard a set of joint measure at most ε (the rest is our set Y_ε as in Definition 2.2). But with such choice of Y_ε we easily see that $Q_1 \cap Y_\varepsilon = S_1 \cap Y_\varepsilon$ and $Q_2 \cap Y_\varepsilon = S_2 \cap Y_\varepsilon$, which implies $\mathcal{R} \succ_\varepsilon \mathcal{Q}$. ■

Proof of Theorem 3.5. Let $k = \mathsf{rank}(\mu)$ and $k_i = \mathsf{rank}(\mu_i)$ $(i = 1, 2)$. Fix Q_1 and Q_2 so that $Q_1 \cap Q_2 = \emptyset$ and $\mu_1(Q_1) = \mu_2(Q_2) = 1$ (hence $Q_1 \cup Q_2 = X \bmod \mu$). By replacing Q_i by $\bigcap_{k \in \mathbb{N} \cup \{0\}} T^{-k} Q_i$, we can assume that Q_i are (forward) invariant. If $\{\mathcal{P}_m^{(i)}\}_{m \ge 1}$ are sequences of k_i-tower partitions of Q_i satisfying the conditions in Definition 3.2 for μ_i and k_i, respectively, then it is easy to see that $\{\mathcal{P}_m^{(1)} \cup \mathcal{P}_m^{(2)}\}_{m \ge 1}$ is a sequence of (k_1+k_2)-tower partitions of $Q_1 \cup Q_2$ which satisfies these conditions for μ and k implying $k \le k_1 + k_2$.

Now suppose that $k < k_1 + k_2$ (notice that this implies in particular $k < \infty$, hence the entropy of μ is zero and hence the map T is invertible μ-almost everywhere). Let $\{\mathcal{P}_m\}_{m\ge1}$ be a sequence of k-tower partitions with remainders R_m satisfying the conditions in Definition 3.2 for μ and k. Let $\varepsilon_m \to 0$ be such that $\mu(R_m) < \varepsilon_m$ and $\mathcal{P}_m \succ_{\varepsilon_m} \mathcal{Q}$ for each m, where $\mathcal{Q} = \{Q_1, Q_2\}$. Temporarily fix some m. By Lemma 3.6, the special two-element partition $\mathcal{R}$ (whose element S_1 is the union of the elements of $\mathcal{P}_m$ with larger part in Q_1 than in Q_2, and S_2 is the rest) also refines $\mathcal{Q}$ up to ε_m. But by invariance of Q_1 and Q_2 and by invertibility of T, the proportion of parts is constant throughout all levels of one tower. This implies that every tower of $\mathcal{P}_m$ is entirely contained in one element of $\mathcal{R}$. For $i = 1, 2$ let k'_i denote the number of towers contained in S_i. These towers can be thought of as a k'_i-tower partition (which we now denote by $\mathcal{P}^{(i)}_m$) of the space $(X, \mathcal{B}, \mu_i)$. The remainder of this partition is contained in the union of three sets: Q_{3-i} (which has measure μ_i equal to zero), the remainder R_m whose measure μ_i is smaller than $\frac{\varepsilon_m}{\min\{p,q\}}$, and the complement of the set Y_{ε_m} (as defined in Lemma 3.6) which also has measure μ_i smaller than $\frac{\varepsilon_m}{\min\{p,q\}}$. So, the measure μ_i of the remainder of $\mathcal{P}^{(i)}_m$ tends to zero. Since $k'_1 + k'_2 = k < k_1 + k_2$, we have $k'_i < k_i$ for either $i = 1$ or $i = 2$. By restricting to a subsequence of $\{m\}$ we can assume that the index i and the number of towers k'_i $(< k_i)$ are common for all m. Combining the two facts:

- since the partitions $\{\mathcal{P}_m\}_{m\ge1}$ sequentially generate for μ, the partitions $\{\mathcal{P}_m|_{Q_i}\}_{m\ge1}$ sequentially generate for μ_i,
- the partition $\mathcal{P}^{(i)}_m$ differs from $\mathcal{P}_m|_{Q_i}$ only on the set Y_{ε_m} (of measure decreasing to zero with m),

we deduce that the k'_i-tower partitions $\{\mathcal{P}^{(i)}_m\}$ also sequentially generate for μ_i, which yields $\mathsf{rank}(\mu_i) \le k'_i < k_i$, a contradiction. ■

