Anomalous diffusion with transient subordinators: A link to compound relaxation laws

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(Received 19 November 2013; accepted 16 January 2014; published online 6 February 2014)

This paper deals with a problem of transient anomalous diffusion which is currently found to emerge from a wide range of complex processes. The nonscaling behavior of such phenomena reflects changes in time-scaling exponents of the mean-squared displacement through time domain – a more general picture of the anomalous diffusion observed in nature. Our study is based on the identification of some transient subordinators responsible for transient anomalous diffusion. We derive the corresponding fractional diffusion equation and provide links to the corresponding compound relaxation laws supported by this case generalizing many empirical dependencies well-known in relaxation investigations. © 2014 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4863995]

I. INTRODUCTION

Anomalous diffusion with the mean-squared displacement (MSD) being nonlinear in time is an active area of research in physics. The MSD is linear for normal diffusion, has form of a power law MSD(t) = t^α, 0 < α < 1 for pure anomalous subdiffusion, and MSD(t) = t^α for t ≪ t_cr, and MSD(t) = t for t ≫ t_cr, for so called transient anomalous subdiffusion. Here, t is time, α is anomalous diffusion exponent, and t_cr is the crossover time. We study a problem of general interest since transient anomalous diffusion is currently found to emerge from a wide range of complex processes, biological, economical, sociological, as well as physical, see Refs. 1 and 2.

For classical normal diffusion, the MSD of a stochastic process Var Y(t) = (⟨Y(t)⟩ − ⟨Y(t)⟩)^2, representing the diffusive particle’s position at time t, is linear. This simplest case is observed in many physical systems, but not always. Anomalous diffusion processes correspond to a more general situation. In particular, experimentalists often fit data to a subdiffusive power law t^α (with 0 < α < 1) of the MSD. However, the homogeneous time scaling proves to be only a simplified picture of complex phenomena. The complex systems rather demonstrate nonscaling time behavior of their MSD, although there are chances to separate finite time intervals where one time scale of diffusion and one exponent αj dominate. We will call this kind of diffusion transient. It is clear why, if one assumes two different short- and long-time scaling of the MSD. Then, the long-time exponent α∞ will accelerate (∝ < α∞) or retard (∝ > α∞) the diffusive evolution starting with the short-time scaling under the exponent α0. Extension of the diffusion concepts was investigated in many works (see, for example, Refs. 3–13).

Starting with assumption that the stochastic process Y(t) corresponds to the probability density function (PDF) obeying the distributed-order time-fractional diffusion equation, the serious attempt to analyze properties of the process Y(t) itself has been carried out recently in Ref. 13 as applied to the so-called accelerating diffusion, introduced and studied in Refs. 7 and 9. Further development of the stochastic representation for a wider class of the anomalous diffusion phenomena with a more arbitrary nonlinear MSD is of great interest. Moreover, it is a very common phenomenon.14–19

Replacing the time variable in a normal diffusion by an independent inverse tempered α-stable subordinator S_α,δ(t) of index 0 < α < 1, 0 < δ yields a useful stochastic model for the transient anomalous diffusion. For example, if Y(t) = B(t) is a standard Brownian motion, independent of S_α,δ(t), it was shown in Ref. 20 that the MSD of the subordinated process Y(S_α,δ(t)) is proportional to t^α / t^α, as t → 0; and proportional to t/δα, as t → ∞. Hence, the process B(S_α,δ(t)) occupies an intermediate place between pure subdiffusion, in which the second moment grows like t^α, and normal diffusion, where the second moment is proportional to t. For a more general d-dimensional version of this result, see Ref. 21.

The present paper is just devoted to the study of this type of dynamics or transport. In Sec. II, our consideration starts with a stochastic representation of random processes responsible for anomalous diffusion with arbitrary subordinator. This allows us to calculate the MSD of this subordinated process in Sec. III and to derive the corresponding diffusive equation in Sec. IV in a general form. Some other examples of transient subordinators support our consideration in Sec. V. Using the results, we discuss a relationship with the frequency-domain relaxation function in Sec. VI and explicitly identify under- and overshooting transient subordinators important for the analysis of relaxation data in complex phenomena. We also show a link to compound

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responses of glass-forming liquids. Finally, the conclusions are drawn.

