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Project: *Drying of a sausage*

Report

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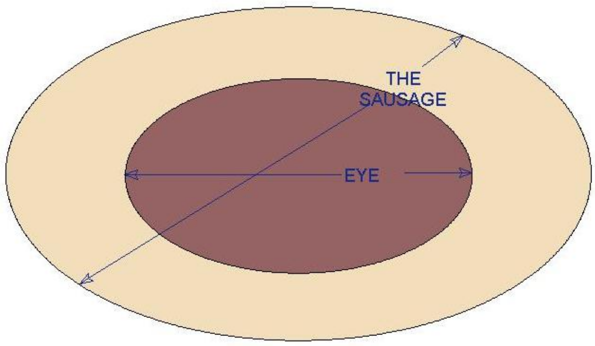
1 The problem description

- We are drying the sausage - reducing amount of water from 70% to 60%.
- Sometimes we have a sharp separation between **wet** and **dry** regions - the 'EYE' problem.



  - The humidity inside of the eye - **70%**.
  - The humidity above the eye (inside of the sausage) - **60%**.
  - The diameter of the eye - **5 cm**.
  - The diameter of the sausage - **6 cm** (after drying).
- "Perfect sausage" - the sausage **without of the eye** - our target.
  - The drying time has to be no longer than **4 days**.
  - **External humidity** and **external temperature** can be controlled.
- In each model a good calibration is impossible, because of
  - We have too many unknown parameters ($4 \div 5$): permeability, porosity, etc.
  - We have only one experiment.

2 The general model

We consider the problem in isothermal condition, that is we assume the temperature T to be constant. The general model is governed by the diffusion equation for porous media based on Darcy's law. Such an equation reads as:

$$\frac{1}{T_c}\frac{\partial}{\partial t}\Big\{\rho_w \Phi S_w + \frac{M_v}{\hat{R}T} p_v^{sat} \Phi \exp\big[-\frac{M_v}{\hat{R}T} p_c^* p_c(S_w)\big](1-S_w)\Big\} =$$

$$= \Big\{\rho_w \frac{k_w^{sat}}{\mu_w} p_c^* \frac{1}{R^2}\Big\} \frac{\partial}{\partial r}\Big\{k_w(S_w)\frac{dp_c}{dS_w}\frac{\partial S_w}{\partial r}\Big\} +$$

$$+ \Big\{\rho_w \frac{k_w^{sat}}{\mu_w} p_v^{sat} \frac{1}{R^2}\Big\} \frac{\partial}{\partial r}\Big\{k_w(S_w)\frac{dp_v}{dS_w}\frac{\partial S_w}{\partial r}\Big\} +$$

$$- \Big\{p_v^{sat} \frac{M_v}{\hat{R}T} \frac{k_w^{sat}}{\mu_g} \frac{1}{R^2}\Big\} \frac{\partial}{\partial r}\Big\{K_g(1-S_w)\frac{dp_v}{dS_w} \exp\big[-\frac{M_v}{\rho_w \hat{R}T} p_c(S_w)\big] \frac{M_v}{\hat{R}T}\frac{\partial S_w}{\partial r}\Big\}, \tag{1}$$

where: R – radius of the sausage, $\hat{R}$ – gas constant,
T – temperature, Φ – porosity of the sausage, $S_w(t, r)$ – water saturation,
p – pressure, μ – viscosity, K – hydraulic conductivity, k – permeability,
p_c^* – maximum value of the function $p_c(S_w)$, $T_c \approx 86400$ s (one day).
Subscripts and superscripts: v – vapour, w – water, c – capillary,
g – vapour + air.

Equation (1) can be written in the form:

$$\begin{aligned} A_1 \frac{\partial S_w}{\partial t} &+ A_2 \exp\big[-\frac{M_v}{\hat{R}T} p_c^* p_c(S_w)\big] \frac{\partial (1-S_w)}{\partial t} \\ &= -A_3 \frac{\partial}{\partial r}\Big[\Big(k_w(S_w)\frac{dp_c}{dS_w}\Big)\frac{\partial S_w}{\partial r}\Big] + +A_4 \frac{\partial}{\partial r}\Big\{k_w(S_w)\frac{dp_v}{dS_w}\frac{\partial S_w}{\partial r}\Big\} \\ &\quad - A_5 \frac{\partial}{\partial r}\Big\{K_g(1-S_w)\frac{dp_v}{dS_w} \exp\big[-\frac{M_v}{\rho_w \hat{R}T} p_c(S_w)\big] \frac{M_v}{\hat{R}T}\frac{\partial S_w}{\partial r}\Big\}, \end{aligned} \tag{2}$$

where the following quantities have been introduced:

$$A_1 := \frac{\rho_w \Phi}{T_c}, \ \ A_2 := \frac{1}{T_c}\Phi p_v^{sat}\frac{M_v}{\hat{R}T}, \ \ A_3 := \frac{1}{R^2}\frac{\rho_w k_w^{sat}}{\mu_w} p_c^*,$$

$$A_4 := \frac{\rho_w k_w^{sat} p_v^{sat}}{\mu_w R^2}, \ \ A_5 := p_v^{sat}\frac{M_v k_w^{sat}}{\hat{R}T \mu_g} p_v^{sat}\frac{1}{R^2}$$

