Monte Carlo Pricing of Callable Derivatives

Weierstraß Institute Berlin

Berlin, 28 October 2007

Overview

1. Bermudan pricing problem
   - Bermudan derivatives
   - Snell Envelope Process
   - Backward Dynamic Programming
   - Exercise policies and lower bounds
2. Upper Bounds
   - Dual Upper Bounds
   - Riesz upper bounds
3. Fast upper bounds
   - Doob-Meyer Martingale
   - Martingale Representation
   - Projection Estimator
4. Applications
5. Dimension Reduction

Bermudan derivatives

Let $L(t) \in \mathbb{R}^D$ be an underlying and $\mathbb{T} := \{T_0, T_1, \ldots, T_{\mathcal{J}}\}$ be a set of exercise dates.

Bermudan derivative: an option to exercise a cashflow $C(T_\tau, L(T_\tau))$ at a future time $T_\tau \in \mathbb{T}$, to be decided by the option holder.

Example

The ***callable snowball note*** *pays semi-annually a constant coupon I over the first year and in the forthcoming years*

$$(\textit{Previous coupon} + A - \textit{Libor})^+,$$

semi-annually, where A increases on a regular basis.

Call feature: *the issuer has the right to call the note at 100% on each coupon payment date*

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Bermudan derivatives

NEUEMISSION

Zürcher Kantonalbank

ZKB 5-Jahres Callable Snowball Note in CHF 100 % Kapitalschutz

12.07.2005 – 12.07.2010

Produktbeschreibung	5-Jahres Callable Snowball Note in CHF mit Zinszahlungen in Verbindung mit dem CHF 6-Monats Libor Satz. Die Callable Snowball Note in CHF stellt eine Alternative im Bereich der Fixed Income Anlagen dar. Tendiert der Basiswert seitwärts oder nur leicht höher, resultiert für dieses Produkt ein im Vergleich zu herkömmlichen Obligationen-Anlagen attraktiveres Renditeprofil. Ein Ansteigen des Basiswertes hat tiefere Couponszahlungen zur Folge. Ein sinkender Basiswert sowie eine Verflachung der Zinskurve erhöhen die Wahrscheinlichkeit einer vorzeitigen Rückzahlung durch die Emittentin. Die Note ist per Verfall oder per vorzeitigem Kündigungsdatum zu 100 % des Nominalbetrages kapitalgeschützt.
Emittent	Zürcher Kantonalbank Finance (Guernsey) Limited, Guernsey
Keep-Well Agreement	mit der Zürcher Kantonalbank, Zürich
Lead Manager	Zürcher Kantonalbank, Zürich
Emissionsbetrag	CHF 50'000'000
Basiswert	CHF 6-Monats Libor
Währung	CHF
Stückelung	CHF 10'000 Nominal
Emissionspreis	100.00%
Zeichnungsfrist	bis 07. Juli 2005, 17.00 Uhr
Liberierung	12. Juli 2005
Rückzahlungsdatum	12. Juli 2010

Zinsstabelle

Zinsperiode	Zinscoupon p.a.
Juli 2005 – Januar 2006	2.00 % fix
Januar 2006 – Juli 2006	2.00 % fix
Juli 2006 – Januar 2007	vorhergehender Cp +1.00 % - 6M LIBOR CHF in arrears
Januar 2007 – Juli 2007	vorhergehender Cp +1.25 % - 6M LIBOR CHF in arrears
Juli 2007 – Januar 2008	vorhergehender Cp +1.50 % - 6M LIBOR CHF in arrears
Januar 2008 – Juli 2008	vorhergehender Cp +1.75 % - 6M LIBOR CHF in arrears
Juli 2008 – Januar 2009	vorhergehender Cp +2.00 % - 6M LIBOR CHF in arrears
Januar 2009 – Juli 2009	vorhergehender Cp +2.25 % - 6M LIBOR CHF in arrears
Juli 2009 – Januar 2010	vorhergehender Cp +2.50 % - 6M LIBOR CHF in arrears
Januar 2010 – Juli 2010	vorhergehender Cp +2.75 % - 6M LIBOR CHF in arrears

Der minimale Zinscoupon pro Periode beträgt 0 %.

Zinszahlungskonvention	30/360 , modified following adjusted, Zürcher Handelstage für Zahlungen, Londoner Handelstage für die Liborfixierung
Zinsperiode	Die erste Zinsperiode beginnt mit dem Liberierungstag und endet einen Tag vor dem ersten Zinszahlungstag. Die nachfolgenden Zinsperioden beginnen jeweils mit dem Zinszahlungstag und enden einen Tag vor dem nächsten Zinszahlungstag.