II. ANOMALOUS DIFFUSION AND SUBORDINATORS

The well-known fractional Fokker-Planck (FP) equation

\[ p_\alpha(x,t) = q(x) + \frac{1}{\Gamma(\alpha)} \int_0^t d\tau (t-\tau)^{\alpha-1} \hat{L}(x) p_\alpha(x,\tau), \]

where \( \hat{L}(x) \) is a time-independent FP operator (for example, \( -\frac{2}{\delta^2} F(x) + D \frac{\partial^2}{\delta x^2} \) with a force \( F \)), and \( q(x) \) defines an initial condition, describes the PDF \( p_\alpha(x,t) \) with one-parameter time scaling of the MSD. In a general case, the operator \( \hat{L} \) can be multidimensional and/or fractional in space, whose exact form is not important here. As it is well known,\(^{22,23}\) the stochastic process \( Y(t) \) with the PDF of Eq. (1) can be represented as a rescaled diffusive motion \( X(\tau) \) subordinated by an inverse \( \alpha \)-stable Lévy process \( S_\alpha(t) \), independent of the former. In this case, the stochastic process \( X(\tau) \) has the PDF \( h(x,\tau) \) evolving according to the following FP equation, namely:

\[ \partial h(x,\tau)/\partial \tau = \hat{L}(x)h(x,\tau). \]

It describes evolution of a particle subject to the operational time \( \tau \). The subordination of \( X(\tau) \) by the process \( S_\alpha(t) \) means that \( Y(t) = X[S_\alpha(t)] \) consists of two random processes such that one of them, \( X(\tau) \), is a parent, and another introduces a new operational time to the system. In fact, the process \( Y(t) \) is a continuous time random walk. We will consider the processes \( X(\tau) \) and \( S_\alpha(t) \) independent of each other. Generally speaking, they can be also dependent, for example, in the case of Ref. \(^{24}\). The operational time \( \tau = S_\alpha(t) \) is defined from a strictly increasing \( \alpha \)-stable Lévy process \( T_\alpha(t) \). Recall that the subordinator \( T_\alpha(t) \) is nothing else, but the continuous limit of a sequence \( T_{\alpha,i} \), \( i = 1, 2, \ldots \) of non-negative, independent, identically distributed (iid) random variables (obeying an \( \alpha \)-stable Lévy PDF) which represent waiting-time intervals between subsequent jumps of a walker. If \( g_\alpha(t,\tau) \) is the PDF of \( T_\alpha(t) \), the mean \[ e^{-uT_\alpha(t)} \] is the Laplace transform of \( g_\alpha(t,\tau) \) equal to

\[ \langle e^{-uT_\alpha(t)} \rangle = \int_0^\infty e^{-ut} g_\alpha(t,\tau) d\tau = e^{-\sigma u}, \quad 0 < \alpha < 1. \]

Then the operational time \( S_\alpha(t) \) as an inverse process to \( T(\tau) \) fulfills the following first passage time relation:

\[ S_\alpha(t) = \inf\{\tau > 0 : T_\alpha(\tau) > t\}. \]

Here, it should be emphasized that Eq. (1) is not sufficient to define the stochastic process \( Y(t) \) completely. This equation determines only the PDF of \( Y(t) \) that is not enough to examine all properties of anomalous diffusion without any exception. The same relates to the transient anomalous diffusion.

Let us denote the corresponding stochastic process of the transient anomalous diffusion by \( Y_\alpha(t) \). As for the pure subdiffusion considered above, we will find the description of the process \( Y_\alpha(t) \) in the subordination form where the parent process is \( X(\tau) \) as before, i.e., \( Y_\alpha(t) = X[S_\alpha(t)] \). The operational time \( S_\alpha(t) \) independent of \( X(\tau) \), written as

\[ S_\alpha(t) = \inf\{\tau > 0 : T_\alpha(\tau) > t\} \]

is inverse to the subordinator \( T_\alpha(\tau) \). We define this subordinator through a sum of iid random variables with an infinitely divisible PDF. There exist many examples of such PDFs. Among them well-known are: Gaussian, inverse Gaussian, \( \alpha \)-stable, tempered \( \alpha \)-stable, exponential, gamma, compound Poisson, Pareto, Linnik, Mittag-Leffler (ML) and others, including right-skewed \( \alpha \)-stable distributions.\(^{25}\)