Numerical estimations lead to: $A_1 = O(10^{-2})$, $A_2 = O(10^{-7})$, $A_3 = O(1)$, $A_4 = O(10^{-3})$, $A_5 = O(10^{-7})$.
Therefore, the scaling process turns out that we are dealing with a multi-scale problem.

The shortage of time to full numerical treatment of the equation forced considering constants with highest order. However assuming only $A_3 \neq 0$ gives the constant solution (i.e. the function S_w is constant). Taking into account the terms with lower order reveals the existence of the boundary layer in the sausage. In the drying sausage function S_w is constant till the boundary layer, where S_w decreases. The point where it happens might be a singular point for S_w, since S_w can be nondifferentiable there.

To solve the problem (i.e. to dry the sausage without an "eye" left) mathematically means to lower S_w below the level under which the sausage is considered to be dry.

3 The simplified model

3.1 An estimate of the *drying time*

In this very simple model we roughly estimate the drying time of the sausage. We make the following assumptions:

- The sausage **does not shrink** during the drying process.
- There is only water in the inner part of the sausage (wet region) and the evaporation takes place at $r_{evap} = R - \epsilon$. Close to the skin, i.e. for $r \in (R - \epsilon, R)$, water and vapour coexist.
- The saturation of water in the wet region is constant along the radius of the sausage, but can change in time, i.e.

$$S_w(t, r) = S_{evap}(t) \quad \text{for } r \in (0, R - \epsilon).$$

- The saturation of water close to the skin is given by a linear function

$$S_w(t, r) = \frac{S_{evap}(t) - S_{res}}{\epsilon}(R - r) + S_{res} \quad \text{for } r \in (R - \epsilon, R),$$

  where S_{res} denotes the residual saturation of water at the skin. This means that we impose a linear solution in the boundary layers, instead of solving the full problem there. This approximation, although quite rough, seems to be reasonable.The saturation of vapour close to the skin is given by

$$S_v = 1 - S_w.$$

- The flux of water close to the skin is constant

$$q_w = C \quad \text{in } (R - \epsilon, R).$$

We define here the drying time T_d to be the time after which the saturation S_{evap} is equal to that of the "dried sausage".

In view of the above assumptions we neglect the problem in the wet region and consider only the region close to the skin. Hence we have to solve the following mass balance equation

$$\Phi \frac{\partial}{\partial t}\big(\rho_w S_w + \rho_v(1 - S_w)\big) = \frac{\partial}{\partial r}\left(\rho_v q_v\right) \qquad \text{for } r \in (R-\epsilon, R),$$

where ε is the thickness of the boundary layer.

To estimate the drying time we first integrate the above equation over our domain $(R-\epsilon, R)$. This procedure can be interpreted as averaging the amount of water in the domain. We get

$$\epsilon\Phi \frac{\partial}{\partial t}\left(< \rho_w S_w + \rho_v(1 - S_w) >\right) = \rho_v q_v|_R - \rho_v q_v|_{R-\epsilon},$$

where

$$< \cdot >:= \frac{1}{\epsilon}\int_{R-\epsilon}^{R} (\cdot)\, dr.$$

Next we use the boundary condition at the skin $\rho_v q_v|_R = \nu(1 - H_{ext})$, where ν is a mass flux across the skin, and the fact that at $r = R-\epsilon$ the saturation of water is equal to S_{evap}. Moreover we neglect the term $\rho_v(1 - S_w)$ (since the vapour density is much smaller than the water density) and use the assumption on the form of S_w to get

$$\frac{\epsilon\Phi\rho_w}{2}\frac{\partial}{\partial t}S_{evap} = \nu(1 - H_{ext}) - \rho_v(S_{evap})q_v(S_{evap}).$$

Now we discretize the derivative (assuming the initial saturation equal to 1)

$$\frac{\partial}{\partial t}S_{evap} \approx \frac{1 - S_{dry}}{T_d}$$

and we get the estimate of the drying time T_d:

$$T_d = \frac{\epsilon\Phi\rho_w(S_0 - S_{dry})}{2\big(\nu(1 - H_{ext}) - \rho_v(S_{dry})q_v(S_{dry})\big)}.$$

3.2 Simulations

We run numerical simulations to present the dependence of the drying time on the external humidity, the temperature and the value of saturation for the dried sausage obtained from the considered simple model. In all the simulations we set $\epsilon = 0.03R$.