Valuation

Let N, with $N(0) = 1$, be a numeraire and $\mathbb{P}$ be the associated pricing measure. Define a deflated cash flow via

$$Z_\tau := C(T_\tau, L(T_\tau))/N(T_\tau).$$

The price of the Bermudan derivative is given by the solution of the **optimal stopping problem**

$$V_0 = \sup_{\tau \in \{0,\ldots,\mathcal{J}\}} E^{\mathcal{F}_0} Z_\tau,$$

where the supremum runs over all stopping times $\tau \in \{0, ..., \mathcal{J}\}$.

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Optimal stopping

- Mathematical problem:

  Optimal stopping (calling) of a reward (cash-flow) process Z depending on an underlying (e.g. interest rate) process L

- Typical difficulties:
  - L is usually **high dimensional**, for Libor interest rate models, $D = 10$ and higher, so PDE methods do not work in general
  - Z may only be virtually known, e.g. $Z_i = E^{\mathcal{F}_i} \sum_{j \geq i} C(L_j)$ for some pay-off function C, rather than simply $Z_i = C(L_i)$
  - Z may be **path-dependent**

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Snell Envelope Process

At a future time point t, when the option is not exercised before t, the Bermudan option value is given by

$$V_t = N(t) \sup_{\tau \in \{\kappa(t),\dots,\mathcal{J}\}} E^{\mathcal{F}_t} Z_\tau$$

with $\kappa(t) := \min\{m : T_m \geq t\}$.

The process

$$Y_t^* := \frac{V_t}{N(t)}$$

is called the **Snell-envelope** process and is a supermartingale, i.e.

$$E^{\mathcal{F}_s} Y_t^* \leq Y_s^*, \quad t \geq s.$$

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Backward Dynamic Programming

Set $Y_j^* := Y_{T_j}^*$, $L_j = L(T_j)$, $\mathcal{F}_j := \mathcal{F}_{T_j}$. At the last exercise date $T_\mathcal{J}$

$$Y_\mathcal{J}^* = Z_\mathcal{J}$$

and for $0 \leq j < \mathcal{J}$,

$$Y_j^* = \max\left(Z_j, E^{\mathcal{F}_j} Y_{j+1}^*\right).$$

Observation

Nested Monte Carlo simulation of the price Y_0^* would require $N^\mathcal{J}$ samples when conditional expectations are computed with N samples

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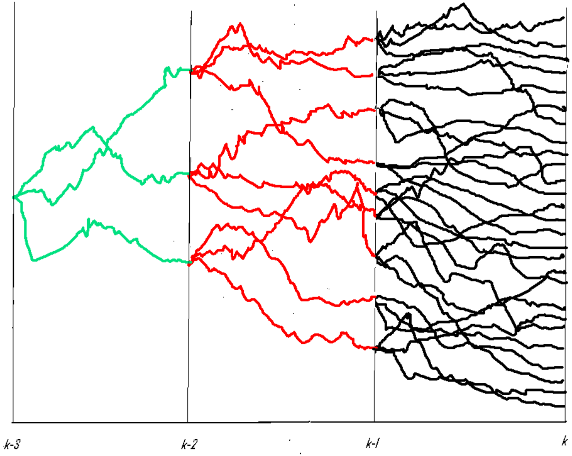

Construction of Lower Bounds

Any stopping family (policy) (τ_j) satisfying

$$j \le \tau_j \le \mathcal{J},\ \tau_{\mathcal{J}} = \mathcal{J}, \quad \tau_j > j \Rightarrow \tau_j = \tau_{j+1}, \qquad 0 \le j < \mathcal{J},$$

leads to a **lower bound** Y for the Snell envelope Y^*

$$Y_i := E^{\mathcal{F}_i} Z_{\tau_i} \le Y_i^*.$$

Example

The policy

$$\tau_i := \inf\{j \ge i : L_j \in G \subset \mathbb{R}^D\} \wedge \mathcal{J}$$

exercises when the underlying process L enters a certain region G.