4 Rank in topological systems

Throughout, by a *topological dynamical system* we will mean a pair (X, T), where X is a metric space (with the metric denoted by d) and $T : X \to X$ is continuous. We will denote by $\mathcal{M}_T(X)$ the set of all T-invariant Borel probability measures on X and by $\mathsf{ex}\,\mathcal{M}_T(X) \subset \mathcal{M}_T(X)$ the set of ergodic measures. It is well known that both sets are nonempty and the former set equals the simplex whose extreme points constitute the latter set. For $\mu \in \mathcal{M}_T(X)$ we obtain a measure-theoretic dynamical system $(X, \mathcal{B}_\mu, \mu, T)$, where $\mathcal{B}_\mu$ denotes the sigma-algebra of Borel sets completed with respect to μ. So, there is a well defined *rank function* $\mathsf{rank} : \mathcal{M}_T(X) \to \mathbb{N} \cup \{\infty\}$. We are going to investigate convex and topological properties of this function.

The first observation is that the "additive rule" of Theorem 3.5 (also in form of Theorem 3.4) hold in this context. A small issue we need be careful about is that now the sigma-algebra formally depends on the measure. But in topological systems rank does not depend on the completion: if the rank computed for the completed sigma-algebra is infinite then clearly it is also infinite for the Borel sigma-algebra. Otherwise the map is invertible mod μ and then any tower is equal mod μ to a tower with Borel level sets. It suffices to replace the base by its subset (of equal measure) of type F_σ, and note that type F_σ is preserved by forward images of continuous maps on compact spaces. By invertibility, the forward images of the discarded part of the base have measure zero and can be discarded from the level sets.

The main goal of this section is proving the following theorem:

Theorem 4.1. *In any topological dynamical system* (X, T) *the rank function* $\mathsf{rank} : \mathcal{M}_T(X) \to \mathbb{N} \cup \{\infty\}$ *is of Young class LU.*

In order to prove the theorem we define an approximate notion of rank applicable to measure-theoretic dynamical systems $(X, \mathcal{B}, \mu, T)$ equipped with a metric d.

Definition 4.2. We are going to define *ε-rank* of the measure μ which we will denote by $\mathsf{rank}_\varepsilon(\mu)$. Let $k \in \mathbb{N}$. We say that $\mathsf{rank}_\varepsilon(\mu) \leq k$ if there exist a measurable k-tower partition $\mathcal{P}$ of X whose remainder satisfies $\mu(R) < \varepsilon$, and a measurable set X_ε such that $\mu(X_\varepsilon) > 1-\varepsilon$ and all elements of $\mathcal{P}|_{X_\varepsilon}$ have diameters smaller than ε. Otherwise we say that $\mathsf{rank}_\varepsilon(\mu) > k$. $\mathsf{rank}_\varepsilon(\mu) = k$ if $\mathsf{rank}_\varepsilon(\mu) \leq k$ and $\mathsf{rank}_\varepsilon(\mu) > k - 1$. $\mathsf{rank}_\varepsilon(\mu) = \infty$ if $\mathsf{rank}_\varepsilon(\mu) > k$ for all natural numbers k.

Remark 4.3. The definition does not imply existence of a k-tower partition whose all sets except the remainder (of small measure) have diameters smaller than ε. To achieve small diameters, we may need to discard large (in relative measure) parts from some level sets scattered along the tower. This destroys the tower structure on a globally large set.

Observe that if $\varepsilon < \varepsilon'$ then $\mathsf{rank}_\varepsilon \geq \mathsf{rank}_{\varepsilon'}$.

Lemma 4.4. *Let* (X, T) *be a topological dynamical system. Let* $\mu \in \mathcal{M}_T(X)$*. Then* $\mathsf{rank}(\mu) \leq k$ *if and only if* $\mathsf{rank}_\varepsilon(\mu) \leq k$ *for all* $\varepsilon > 0$.

In other words (by monotonicity)

$$\mathsf{rank}(\mu) = \lim_{m} \uparrow \mathsf{rank}_{\varepsilon_m}(\mu),$$

whenever $\varepsilon_m \searrow 0$.