In Figure 1 we show the drying time as a function of the external humidity and the temperature for $S_{dry} = 0.8$ and $S_{dry} = 0.9$. These values of the saturation correspond to 56% and 63% content of water in the dried sausage.

In Figures 2-4 we present the dependence of the drying time on the temperature, the saturation of the dried sausage and the external humidity. In this very simple model the dependence of the drying time on the temperature is very low and the drying time is longer at the higher external humidity.

4 The model with the free boundary

In this section, we propose another simplified model for the sausage drying process in which we introduce the free boundary. The assumptions concerning this model are the following:

- There is only water in the wet region.
- There is a *sharp interface* (free boundary) between wet and dry region.
- We neglect the problem in the dry region. That is, we assume that the dry region is a zone completely saturated by vapour. Therefore, the flux is constant and equal to the one at the external boundary.
- Therefore: all the evaporation takes place at the evaporation front.
- We assume a *shrinkage*, but the shrinkage appears only as a reduction of radius, without influencing the mechanical structure of the medium. Therefore, the porosity is constant. We have a moving boundary, which actually is another free boundary.

4.1 The free boundary problem

Let $S(r,t)$ denotes the saturation of water at position $r \in (0,R)$ and at time $t \geq 0$. We assume that a position of the evaporation front σ is a function of time, and the condition at $\sigma(t)$ is given by

$$S(\sigma(t),t) = S_{evap}. \tag{3}$$

Futhermore, we assume that the velocity of the external front R (the skin of a sausage) is proportional to the velocity of water lost, and reads

$$\dot{R}(t) = -\gamma R_0 \phi \left(\left. \frac{\partial S}{\partial t} \right|_{r=\sigma} \right) . \tag{4}$$

Here R_0 denotes the initial radius of a sausage, ϕ is the porosity and γ is a positive constant.