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Construction of Lower Bounds

An exercise policy τ can be constructed via

$$\begin{aligned} \tau^{\mathcal{J}} &= \mathcal{J}, \\ \tau^{j} &= j\chi_{\{\widehat{C}_j(L_j)\le Z_j\}} + \tau^{j+1}\chi_{\{\widehat{C}_j(L_j)>Z_j\}}, \quad j<\mathcal{J}, \end{aligned}$$

where $\widehat{C}_j$ is an approximation for the continuation value

$$C_j(L_j) := E^{\mathcal{F}_j} Y^*_{j+1}, \quad j < \mathcal{J}.$$

Remark

$C_j(L_j)$ can be first approximated by $E^{\mathcal{F}_j} Z_{\tau^{j+1}}$ with previously constructed τ^{j+1}

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Regression Methods

The conditional expectation can be found by a **linear regression**:

$$C_j(x) \approx \sum_{r=1}^{R} \beta_{jr}\psi_r(x), \quad j = 0, 1, \ldots, \mathcal{J} - 1,$$

using a sample from $(L_j, Z_{\tau^{j+1}})$ and a set of basis functions $\{\psi_r\}_{r=1}^{R}$.

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The choice of basis functions is of crucial importance, especially in the case of large D.

Question

Is the policy τ a good one ?

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Dual upper bounds

Consider a discrete martingale $(M_j)_{j=0,\ldots,\mathcal{J}}$ with $M_0 = 0$ with respect to the filtration $(\mathcal{F}_j)_{j=0,\ldots,\mathcal{J}}$. Following Rogers, Haugh and Kogan, we observe that

$$Y_0 = \sup_{\tau \in \{0,,\ldots,\mathcal{J}\}} E^{\mathcal{F}_0} [Z_\tau - M_\tau] \leq E^{\mathcal{F}_0} \max_{0 \leq j \leq \mathcal{J}} \left[Z_j - M_j \right].$$

Hence the r.h.s. with an arbitrary martingale gives an upper bound for the Bermudan price Y_0.

Question

What martingale does lead to equality ?

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Dual upper bounds

Theorem (Rogers (2001), Haugh & Kogan (2001))

Let M^* be the (unique) Doob-Meyer martingale part of $\left(Y_j^*\right)_{0\leq j\leq \mathcal{J}}$, i.e. M_j^* is an $(\mathcal{F}_j)$-martingale which satisfies

$$Y_j^* = Y_0^* + M_j^* - A_j^*, \quad j = 0, ..., \mathcal{J}$$

with $M_0^* := A_0^* := 0$ and A_j^* being $\mathcal{F}_{j-1}$ measurable. Then

$$Y_0^* = E^{\mathcal{F}_0} \max_{0\leq j\leq \mathcal{J}} \left[Z_j - M_j^*\right].$$

Riesz upper bounds

Doob-Meyer decomposition

$$Y_j^* = Y_0^* + M_j^* - A_j^*, \quad j = 0, ..., \mathcal{J},$$

and $Y_{\mathcal{J}}^* = Z_{\mathcal{J}}$ imply **Riesz decomposition**

$$Y_j^* = E^{\mathcal{F}_j} Z_{\mathcal{J}} + E^{\mathcal{F}_j}(A_{\mathcal{J}}^* - A_j^*)$$

Since $A_{i+1}^* - A_i^* = Y_i^* - E^{\mathcal{F}_i} Y_{i+1}^* = [Z_i - E^{\mathcal{F}_i} Y_{i+1}^*]^+$,

$$Y_j^* = E^{\mathcal{F}_j} Z_{\mathcal{J}} + E^{\mathcal{F}_j} \sum_{i=j}^{\mathcal{J}-1} [Z_i - E^{\mathcal{F}_i} Y_{i+1}^*]^+.$$

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Riesz upper bounds

Theorem (Belomestny & Milstein (2005))

If Y_i is a lower approximation for Y_i^*, then

$$Y_j^{up} = E^{\mathcal{F}_j} Z_{\mathcal{J}} + E^{\mathcal{F}_j} \sum_{i=j}^{\mathcal{J}-1} [Z_i - E^{\mathcal{F}_i} Y_{i+1}]^+$$

is an upper approximation for Y_j^*, that is

$$Y_j \le Y_j^* \le Y_j^{up}, \quad j = 0, \ldots, \mathcal{J}.$$

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Properties

- ***Monotonicity***

$$\widetilde{Y}_i \geq Y_i \longrightarrow \widetilde{Y}_i^{up} \leq Y_i^{up}$$

- ***Locality***

Let $\{Y_i^\alpha, \alpha \in I_i\}$ be a family of local lower bounds at i, then

$$Y_j^{\alpha,up} = E^{\mathcal{F}_j} Z_{\mathcal{J}} + E^{\mathcal{F}_j} \sum_{i=j}^{\mathcal{J}-1} [Z_i - \max_{\alpha \in I_{i+1}} E^{\mathcal{F}_i} Y_{i+1}^\alpha]^+$$

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  is an upper bound.