Following the Lévy-Khintchine formula, the case of non-negative infinitely divisible random variables is fully characterized by the exponentially weighted function

\[ \langle e^{-uT_\alpha(\tau)} \rangle = e^{-\psi(u)}, \]

where \( \psi(u) \) is called the Laplace exponent. So, the transient subordinator can be written also as \( T_\alpha(\tau) = T_\psi(\tau) \), since it is fully determined by the corresponding Laplace exponent \( \psi(u) \). Recall that infinitely divisible distributions were introduced by de Finetti in 1929 and studied intensively by Kolmogorov, Lévy, and Khintchine later.\(^{26}\)

Comparing with the pure anomalous diffusion, the development formally looks like a simple substitution \( u^\alpha \rightarrow \psi(u) \), but actually we extend a class of random processes which can be naturally used as subordinators. Notice that the subordinator \( T_\psi(\tau) \) should be a nondecreasing stochastic process but it can have positive jumps. For being a well-defined subordinator with an infinitely divisible distribution, the function \( \psi(u) \) should be nonnegative with \( \psi(0) = 0 \) and a complete monotone first derivative.\(^{25}\) This class of functions has a special name – Bernstein functions.\(^{27}\) It would be useful to recall here\(^{28}\) that a positive function \( \xi(t) \) defined on \( t \in (0, \infty) \) is complete monotone if its derivatives of all orders alternates in sign as \((-1)^k \xi^{(k)}(t) > 0, k = 0, 1, 2, \ldots \) In this case, the inverse subordinator \( S_\alpha(t) \) has nondecreasing continuous trajectories and can be used as a time arrow. If \( g(t,\tau) \) is the PDF of \( T_\psi(\tau) \), then the PDF \( f(t,\tau) \) of its inverse \( S_\alpha(t) \) can be represented as

\[ f(t,\tau) = \int_{-\infty}^\infty g(t',\tau) dt'. \]

Taking the Laplace transform of \( f(t,\tau) \) with respect to \( t \), we get

\[ \tilde{f}(\tau,u) = \frac{\psi(u)}{u} e^{-\tau\psi(u)}. \]

In particular, when \( \psi(u) = u^\alpha \) (\( \alpha \)-stable subordinator), Eq. (3) becomes

\[ \tilde{f}(\tau,u) = u^{\alpha-1} e^{-\tau u^\alpha}, \]

which is the Laplace image of an inverse \( \alpha \)-stable PDF typical for pure anomalous subdiffusion.\(^{29}\) If we consider \( \psi(u) = (u + \delta)^\alpha - \delta^\alpha \), where \( \alpha \) and \( \delta \) are the constant parameters (tempered \( \alpha \)-stable subordinator), Eq. (3) describes the Laplace image of an inverse tempered \( \alpha \)-stable PDF.\(^{20}\) Thus, tempering of the \( \alpha \)-stable PDF simply yields the transient anomalous diffusion.

III. MEAN-SQUARED DISPLACEMENT

Let \( h(x,\tau) \) be the PDF of the parent process \( X(\tau) \). Then the PDF of the subordinated process \( X[S_\alpha(t)] \) obeys the integral relationship between the probability densities of the
parent and directing processes, $X(\tau)$ and $S_n(\tau)$, respectively,
\[ p(x, t) = \int_0^\infty h(x, \tau) f(\tau, t) d\tau. \] (4)

Taking into account Eq. (3), the Laplace transform of Eq. (4) with respect to $\tau$ gives
\[ \tilde{p}(x, u) = \frac{\Psi(u)}{u} \tilde{h}(x, \Psi(u)). \] (5)

If the moments of the process $X(\tau)$ are known, it is not difficult to find the moments of the process $X[S_n(\tau)]$. For example, for the Gaussian process $B(\tau)$ the second moment is $\langle B^2(\tau) \rangle = D \tau$, where $D$ is a diffusive constant. Then the mean-squared displacement of $B[S_n(\tau)]$ can be written as
\[ \langle B^2[S_n(\tau)] \rangle = \int_0^\infty \langle B^2(\tau) \rangle f(\tau, t) d\tau. \]