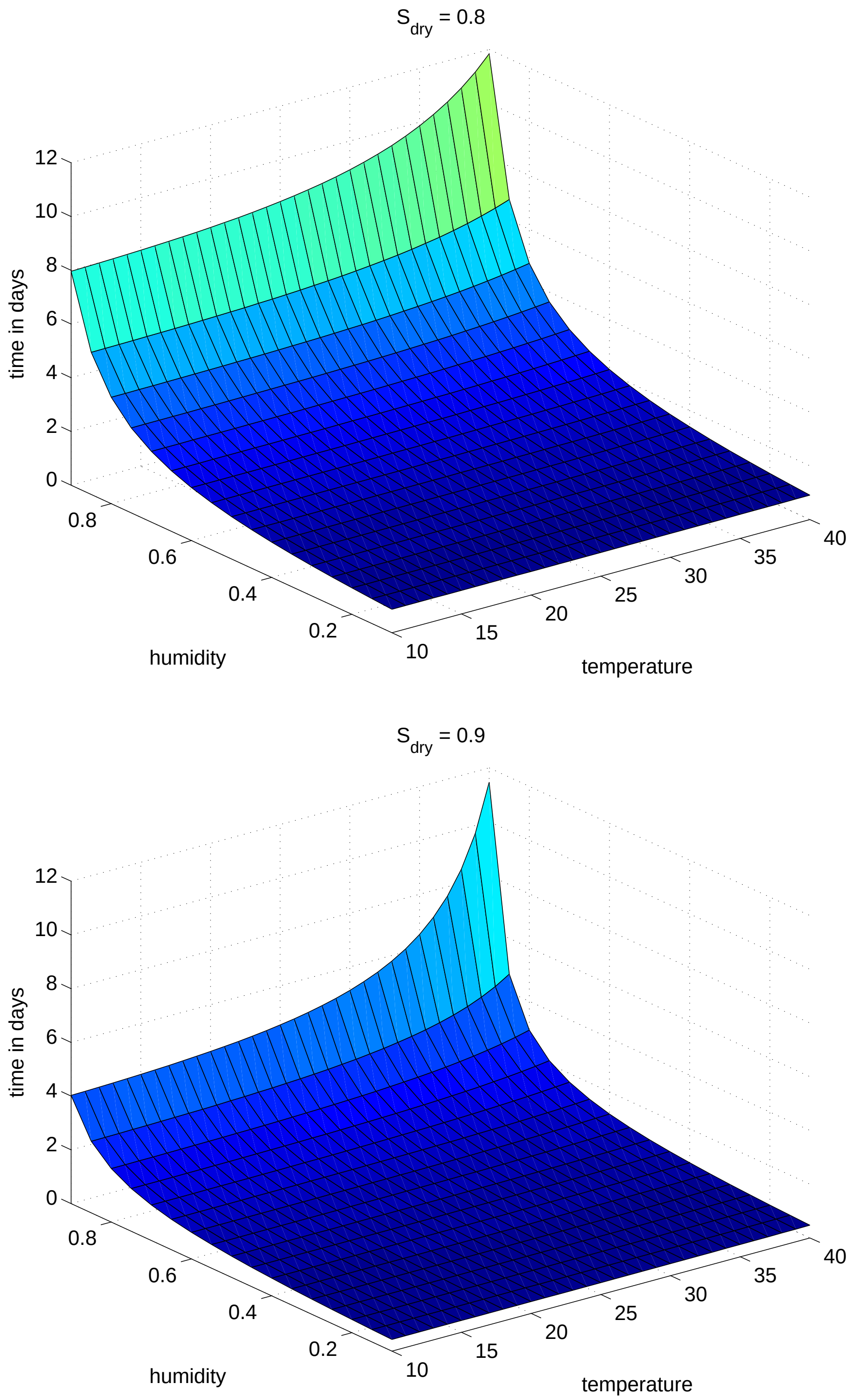


Figure 1: Time of drying for $S_{dry} = 0.8$ and $S_{dry} = 0.9$.

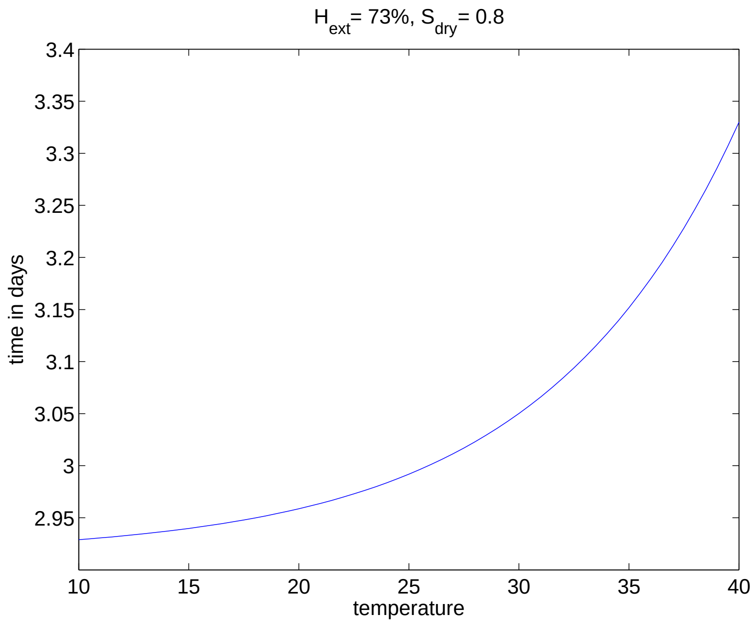


Figure 2: Dependence of drying time on the temperature for $S_{dry} = 0.8$ and $H_{ext} = 73\%$.

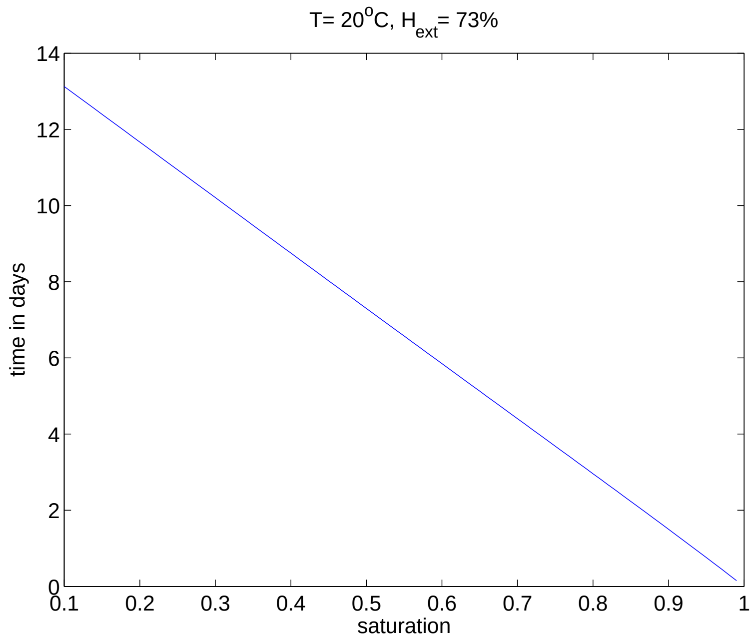


Figure 3: Dependence of drying time on the saturation for the dried sausage for $T = 20\,^\circ$C and $H_{ext} = 73\%$.