Doob-Meyer Martingale

For any martingale M_{T_j}, starting at $M_0 = 0$,

$$Y_0^{up}(M) := E^{\mathcal{F}_0}\left[\max_{0\leq j\leq \mathcal{J}}(Z_{T_j} - M_{T_j})\right]$$

is an **upper bound** for the price of the Bermudan option with the deflated cash-flow Z_{T_j}.

Exact Bermudan price is attained at the martingale part M^* of the Snell envelope:

$$Y_{T_j}^* = Y_{T_0}^* + M_{T_j}^* - A_{T_j}^*,$$

where $M_{T_0}^* = A_{T_0}^* = 0$ and $A_{T_j}^*$ is $\mathcal{F}_{T_{j-1}}$ measurable.

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Doob-Meyer Martingale

Assume that $Y_{T_j} = u(T_j, L(T_j))$ is an approximation for the Snell envelope $Y^*_{T_j}$ with the Doob decomposition

$$Y_{T_j} = Y_{T_0} + M_{T_j} - A_{T_j}.$$

It then holds:

$$M_{T_{j+1}} - M_{T_j} = Y_{T_{j+1}} - E^{T_j}[Y_{T_{j+1}}]$$

Observation

The computation of M_{T_j} by MC leads to quadratic Monte Carlo for Y_0^{up}

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Martingale Representation

If process L satisfies

$$\begin{aligned} dL(t) &= a(t,L)dt + b(t,L)dW_t, \\ L(0) &= l, \end{aligned}$$

then due to the **martingale representation theorem**

$$\begin{aligned} M_{T_j} &=: \int_0^{T_j} H_t dW_t \\ &=: \int_0^{T_j} h(t,L(t))dW_t, \quad j = 0,\ldots,\mathcal{J}, \end{aligned}$$

where H_t is a square integrable and previsible process.

Observation

For any function $h(\cdot,\cdot)$ with $h(t,L(t)) \in L_2$ we get a martingale

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Projection Estimator

We are going to estimate H_t on partition $\pi = \{t_0, \ldots, t_{\mathcal{I}}\}$ with $t_0 = 0$, $t_{\mathcal{I}} = T$, and $\{T_0, \ldots, T_{\mathcal{J}}\} \subset \pi$.

Write formally,

$$Y_{T_{j+1}} - Y_{T_j} \approx \sum_{t_l \in \pi; T_j \le t_l < T_{j+1}} H_{t_l} \cdot (W_{t_{l+1}} - W_{t_l}) + A_{T_{j+1}} - A_{T_j}.$$

By multiplying both sides with $(W^d_{t_{i+1}} - W^d_{t_i})$, $T_j \le t_i < T_{j+1}$, and taking $\mathcal{F}_{t_i}$-conditional expectations, we get by the $\mathcal{F}_{T_j}$-measurability of $A_{T_{j+1}}$,

$$H^d_{t_i} \approx \widehat{H}^d_{t_i} := \frac{1}{t_{i+1} - t_i} E^{\mathcal{F}_{t_i}} \left[(W^d_{t_{i+1}} - W^d_{t_i}) \cdot Y_{T_{j+1}} \right].$$

Projection Estimator

The corresponding approximation of the martingale M is

$$M^{\pi}_{T_j} := \sum_{t_i \in \pi; 0 \le t_i < T_j} \widehat{H}_{t_i} \cdot \Delta^{\pi} W_i,$$

with $\Delta^{\pi} W^d_i := W^d_{t_{i+1}} - W^d_{t_i}$.

Theorem (Belomestny, Bender, Schoenmakers (2006))

$$\lim_{|\pi| \to 0} E\left[\max_{0 \le j \le \mathcal{J}} |M^{\pi}_{T_j} - M_{T_j}|^2\right] = 0,$$

where $|\pi|$ denotes the mesh of π.

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Projection Estimator

In fact, for $T_j \leq t_i < T_{j+1}$

$$\widehat{H}_{t_i} = \widehat{h}(t_i, L(t_i)) = \frac{1}{\Delta_i^\pi} E^{\mathcal{F}_{T_i}} \left[(\Delta^\pi W_i)^\top u(T_{j+1}, L(T_{j+1})) \right]$$

and the expectation can be computed by a linear regression.

