Multiorder in countable amenable groups

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based on a joint work with

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These two properties imply (in ergodic theory) that:

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The Pinsker sigma-algebra of this process is characterized by the formula

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where $\mathcal{P}^{(-\infty,-n]} = \bigvee_{i=n}^{\infty} T^i(\mathcal{P})$ is called the nth *remote past* of the process (analogously, $\mathcal{P}^{[n,\infty)}$ in the nth *remote future*).

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The story goes on: one can prove that positive entropy implies Li–Yorke chaos (Blanchard–Glasner–Kolyada–Maass, 2002) and even mean Li–Yorke chaos (also known as distributional chaos DC2). (D., 2011)

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1. some kind of invariance: $g_1 < g_2 \iff g_1 g < g_2 g$,
2. the order intervals $[g_1, g_2]$ (where $g_1 < g_2$) should be finite and form a Følner sequence.

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Our invention has the form of a family of orders of type $\mathbb{Z}$ (bijections $\mathbb{Z} \to G$) on which G has a natural action and which carries an invariant measure of entropy zero.

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There is no requirement that the orders are of type $\mathbb{Z}$ or that the order intervals form a Følner sequence.

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Definition 1

Let G be a countable set. A total order $\prec$ on G is *of type* $\mathbb{Z}$ if every order interval $[g_1, g_2]^{\prec} = \{g : g = g_1 \text{ or } g = g_2 \text{ or } g_1 \prec g \prec g_2\}$ (where $g_1 \prec g_2$) is finite and there is no minimal and no maximal element of G.

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Let G be a countable group. The group acts by homeomorphisms on $\mathcal{O}_G$ as follows:

$$a\, g(\prec)\, b \iff ag \prec bg, \tag{0.1}$$

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Let G be a countable amenable group. A multiorder $\tilde{\mathcal{O}}$ is *uniformly Følner* if for any finite set $K \subset G$ and any $\varepsilon > 0$ there exists n such that for any $\prec \in \tilde{\mathcal{O}}$, any order interval $[a, b]^{\prec}$ of length at least n is (K, ε)-invariant.

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Multioder can be viewed as a family of bijections $\mathbf{bi} : \mathbb{Z} \to G$ such that $\mathbf{bi}(0) = e$. The action of G on such bijections is a bit more complicated:

$$\big(g(\mathbf{bi})\big)(n) = \mathbf{bi}(n + k) \cdot g^{-1}, \text{ where } k \text{ is such that } g = \mathbf{bi}(k). \quad (0.2)$$

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Theorem 1

The assignment $\prec \mapsto \mathbf{bi}_\prec$ is a topological conjugacy between the action of G on $\mathcal{O}_G$ given by (0.1) and the collection of all anchored bijections from $\mathbb{Z}$ to G equipped with the action given by (0.2).

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To complete the proof we need to show that

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for all $i \in \mathbb{Z}$.

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By (0.2), the latter expression equals $(g(\mathbf{bi}_{\prec}))(i)$, and we are done. □

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$$\mathcal{P}^D = \bigvee_{g \in D} g^{-1}(\mathcal{P}), \text{ for example } \mathcal{P}^-_\prec = \mathcal{P}^{(-\infty,-1]^\prec} = \bigvee_{g \preccurlyeq (-1)^\prec} g^{-1}(\mathcal{P})$$

("random past").

Key theorem

Theorem 2

Let G be a countable amenable group. There exists a uniformly Følner multiorder $\tilde{\mathcal{O}}$ which supports at least one invariant measure and all invariant measures it supports have entropy zero.

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Definition 5

By a *multiordered dynamical system* (X, μ, G, φ) we will mean (X, μ, G) with a fixed factor map $\varphi : X \to \tilde{\mathcal{O}}$, where $(\tilde{\mathcal{O}}, \nu, G)$ is a multiorder. By $\{\mu_\prec : \prec \in \tilde{\mathcal{O}}\}$ we will denote the disintegration of μ with respect to ν.

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By a *multiordered dynamical system* (X, μ, G, φ) we will mean (X, μ, G) with a fixed factor map $\varphi : X \to \tilde{\mathcal{O}}$, where $(\tilde{\mathcal{O}}, \nu, G)$ is a multiorder. By $\{\mu_{\prec} : \prec \in \tilde{\mathcal{O}}\}$ we will denote the disintegration of μ with respect to ν.

Theorem 4

Let (X, μ, G, φ) be a multiordered dynamical system. For any finite partition $\mathcal{P}$ of X the following equality holds:

$$h(\mu, \mathcal{P}|\tilde{\mathcal{O}}) = \int H(\mu_{\prec}, \mathcal{P}|\mathcal{P}^{-}_{\prec})\, d\nu(\prec) = \int H(\mu_{\prec}, \mathcal{P}|\mathcal{P}^{+}_{\prec})\, d\nu(\prec).$$

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In particular, (0.4) seems to *really need* the uniform Følner property.

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Conjecture 1

Let (X, μ, G, φ) be a multiordered dynamical system. Assume that the underlying multiorder $(\tilde{\mathcal{O}}, \nu, G)$ has entropy zero (which is possible if the Pinsker factor is free). Let $\mathcal{P}$ be a finite partition of X. Then the Pinsker sigma-algebra $\Pi_{\mathcal{P}}$ of the process generated by $\mathcal{P}$ is characterized by

$$A \in \Pi_{\mathcal{P}} \iff \forall_{\prec \in \tilde{\mathcal{O}}} \ A \cap \varphi^{-1}(\prec) \in \bigcap_{n \geq 1} \mathcal{P}^{(-\infty,-n]^{\prec}}.$$

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Compare it with the classical formula for $\mathbb{Z}$-actions:

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Conjecture 2

Let (Y, G) be topological dynamical system. Then the system has topological entropy zero if and only if it is a topological factor of a multiordered system (X, G, φ) with no φ-asymptotic pairs.

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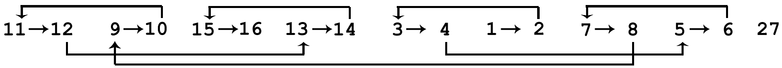

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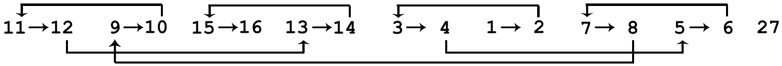

Nonetheless, these orders allow to compute (in a nonstandard way) the entropy and (hopefully) the Pinsker factor, for example in Toeplitz systems.

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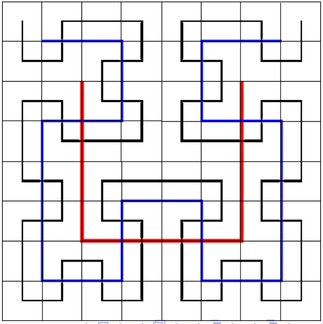

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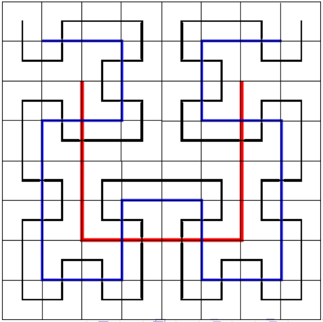

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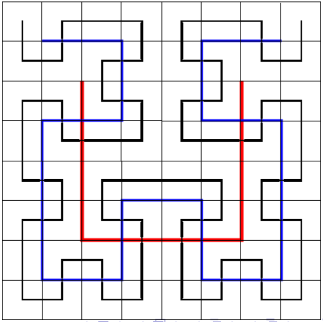

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These orders have the (rare) property "successor is a neighbor".

THANK YOU!