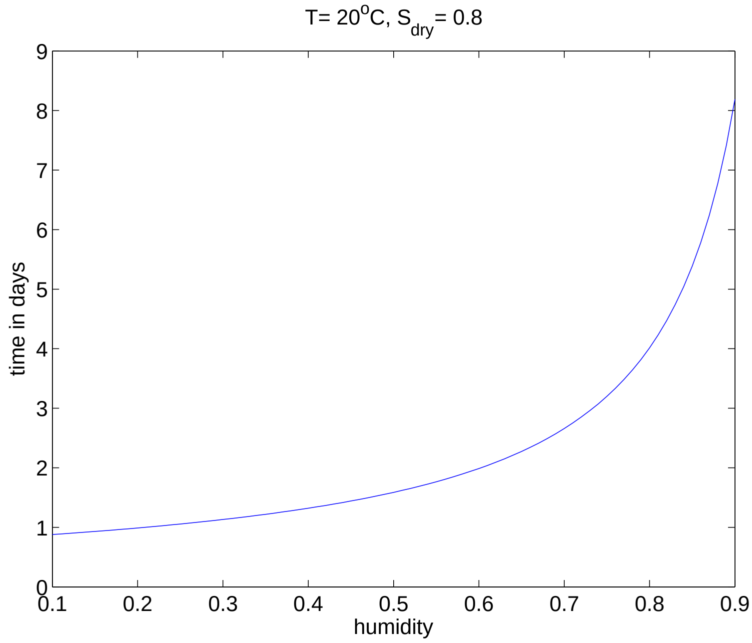


Figure 4: Dependence of drying time on the external humidity for $S_{dry} = 0.8$ and $T = 20\,^\circ$C.

The free boundary model considered in this section is the one corresponding to the Darcy flow in the wet region, and is given by

$$\frac{\partial}{\partial t}\left(\rho_w \phi S\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{k_{sat}k(S)}{\mu}\left(\frac{dp}{dS}\right)\frac{\partial S}{\partial r}\right], \quad r \in (0, \sigma(t)) \tag{5}$$

$$S(r, 0) = 1, \tag{6}$$

$$\frac{\partial S}{\partial r}(0, t) = 0, \tag{7}$$

$$S(\sigma(t), t) = S_{evap}. \tag{8}$$

In next two subsections, we derive some estimates allowing of qualitative analysis of the sausage drying process.

4.2 The estimate of the *no eye condition*

In this subsection, we introduce the condition that has to be satisfied so that the *eye* would not appear in a sausage. In order to avoid this undesirable effect, we observe that two fronts σ and R should move with the same velocity, what implies that the ratio between $\dot{\sigma}$ and $\dot{R}$ must be equal 1.

In order to find an explicit formula for $\dot{\sigma}(t)$, first, we differentiate (3) w.r.t. time, obtaining

$$\left(\frac{\partial S}{\partial r}|_\sigma\right)\dot{\sigma}(t) + \frac{\partial S}{\partial t}|_\sigma = 0\,,$$

what yields to

$$\dot{\sigma}(t) = -\frac{1}{\frac{\partial S}{\partial r}|_\sigma} \frac{\partial S}{\partial t}|_\sigma . \tag{9}$$

Next, we need to express the condition at the boundary in a different way. To that end, we assume that the mass flux at the evaporation front is proportional to the difference between the mass at the front and the one in the external part. Next, we use the condition of the external humidity, and so we have

$$\rho_w q = \nu \rho_v^{sat}(T)(1 - H_{ext}).$$

where H_{ext} is the external humidity, $\rho_v^{sat}(T)$ is the saturated vapour pressure, and ν is a positive constant.

Therefore, comparing the above formula for the flux q with the one obtained from the Darcy law, we get the equality

$$\left[\frac{k_{sat}k(S)}{\mu}\left(\frac{dp}{dS}\right)\frac{\partial S}{\partial r}\right] = \nu \frac{\rho_v^{sat}(T)}{\rho_w}(1 - H_{ext}), \tag{10}$$

which has to be used with $S = S_{evap}$.

Therefore, combining (10) with (9) and using the formula (4), we have

$$V = \left|\frac{\dot{\sigma}}{\dot{R}}\right| = \frac{1}{\gamma\phi R_0} \frac{\rho_w}{\nu \rho_v^{sat}(T)(1 - H_{ext})} \left[\frac{k_{sat}k(S_{evap})}{\mu}\left(\frac{dp}{dS}(S_{evap})\right)\right]$$

Imposing $V = 1$, we find the optimal dependence of temperature on external humidity H_{ext} that is given by the formula

$$H_{ext} = 1 - \frac{1}{\gamma\phi R_0} \frac{\rho_w}{\nu \rho_v^{sat}(T)} \left[\frac{k_{sat}k(S_{evap})}{\mu}\left(\frac{dp}{dS}(S_{evap})\right)\right]. \tag{11}$$

Therefore, we conclude that once the temperature T has been fixed, the relative humidity given by (11) represents the best condition (according to our model) to avoid the eye formation.

4.3 An estimate of the *drying time*

We define the drying time as that time T_d such that

$$< S > (T_d) = S_{dry},$$

where

$$< \cdot > := \frac{1}{\sigma(t)} \int_0^{\sigma(t)} (\cdot) dr,$$

and S_{dry} is the degree of saturation (to be fixed) at which we say that "the sausage is dry". Thus, $< S >$ denotes the average value of the saturation on the interval $(0, \sigma(t))$.