1. Take basis functions

$$\psi(t_i, \cdot) = (\psi_r(t_i, \cdot),\ r = 1, \ldots, R)$$

2. Simulate N independent samples

$$(t_i, {}_nL(t_i)),\ n = 1, \ldots, N$$

from $L(t_i)$ using the Brownian increments $\Delta_n^\pi W_i$.

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3. Construct the matrix $A^{\oplus}_{t_i} := (A^{\top}_{t_i} A_{t_i})^{-1} A^{\top}_{t_i}$, where

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4. Define

$$\widehat{h}(t_i, x) = \psi(t_i, x) A^{\oplus}_{t_i} \left(\frac{\Delta^{\pi}_{\cdot} W_i}{\Delta^{\pi}_i} . Y_{T_{j+1}} \right) =: \psi(t_i, x) \widehat{\beta}_{t_i},$$

where $\widehat{\beta}_{t_i}$ is the $R \times D$ matrix of estimated regression coefficients at time t_i.

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Fast MC Upper Bound

Finally construct

$$\widehat{Y}_0^{up} = \frac{1}{\widetilde{N}} \sum_{n=1}^{\widetilde{N}} \max_{0 \le j \le \mathcal{J}} \left[{}_n\widetilde{Z}_{T_j} - \widetilde{M}_{T_j} \right],$$

with

$$\widetilde{M}_{T_j} = \sum_{t_i \in \pi; 0 \le t_i < T_j} \widehat{h}(t_i, \widetilde{L}(T_j)) \cdot (\Delta^{\pi} \widetilde{W}_i)$$

by simulating new paths $({}_n\widetilde{Z}_{T_j}, \Delta_n^{\pi} \widetilde{W}_i)$, $n = 1, \ldots, \widetilde{N}$.

Observation

$\widetilde{M}_j$ is always a martingale, so the upper bound is true!

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Max Call on D assets

Black-Scholes model:

$$dX_t^d = (r - \delta)X_t^d dt + \sigma X_t^d dW_t^d, \quad d = 1, ..., D,$$

Pay-off:

$$Z_t := z(X_t) := (\max(X_t^1, ..., X_t^D) - \kappa)^+.$$

$T_{\mathcal{J}} = 3$yr, $\mathcal{J} = 9$ (ex. dates), $\kappa = 100$, $r = 0.05$, $\sigma = 0.2$, $\delta = 0.1$, $D = 2$ and different x_0

D	x_0	Lower Bound Y_0	Upper Bound $Y_0^{up}(\widehat{M}^\pi)$	A&B Price Interval
	90	8.0242±0.075	8.0891±0.068	[8.053, 8.082]
2	100	13.859±0.094	13.958±0.085	[13.892, 13.934]
	110	21.330±0.109	21.459±0.097	[21.316, 21.359]

Dimension Reduction

Let $a(\cdot,\cdot), \sigma_r(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and

$$\begin{aligned} dL(t) &= a(t,L)dt + \sum_{r=1}^{q} \sigma_r(t,L)dW_r(t), \\ L(0) &= l, \end{aligned}$$

where $(W_1, \ldots, W_q)$ are independent Brownian motions and $q \leq d$.

We assume that coefficients a and b are almost affine, that is

$$a(t,x) = x \circ \zeta_a(t,x), \quad \sigma(t,x) = x \circ \zeta_{\sigma,r}(t,x),$$

where $\zeta_a(t,x)$ and $\zeta_{\sigma,r}(t,x)$ are slow varying functions in x.

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Dimension Reduction

Let $f(\cdot)$ be a function of the form $f(x) = \phi(\beta^\top x)$, $x \in \mathbb{R}^d$, then

$$\mathrm{E}[f(L(t+h))|L(t) = L] =$$
$$\int_{\mathbb{R}^q} \phi\left([\beta + h\beta \circ \zeta_a(t,L)]^\top L + \sum_{r=1}^{q} \sqrt{h}[\beta \circ \zeta_{\sigma,r}(t,L)]^\top L \xi_r \right) dP(\xi) + O(h)$$
$$=: g(BL) + O(h)$$

with $(q+1) \times n$ matrix B defined as

$$B := (\beta + h\beta \circ \zeta_a(t,L), h^{1/2}\beta \circ \zeta_{\sigma,1}(t,L), \ldots, h^{1/2}\beta \circ \zeta_{\sigma,q}(t,L))^\top$$

and $g(\cdot) : \mathbb{R}^{q+1} \mapsto \mathbb{R}$

$$g(x_0, \ldots, x_q) := \int_{\mathbb{R}^q} \phi(x_0 + x_1\xi_1 + \ldots + x_q\xi_q)\, dP(\xi).$$

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