Proof. Assume that $\mathsf{rank}_\varepsilon(\mu) \le k$ for all $\varepsilon > 0$. This means that there is a sequence of measurable k-tower partitions $\{\mathcal{P}_m\}_{m\ge1}$ satisfing the conditions of Definition 4.2 for a decreasing to zero sequence ε_m and with sets X_{ε_m} (which we will abbreviate as X_m). We will show that this sequence sequentially generates, which will imply $\mathsf{rank}(\mu) \le k$. Fix a set A and an $\varepsilon > 0$. The measure μ (being a Borel measure on a metric space) is regular, therefore there exist an open set $U \supset A$ with $\mu(U \setminus A) < \varepsilon$. Using continuity of μ from below, we find $\delta > 0$ so that $\mu(U^{-\delta}) > \mu(U) - \varepsilon$, where

$$U^{-\delta} := \{x \in U : \, d(x, X \setminus U) \ge \delta\}.$$

Let m be such that $\varepsilon_m < \min\{\delta, \varepsilon\}$. Define:

$$C = \bigcup\{P \cap X_m : P \in \mathcal{P}_m, P \cap X_m \subset U\}.$$

We will show that $U \supset C \supset U^{-\delta} \cap X_m$. The first inclusion is obvious. Further, let x belong to the latter set. Clearly x belongs to some $P \in \mathcal{P}_m$ and then it belongs to $P \cap X_m$. This intersection has diameter at most $\varepsilon_m < \delta$ and contains $x \in U^{-\delta}$ which implies that $P \cap X_m \subset U$, which makes it a component in the sum defining C. Because $\mu(X_m) > 1 - \varepsilon_m > 1 - \varepsilon$ we conclude that $\mu(U \setminus C) < 2\varepsilon$. The triangle inequality for the metric $\mu(\cdot \triangle \cdot)$ now gives $\mu(C \triangle A) < 3\varepsilon$. The set

$$A_m = \bigcup\{P \in \mathcal{P}_m : \; P \cap X_m \subset U\}$$

is a union of elements of $\mathcal{P}_m$ and $A_m \cap X_m = C$. Thus A_m differs from C (in measure) by at most ε and hence $\mu(A_m \triangle A) < 4\varepsilon$. This ends the proof of the generating.

The reversed implication will be first handled for nonatomic measures μ. Assume there exists a generating sequence of k-tower partitions $\{\mathcal{P}_m\}_{m\ge1}$ with remainders' measures tending to zero. Let $\varepsilon > 0$. For large enough m all remainders have measures smaller than ε. Since μ is nonatomic, there exists a partition $\mathcal{Q} = \{Q_1, \dots, Q_l\}$ of X whose elements have measures smaller than ε and, moreover, diameters smaller than ε. By Lemma 2.3, we can find m so large that (in addition to the remainder condition) $\mathcal{P}_m \succ_\varepsilon \mathcal{Q}$. This determines a set Y_ε with $\mu(Y_\varepsilon) > 1 - \varepsilon$ and such that $\mathcal{P}_m|_{Y_\varepsilon} \succ \mathcal{Q}|_{Y_\varepsilon}$. Thus the elements of the former partitions have diameters smaller than ε. According to Definition 4.2, we have shown that $\mathsf{rank}_\varepsilon(\mu) \le k$.

If μ has atoms then $\mu = p\mu' + \sum_{i=1}^{n} q_i\mu_i$ ($0 \le p < 1$, $p + \sum q_i = 1$), where μ' is nonatomic and each μ_i is a periodic measure supported by an individual periodic orbit. Since the measures $\mu', \mu_1, \mu_2, \dots, \mu_n$ are mutually singular and the rank of any measure is at least 1, we conclude (using the "additive rule" of Theorem 3.5) that $k \ge k' + n$, where $k' = \mathsf{rank}(\mu')$ (this also explains why the number n of periodic orbits must be finite). Given $\varepsilon > 0$, let $\mathcal{P}$ be the $(k'+n)$-tower partition consisting of the n periodic orbits (each viewed as a tower with singleton level sets) and a k'-tower partition of the rest of the space, satisfying the conditions of Definition 4.2 for μ' (with some set X'_ε), whose existence is established in the preceding paragraph. It is clear that $\mathcal{P}$ fulfills the requirements of Definition 4.2 showing that $\mathsf{rank}_\varepsilon(\mu) \le k' + n \le k$; the set X_ε equals the union of X'_ε and the periodic orbits (then $\mu(X_\varepsilon) > 1 - p\varepsilon > 1 - \varepsilon$). ∎

The following theorem is the key observation of this work. For easier proof we assume invertibility (albeit this need not be necessary). The noninvertible case will be handled later.

Theorem 4.5. *Let (X, T) be an* invertible *(i.e., in which T is a hemoemorphism) topological dynamical system. For any $\varepsilon > 0$ the function* $\mathsf{rank}_\varepsilon(\mu) : \mathcal{M}_T(X) \to \mathbb{N} \cup \{\infty\}$ *is upper semicontinuous.*

Proof. We need to show that for each $t \in \mathbb{R}$, $\mathsf{rank}_\varepsilon(\mu) < t$ holds on an open set of invariant measures. Since $\mathsf{rank}_\varepsilon$ assumes only natural values (or ∞) this set of measures is nonempty only for $t > 1$ and then the condition $\mathsf{rank}_\varepsilon(\mu) < t$ can be equivalently replaced by $\mathsf{rank}_\varepsilon(\mu) \le k$ for some $k \in \mathbb{N}$.