Let us integrate both sides the equation (5) over the interval $(0, \sigma(t))$. Then, we have

$$\rho\phi \frac{\partial < S >}{\partial t} = -\frac{1}{\sigma(t)} \int_0^{\sigma(t)} \rho_w \frac{\partial}{\partial r} q dr,$$

Using the boundary conditions, we get

$$\frac{\partial}{\partial t} < S >= -\frac{\rho_v^{sat}(T)}{\rho\phi\sigma(t)} \nu(1 - H_{ext}). \tag{12}$$

We discretize the derivative by

$$\frac{\partial}{\partial t} < S > \approx \frac{(1 - S_{dry})}{T_d},$$

and we get

$$T_d \approx \frac{(1 - S_{dry})\phi\rho R_0}{\nu(1 - H_{ext})\rho_v^{sat}(T)} \sigma(T_d).$$

Since we do not know $\sigma(t)$, from the above relation we get only an upper estimate for T_d,

$$T_d \leq \frac{(1 - S_{dry})\phi\rho R_0}{\nu(1 - H_{ext})\rho_v^{sat}(T)} R_0. \tag{13}$$

4.4 Simulations

In Figures 5 and 6, we present the depedence (for fixed parameters) of drying time and the ratio V, respectively, on the external humidity H_{ext} and temperature T.

The plot in Figure 7 presents the optimal value of the external humidity H_{ext} for fixed temperature T that should be set in order to avoid the *eye* formation in a sausage. This plot can be used to set the best value for the temperature once the external humidity has been fixed, or viceversa. For instance: assuming a humidity of 0.6 we should set a temperature of almost $27\,C^{\circ}$.

5 Mass transport equation as a model for a process of drying a sausage

In this section we present an alternative modelling approach to the sausage drying process using the mass transport equation.
As in the previous modelling approaches we investigate the one dimensional problem. We consider a mass of liquid $m(x, t)$ inside the sausage. The