The Laplace image $\langle \tilde{B}^2[S_n(\tau)] \rangle$ of $\langle B^2[S_n(\tau)] \rangle$ reads
\[ \langle \tilde{B}^2[S_n(\tau)] \rangle = \frac{D}{u} \Psi(u). \] (6)

By inverting the Laplace transform, we get
\[ \langle B^2[S_n(\tau)] \rangle = D \int_0^\tau K(y) dy, \] (7)

where $K(y)$ is the memory function with the Laplace image $1/\Psi(u)$.\(^{23}\)

Its physical sense will be clarified further. In particular, as applied to the tempered $\alpha$-stable transient anomalous diffusion, the memory function $K(y)$ is of the form $e^{-\beta y} y^\alpha - E_{\alpha, \beta}(\beta^\alpha y^\alpha)$, where
\[ E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta > 0 \]
is the two-parameter ML function well known in math literature.\(^{30}\)

Observation of the MSD evolution is not enough for identifying the stochastic equation behind. Various stochastic processes underlying the transient anomalous diffusion can have the same MSD asymptotic evolution in time (see, for example, Ref. 31), but they obey different PDFs. If the character of the MSD evolution in time for a particular subdiffusion was established, then this does not mean that it will be easy to restore the Laplace exponent of the subordinator responsible for such a diffusion process, and hence the identification of random processes participating in observed phenomena is a separate problem. For simplicity, we consider a normal diffuse process with an arbitrary subordinator. According to the consideration,\(^{32}\) the 2-variation of the standard Brownian motion $B(\tau)$ satisfies $\langle V^2(\tau) \rangle = 2D \tau$. The idea of $p$-variation, generalizing the well-known notion of total variation, is applied in various branches of science (see, for example, Ref. 33). It was successfully used in Ref. 32 as an efficient test for distinguishing between two very popular models of anomalous diffusion: continuous time random walk (CTRW) and fractional Brownian motion (FBM) in experimental data.

For a stochastic process $Z(t)$ observed on $[0, T]$, the 2-variation is calculated as $\langle V^2(\tau) \rangle = \lim_{n \to \infty} V_n^2(\tau)$, where $V_n^2(\tau)$ is the partial sum of increments of $Z(t)$ defined by the following relation:
\[ V_n^2(\tau) = \sum_{i=0}^{n-1} \left| Z \left( \frac{(i+1)T}{2^n} \land \tau \right) - Z \left( \frac{iT}{2^n} \land \tau \right) \right|^2 \]
with $a \land b = \min\{a, b\}$. Then, any transient anomalous diffusion $B[S_n(\tau)]$ can be characterized by the quadratic variation equal to $2DS_n(\tau)$, and each such anomalous diffusion is determined by its unique subordinator. This approach allows one to get the stochastic process $S_n(t)$ from $B[S_n(\tau)]$ directly. Namely, in order to identify various stochastic processes of the transient anomalous diffusion, it is necessary to compare properties of their subordinators. This is a more reliable method than only a simple comparison of the MSD evolution mentioned above. Nevertheless, the knowledge of the MSD can provide parameter estimation for subordinators, for example, $\gamma$ and $\delta$ in the inverse Gaussian case.

IV. TRANSIENT ANOMALOUS DIFFUSION EQUATION

If one remembers that $X(\tau)$ is a solution of the Langevin (Itô) stochastic differential equation
\[ dX(\tau) = F[X(\tau)] d\tau + (2D)^{\frac{1}{2}} dB(\tau), \]
driven by the standard Brownian motion $B(\tau)$, then the transient anomalous diffusion $X[S_n(\tau)]$ is governed by the following equation:\(^{22,33}\)
\[ dX[S_n(\tau)] = F[X[S_n(\tau)]] dS_n(\tau) + (2D)^{\frac{1}{2}} dB[S_n(\tau)], \] (8)
which provides a useful stochastic representation of the transient anomalous diffusion.