So assume $\mathsf{rank}_\varepsilon(\mu) \le k$. That means there exists a measurable k-tower partition $\mathcal{P} = \{T^i B_l, R\}_{l \in \{1,2,\dots,k\}, i \in \{0,1,\dots,n_l-1\}}$ with $\mu(R) < \varepsilon$ and a set X_ε with $\mu(X_\varepsilon) > 1 - \varepsilon$ such that $\mathcal{P}|_{X_\varepsilon}$ consists of sets with diameters smaller than ε. Throughout this proof, every time we refer to a pair of indices (l, i) we assume (without reminding) that $l \in \{1, 2, \dots, k\}$ and $i \in \{0, 1, \dots n_l - 1\}$. We will now explain why we can assume that all level sets of the towers are closed. Choose a positive number ξ such that $\mu(R) + \xi < \varepsilon$ and $\mu(X_\varepsilon) - \xi > 1 - \varepsilon$. By regularity we can find closed subsets sets $B'_l \subset B_l$ so that $\mu(B_l \setminus B'_l) < \frac{\xi}{kn_l}$. Then, for all pairs (l, i) the images $T^i B'_l$ are closed, contained in $T^i B_l$ and

$$\mu(T^i B_l \setminus T^i B'_l) < \tfrac{\xi}{kn_l}$$

(here we use the assumption that T is invertible). Let $\mathcal{P}' = \{T^i B'_l, R'\}$ be the k-tower partition associated with the new (smaller) bases B'_l and a new (larger) remainder set R'. The difference $R' \setminus R$ equals the union of the parts dicarded from the level sets, so its measure is smaller than

ξ. Thus $\mu(R') \leq \mu(R) + \xi < \varepsilon$. Similarly, the set $X'_\varepsilon = X_\varepsilon \setminus (R' \setminus R)$ has measure larger than $\mu(X_\varepsilon) - \xi > 1 - \varepsilon$. Because X'_ε differs from X_ε only within the new remainder, the partition $\mathcal{P}'|_{X'_\varepsilon}$ consists of the sets $T^i B'_l \cap X'_\varepsilon = T^i B'_l \cap X_\varepsilon \subset T^i B_l \cap X_\varepsilon$ (which have diameters smaller than ε) and $R' \cap X'_\varepsilon = R \cap X_\varepsilon$ (also of diameter smaller than ε).

From now we assume that the original k-tower partition $\mathcal{P} = \{T^i B_l, R\}$ has closed level sets. We are going to modify the tower and the set X_ε once more, so that R becomes closed and X_ε open. Once again, choose a positive ξ such that $\mu(X_\varepsilon) - \xi > 1 - \varepsilon$ (the other conditions involving the remainder will not be needed). Find δ such that $\mu(R \setminus R^{-\delta}) < \xi$ (this is possible since R is now open; recall that $R^{-\delta}$ is the closed set "δ-deep" inside R). Let α denote a positive number smaller than half of the smallest distance between two distinct closed sets from the family $\{T^i B_l, R^{-\delta}\}$ (clearly $\alpha \leq \frac{\delta}{2}$). Let $\beta > 0$ be so small that $d(x,y) < \beta \implies d(T^i x, T^i y) < \alpha$ for all $0 \leq i < \max\{n_1, \dots n_k\}$ (clearly $\beta \leq \alpha$). Define $B'_l = B_l^\beta$ (i.e., the open β-neighborhood around B_l).

For every pair (l,i) we have $T^i B'_l \subset (T^i B_l)^\alpha$ which implies that the sets $T^i B'_l$ are pairwise disjoint, hence form a new k-tower partition $\mathcal{P}'$ with a new smaller and closed remainder R'.