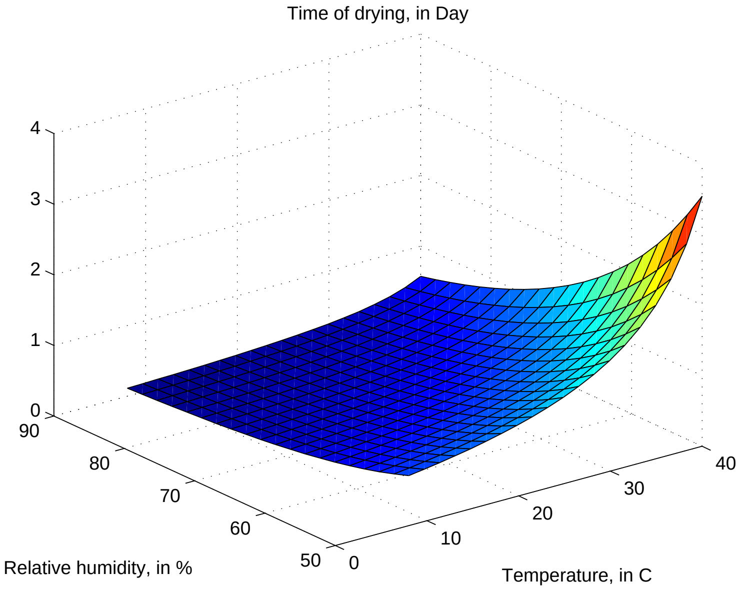


Figure 5: Dependence of drying time on the external humidity and temperature

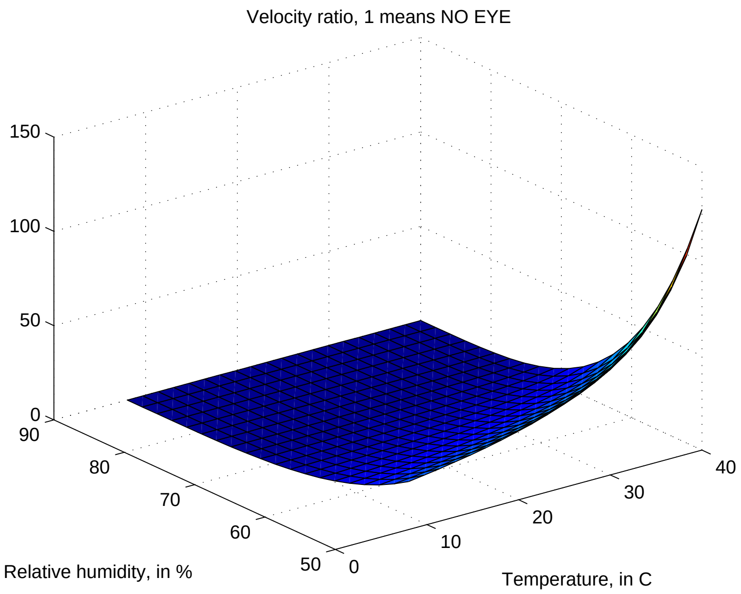


Figure 6: Dependence of ratio V on the external humidity and temperature

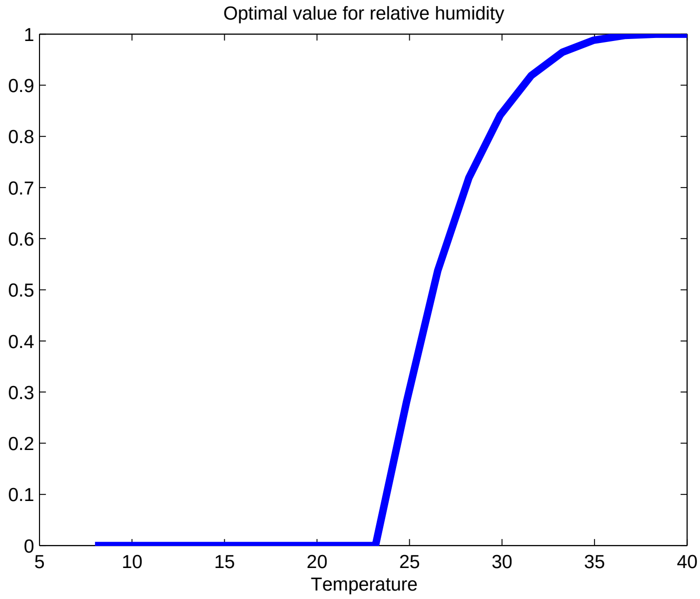


Figure 7: Optimal value for relative humidity

dynamics of the mass of liquid is given by the mass transport equation, where the velocity of transport is the proportional to the gradient of the pressure of the liquid. We assume that the whole outflow of the mass takes place at the sausage casing with the rate Q. This outflow process depends on the external temperature, humidity and the level of liquid at the sausage casing. Thus, the dynamics of the drying process is given by the equation:

$$\frac{\partial m}{\partial t} + \gamma(m)\frac{\partial m}{\partial x} = 0, \tag{14}$$

with the initial condition

$$m(x, 0) = 1,$$

and with the boundary condition

$$\left.\frac{\partial m}{\partial x}\right|_{x=R} = -Q(H, T, m(R, t)),$$

where the velocity of transport is proportional to the gradient of pressure $\gamma(m) \propto \nabla p$, H denotes external humidity and T is external temperature .

In further analysis we consider the $p = \alpha m$, where α is a nonnegative constant. Thus, the investigated model is defined by

$$\frac{\partial m}{\partial t} + \alpha \left(\frac{\partial m}{\partial x}\right)^2 = 0 \tag{15}$$

with the initial condition

$$m(x,0) = 1,$$

and with the boundary condition

$$\left.\frac{\partial m}{\partial x}\right|_{x=R} = -Q(H,T,m(R,t)).$$

We say that the sausage is dry at time t, when $\forall x\ m(x,t) \leq m_{dry}$, where m_{dry} is equal to some threshold. For our experimental data, we know that m_{dry} is equal to 0.8 and the drying process lasts 4 days.

Due to the lack of sufficient experimental data describing the dynamics of $m(x,t)$, we investigate model (15) wheater it exhibts the behevoiur of 'eye' and 'non-eye' sausage after 4 days of drying. We solve the problem (15) using MATLAB software and its numerical procedure `pdepe`. In our numerical experiments we assume that during the drying process developed by the sausage producer the constant α is an optimal one, since most of the drying processes leads to the non-eyed sausage. Of course the factor α is strictly related to the porosity, permeability of the sausage, but we have not enough data to estimate it in a better way. Several numerical experiment resulted in finding the optimal one, namely $\alpha = 0.45$. Then we change the external condition to recover an 'eye' and 'no-eye' drying process, for this purpose we assume that $Q = \beta m(R,t)$. In next stage of our experiments we try to find the value of β which results in 'eye' and 'non-eye' sausage after a fixed time of drying.

In figure 5 we see the dynamics of the drying process which lead to the 'eye' sausage - in this case $\beta = 0.15$. It is seen that after a fixed time t_{end} of drying, the mass of liquid in some region of the sausage is not below the desired threshold m_{dry}, it means that in that region we have an 'eye'. On the other hand, we find the value of $\beta = 0.17$ resulting in 'non-eye' sausage - we see that after the fixed time of drying t_{end} the mass of the liquid inside the sausage is below the desired level m_{dry}.

In further work, as we received experimental data describing the mass of the liquid inside a sausage during the drying process, using for instance the oridinary techniques of data analysis (such as corelation, etc), we can estimate more precisely the function $Q(H,T,m(R,t))$ and the velocity of the transport $\gamma(m(x,t))$. We believe that these estimates can result in more realistic model of drying the sausage.

