Multiorder on countable groups

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based on a joint work with

Piotr Oprocha, Mateusz Więcek and Guohua Zhang

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some of the ideas presented in this particular section were suggested by Tom Meyerovitch

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The action (1) on total orders is Borel measurable (total orders inherit the Borel structure from $\{0, 1\}^{G \times G}$, the space of all relations in G) and preserves type $\mathbb{Z}$.

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Let $\mathcal{O}$ denote the space of all anchored bijections from $\mathbb{Z}$ to G. Then $\mathcal{O}$ inherits a natural Borel structure from $G^{\mathbb{Z}}$ and the correspondence between total orders of type $\mathbb{Z}$ and bijections from $\mathbb{Z}$ to G is a Borel-measurable bijection.

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$$(g(\mathrm{bi}))(i)=\mathrm{bi}(i+k)\cdot g^{-1},\ \text{where } k\in\mathbb{Z} \text{ is such that } g=\mathrm{bi}(k). \tag{2}$$

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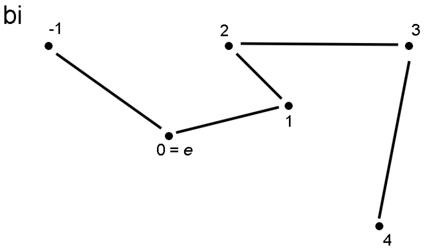

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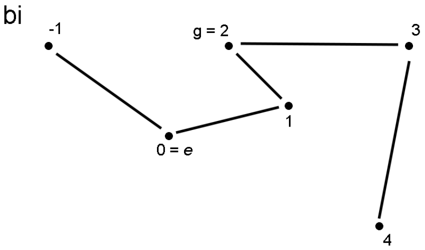

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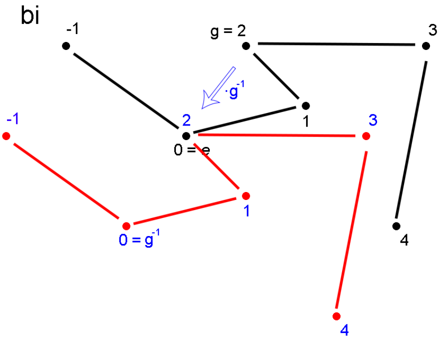

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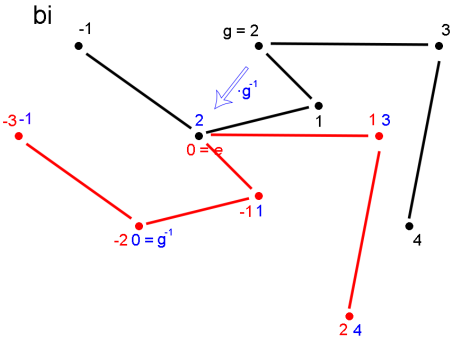

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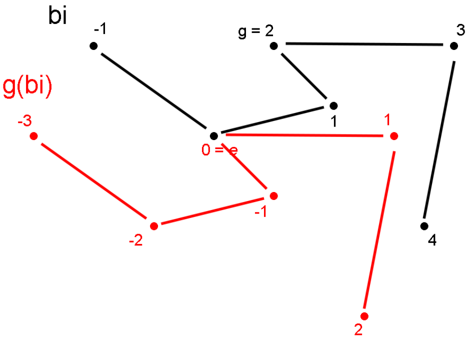

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Let G be amenable. A *multiorder* $(\mathcal{O}, \nu, G)$ on G is *Følner* if, for ν-almost every bijection $\mathsf{bi} \in \mathcal{O}$ the sequence of order intervals $\mathsf{bi}([0, n])$ is a Følner sequence in G.

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Theorem 0

Every multiorder on any amenable group is Følner.

Examples of multiorders

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The first example is completely trivial, but important, because it ensures that all our theorems valid for countable amenable groups apply as well to the classical $\mathbb{Z}$-actions. Here they either reduce to some well known theorems, or sometimes they shed a new light even in this classical setup.

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On $G = \mathbb{Z}$ consider the standard order $\prec = <$. It is easy to verify that the action given by the formula (2) is just shifting, while $<$ is clearly invariant under shifting. We conclude that $g(<) =<$ for every $g \in \mathbb{Z}$, i.e. $<$ is a fixed point of the action. Thus the Dirac measure $\delta_<$ is $\mathbb{Z}$-invariant and $(\{<\}, \delta_<, \mathbb{Z})$ is a (one-element) multiorder. So, whatever we prove to hold for *almost every* order in a multiorder, must hold for the standard order on $\mathbb{Z}$.