Let γ be such that $(T^i B_l)^\gamma \subset T^i B'_l$ for all pairs (l,i) (here we use again that T is a homeomorphism, so the new level sets are all open neighborhoods of the old closed level sets). Clearly, $\gamma \leq \beta \leq \alpha$. We can choose γ also smaller than half of the difference between ε and the largest diameter of an element of $\mathcal{P}|_{X_\varepsilon}$. We can now define the modified *open* version of X_ε:

$$X'_\varepsilon = \bigcup_{(l,i)} (T^i B_l \cap X_\varepsilon)^\gamma \cup (R^{-\delta} \cap X_\varepsilon)^\gamma.$$

Notice that X'_ε contains X_ε except its part contained in $R \setminus R^{-\delta}$ (this is seen even if we disregard the γ-neighborhoods). So the measure of X'_ε has dropped by at most ξ and thus is still larger than $1 - \varepsilon$. Since $(T^i B_l \cap X_\varepsilon)^\gamma \subset T^i B'_l \subset (T^i B_l)^\alpha$ for all pairs (l,i) and $(R^{-\delta} \cap X_\varepsilon)^\gamma \subset (R^{-\delta})^\alpha$, the items of the union defining X'_ε are pairwise disjoint, and each new level set TiB'_l intersects only one of them, namely $(T^i B_l \cap X_\varepsilon)^\gamma$. So,

$$T^i B'_l \cap X'_\varepsilon = (T^i B_l \cap X_\varepsilon)^\gamma.$$

This implies that the last item $(R^{-\delta} \cap X_\varepsilon)^\gamma$ equals the intersection of X'_ε with the remainder R' of the new tower. We have just proved that the items of the union defining X'_ε correspond to the elements of the partition $\mathcal{P}'|_{X'_\varepsilon}$. By the choice of γ (second requirement) (and since $R^{-\delta} \subset R$), the diameters of all these items are smaller than ε.

To summarize, we have shown that if $\mathsf{rank}_\varepsilon(\mu) \leq k$ then we can arrange the partition $\mathcal{P}$ with a closed reminder R, so that the conditions in Definition 4.2 are fulfilled with an open set X_ε. Because the measure of a closed set is an upper semicontinuous function of the measure, we have $\mu'(R) < \varepsilon$ and $\mu'(X \setminus X_\varepsilon) < \varepsilon$ (i.e., $\mu'(X_\varepsilon) > 1 - \varepsilon$) on an open set of measures (containing μ). The same partition $\mathcal{P}$ and the same set X_ε now give that $\mathsf{rank}_\varepsilon(\mu') \leq k$ for all these measures, concluding the proof. ■

We can now prove the main result of this paper.

Proof of Theorem 4.1. If T is a homeomorphism, the result is a direct consequence of the preceding Theorem 4.5 and Lemma 4.4. For noninvertible maps we refer to the notion of a *topological natural extension.* Every topological dynamical system (X, T) can be embedded (as a subsystem) in another, (X', T') such that T' is surjective on X'. Further, the surjective system (X', T') has a topological natural extension (X'', T'') in which T'' is a homeomorphism and every invariant measure μ' of (X', T') lifts to a unique invariant measure μ'' of (X'', T'') such that the measure-theoretic system (X'', μ'', T'') (we neglect marking the obvious sigma-algebra) is isomorphic to the measure-theoretic natural extension of (X', μ', T'). Moreover, the correspondence $\mu' \mapsto \mu''$ is a homeomorphism between $\mathcal{M}_{T'}(X')$ and $\mathcal{M}_{T''}(X'')$ (the details of this construction can be found e.g. in [D], pages 189-190). We will argue that $\mathsf{rank}(\mu') = \mathsf{rank}(\mu'')$. If μ' has entropy zero then T' is invertible modulo μ', and then the system (X', μ', T') is isomorphic to its own natural extension, and thus to (X'', μ'', T''), which obviously implies the desired equality of the ranks. Otherwise both μ' and μ'' have nonzero entropy hence infinite rank. Since we already know that the rank function is of class LU on $\mathcal{M}_{T''}(X'')$, it follows that it is of the same class on $\mathcal{M}_{T'}(X')$ and hence, by restriction, on $\mathcal{M}_T(X)$. ■

5 Final remarks

We have obtained that for any topological dynamical system, the rank function defined on $\mathcal{M}_T(X)$ is of Young class LU and obeys the "additive rule" of Theorem 3.4. In particular, this function is completely determined by its restriction to $\mathsf{ex}\,\mathcal{M}_T(X)$, which obviously is also of class LU. Two natural question arise:

1. Given a simplex K and an LU function on $\mathsf{ex}\,K$, is its extension to all of K by the "additive rule" automatically of class LU?
2. Are these the only "rank obstructions"? I.e., given an LU function on a metrizable Choquet simplex $r : K \to \mathbb{N} \cup \{\infty\}$ which

obeys the "additive rule", does there exist a topological dynamical system (perhaps minimal) realizing r as the rank function on the simplex of invariant measures?

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