6 Conclusions and future work

In this report, we have proposed three different approaches to model the process of a sausage drying. Based on the introduced models, we have derived formulas for the optimal dependence between two parameters (external

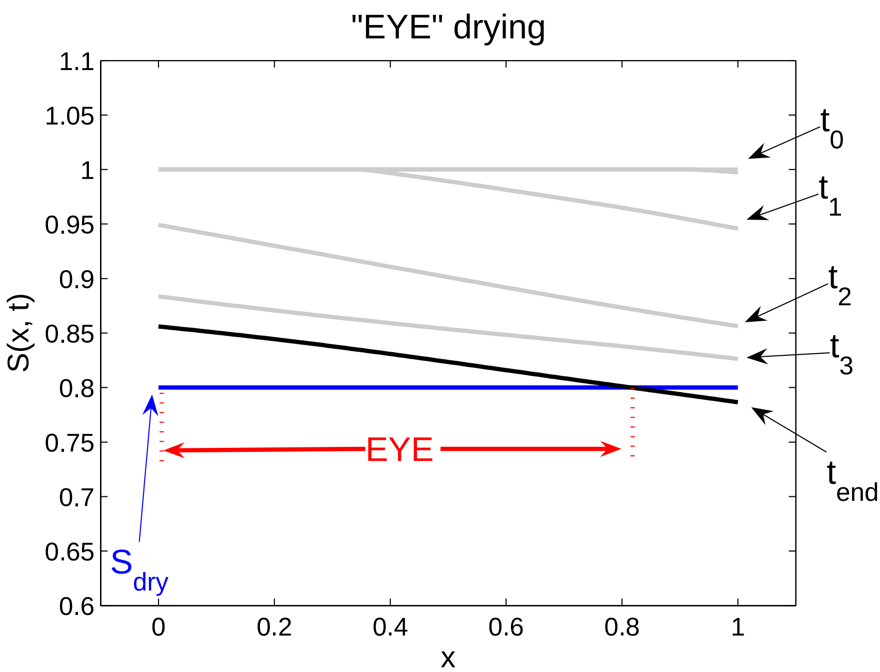


Figure 8: Dynamics of the drying process, which leads to 'eye' sausage. The mass of liquid $m(x,t)$ is presented at 5 time-points: $t_0, t_1, t_2, t_3, t_{end}$.

temperature and humidity) that should be tuned to control the considered process. Obtained qualitative results are satisfactory and corresponds to the expected ones. In order to have quantitative results, more experimental data and more detailed description of the sausage drying process is required.

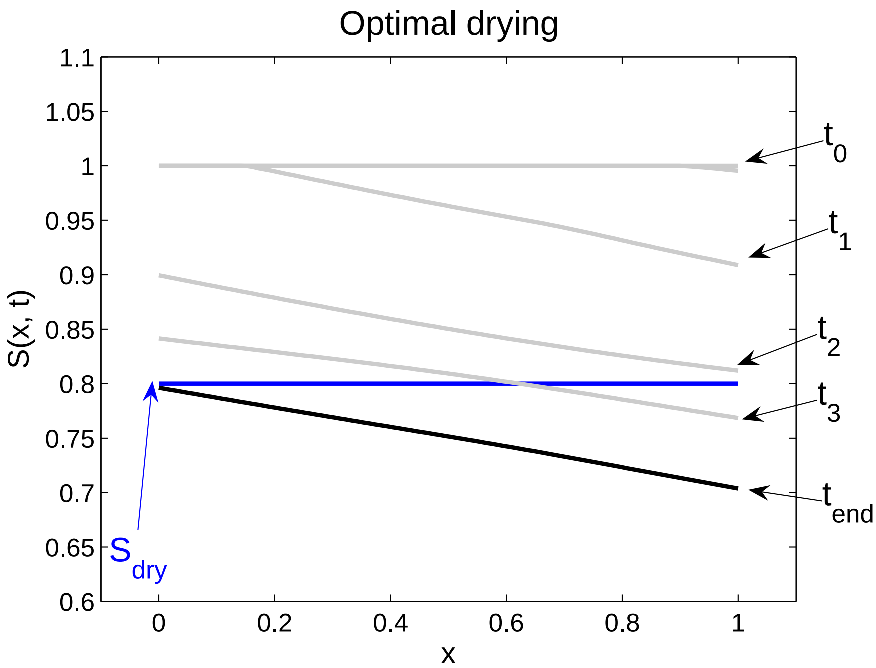


Figure 9: Dynamics of the drying process, which leads to 'no-eye' sausage. The evolution of the mass of liquid $m(x,t)$ is presented at 5 time-points: $t_0, t_1, t_2, t_3, t_{end}$.