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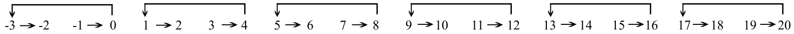

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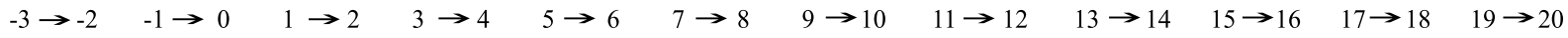

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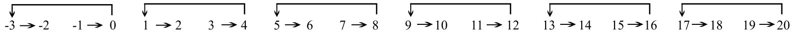
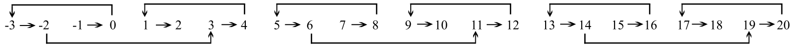


3. either draw an arrow from the head of each even path to the tail of the *following* odd path, or draw an arrow from the head of each odd path to the tail of the *following* even path. You will see connected directed paths consisting of seven arrows. Call every other path "odd" and every remaining one "even".



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   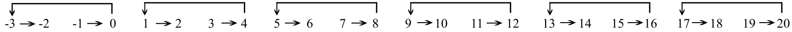

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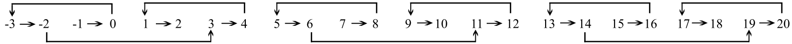
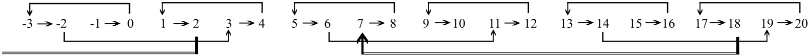
   

4. Proceed in this manner, using alternately "*following*" and "*preceding*".

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Since in each step we have two choices, eventually we will have constructed a binary tree of partial orders which, in the limit, will produce a Cantor set of orders, most of which will be total and of type $\mathbb{Z}$. Namely, if we assume that in each step our two choices have probabilities $\frac{1}{2}, \frac{1}{2}$, and the steps are independent, we will obtain a probability measure ν on the limiting Cantor set. This measure turns out to be invariant under the shift action of $\mathbb{Z}$. Moreover, one can show that the set $\mathcal{O}$ of total orders of type $\mathbb{Z}$ has measure 1. So, we have constructed an object $(\mathcal{O}, \nu, \mathbb{Z})$ that fits the definition of a multiorder. As a matter of fact, it can be shown that $(\mathcal{O}, \nu, \mathbb{Z})$ is isomorphic with the standard dyadic odometer (it is easy to see, that it is an inverse limit of cyclic groups of orders 2^n).
Observe that every order $\prec$ in this multiorder has arbitrarily long arrows, meaning that the distance between an element and its successor is unbouded. By taking the closure of $\mathcal{O}$, we will create partial orders where some element does not have a successor (or predecessor), hence it is not an order of type $\mathbb{Z}$. In other words, the multiorder $\mathcal{O}$ in this example is not closed. The aforementioned Cantor set contains a null set of "bad" elements.

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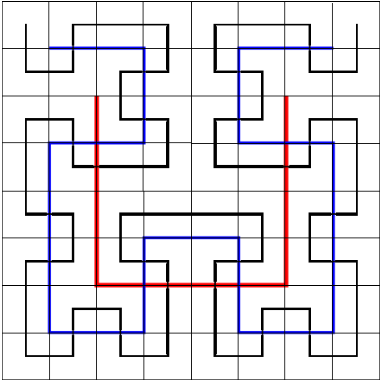

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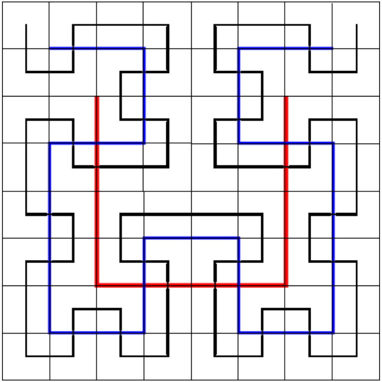

In each step there are four choices to be made. Again, if you make the choices equally likely and independently, you obtain a shift-invariant measure on total orders of type $\mathbb{Z}$, i.e. a multiorder on $\mathbb{Z}^2$. This time the family is closed, because the increments are bounded (the successor of each element is always one of four neighbors).

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Observe that the above bijection is always anchored because $ex = x = e_\Gamma x$.

Multiorder versus orbit equivalence to a $\mathbb{Z}$-action

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Theorem 1

Let (X, μ, G) be a free action on a probability space. Let $(X, \mu, \mathbb{Z})$ be a $\mathbb{Z}$-action orbit equivalent to (i.e. with the same orbits as) (X, μ, G). Let $T = T_1$ be the generating map of this $\mathbb{Z}$-action.

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$$\mathrm{bi}_x(i) = g \iff T^i x = gx, \tag{3}$$

is a measure-theoretic factor map from (X, μ, G) to a multiorder $(\mathcal{O}, \nu, G)$, where $\nu = \theta(\mu)$, and the action of G on $\mathcal{O}$ is given by (2).

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Corollary. Since every action of an *amenable* group is orbit-equivalent to a $\mathbb{Z}$-action, every *free* action of an amenable group has a multiorder as a factor.

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Notation: Suppose $\varphi : X \to \mathcal{O}$ is a measure-theoretic factor map from a measure-preserving G-action (X, μ, G) to a multiorder $(\mathcal{O}, \nu, G)$. The quadruple (X, μ, G, φ) is called a *multiordered G-action*.

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Given $x \in X$, the associated bijection $\mathrm{bi}_x = \varphi(x) \in \mathcal{O}$, and $i \in \mathbb{Z}$, instead of $\mathrm{bi}_x(i)$ we will write i^x (the ith element of G in the order associated to x).
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Theorem 2

Let (X, μ, G, φ) be a multiordered G-action. Then (X, μ, G) is orbit-equivalent to the $\mathbb{Z}$-action generated by the *successor map* defined as follows:

$$Sx = 1^x x. \tag{4}$$

Multiorder versus orbit equivalence to a $\mathbb{Z}$-action

Notation: Suppose $\varphi : X \to \mathcal{O}$ is a measure-theoretic factor map from a measure-preserving G-action (X, μ, G) to a multiorder $(\mathcal{O}, \nu, G)$. The quadruple (X, μ, G, φ) is called a *multiordered G-action*.

Given $x \in X$, the associated bijection $\mathrm{bi}_x = \varphi(x) \in \mathcal{O}$, and $i \in \mathbb{Z}$, instead of $\mathrm{bi}_x(i)$ we will write i^x (the ith element of G in the order associated to x). Note that $i^x \in G$.

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Note that we do not assume the actions (X, μ, G) or $(\mathcal{O}, \nu, G)$ to be free.

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The system (X, μ, S) factors to $(\mathcal{O}, \nu, \tilde{S})$ via the same map φ which serves as a factor map from (X, μ, G) factors to $(\mathcal{O}, \nu, G)$.

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Moreover, for any finite partition $\mathcal{P}$ of X, we have the equality of conditional entropies:

$$h(\mu, G, \mathcal{P}|\Sigma_{\mathcal{O}}) = h(\mu, S, \mathcal{P}|\Sigma_{\mathcal{O}}).$$

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Remark. The last statement follows from a more general theorem of Rudolph and Weiss (Ann. of Math. 2000), but our proof is very different.

Application to Pinsker factors

Theorem 3 allows to identify, in a muliordered system, the Pinsker factor *relative* to the multiorder factor, as follows:

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If the multiorder factor has entropy zero (under the action of G), then we have a formula for the unconditional Pinsker factor:

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If the multiorder factor has *double entropy zero* (i.e. w.r.t. both the action of G and that of $\tilde{S}$), then

$$\Pi_G(X) = \Pi_S(X)$$

(and we can use any formula available for $\mathbb{Z}$-actions).

Application to Pinsker factors

We can prove that on every countable amenable group there exists a multiorder of double entropy zero. However, given a G-action, there is no guarantee that the action factors to a multiorder of entropy zero (let alone double entropy zero). So the range of applicability of the above Corollary is rather limited.

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Theorem 4

Let (X, μ, G) be an arbitrary measure-preserving G-action and let $\mathcal{P}$ be a finite partition of X. Then, for an arbitrary multiorder $(\mathcal{O}, \nu, G)$ on G and ν-almost every order $\prec$ we have

$$\bigcap_{n\geq 1} \mathcal{P}^{(-\infty,-n]^{\prec}} = \Pi_G(\mathcal{P}).$$

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Interpretation: In order to identify the Pinsker factor $\Pi_G(\mathcal{P})$ of the process generated by $\mathcal{P}$ under the action of G follow these three steps:

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Unless you are extremely unlucky (which has probability zero), what you've just found is the desired Pinsker factor.

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THANK YOU