The phenomena with the transient anomalous behavior can also be observed in the framework of a more generalized Langevin equation which is beyond the scope of this paper (for more details, see Ref. 34).

Acting with the operator $\hat{L}(x)$ on the image $\tilde{p}(x, u)$ from Eq. (5), we find
\[ \hat{L}(x) \tilde{p}(x, u) = \Psi(u) \tilde{p}(x, u) - q(x) \frac{\Psi(u)}{u} , \] (9)
where $q(x)$ is an initial condition. Taking the inverse Laplace transform of Eq. (9), the formal integral representation of the FP equation reads
\[ p(x, t) = q(x) + \int_0^t d\tau K(t - \tau) \hat{L}(x) p(x, \tau) , \] (10)
describing the PDF of the transient anomalous diffusion process. Now the physical sense of $K(t)$ becomes completely clear. The function describes nothing else but memory effects. From properties of the Laplace exponent $\Psi(u)$ at once it follows that the memory function $K(t)$ is positive, and its integral (7), i.e., the MSD, grows in time. This means that different types of nonlinear behavior for the MSD of transient anomalous diffusion are possible. Knowing the memory function, we can define the covariance function
\[ \langle Y(t) Y(s) \rangle = D \int_0^{\text{min}(t,s)} K(y) dy \] (11)
pursuant to Refs. 23 and 35, but if we cannot find out the integrals (7) and (11) in any analytical form, the derivations require numerical approximations.

V. OTHER TRANSIENT SUBORDINATORS

In this section, we examine properties of the transient anomalous diffusion in Eq. (10) with new types of the inverse subordinator distribution in addition to \( \alpha \)-stable (pure anomalous subdiffusion case) and tempered stable ones mentioned above. They are of interest in the first place because their Laplace image \( 1/\Psi(u) \) can be inverted analytically. It should be pointed out that the anomalous diffusion with the inverse gamma subordinator was considered recently in Ref. 35.

A. Continuous “mixture” of \( \alpha \)-stable subordinators

The subordinator whose Laplace exponent is a power law (see, Eq. (3)) leads to a model of the pure anomalous diffusion, for which the MSD grows as a power law in time. The subordinator was generalized in Ref. 36. Its Laplace exponent has the following form:

\[
\Psi(u) = \int_0^1 u^{\alpha} r(\alpha) d\alpha,
\]

where \( r \) is a probability density on (0, 1). To take the choice \( r(\alpha; \alpha_1, \ldots, \alpha_n) = C_1 \delta(\alpha - \alpha_1) + \ldots + C_n \delta(\alpha - \alpha_n) \), where \( \delta(\cdot) \) denotes the Dirac \( \delta \)-function, and \( C_1 + \ldots + C_n = 1 \), we come to the model of accelerating diffusion described in Ref. 13. For \( n = 2 \), it reduces to double-order time fractional diffusion studied in Ref. 9. For more general distributed-order time fractional diffusion, see the consideration in Refs. 4 and 7–9. Moreover, a general FP equation, leading to a Sinai-like diffusion, was studied in Ref. 37.

The other interesting case, which was called as the “uniform mix” in Refs. 38 (a similar construction also has been mentioned in Refs. 4 and 5), corresponds to the choice \( r(\alpha) = 1 \) on (0, 1). Using considerations of Ref. 38, let us find the memory function of such a diffusion process. In this case, the Laplace exponent reads

\[
\Psi(u) = \int_0^1 u^{\alpha} d\alpha = \frac{u - 1}{\ln u}.
\]

According to Eq. (29.3.100) of the book in Ref. 39, the Laplace inverse of \( 1/\Psi(u) \) gives

\[
K(t) = L^{-1} \frac{1}{\Psi(u)} = e^t E_1(t),
\]

where \( E_1(t) = \int_0^\infty e^{-z} \frac{dz}{z} \) is the exponential integral. The character of this kernel determines the “ultraslow” diffusion (here and next \( D = 1 \)) judged from the exact form of the MSD

\[
\langle Y^2(t) \rangle = C_\gamma + e^t E_1(t) + \ln t
\]

where \( C_\gamma = 0.57721 \ldots \) is the Euler-Mascheroni constant. Based on \( \lim_{\tau \to \infty} t e^{t E_1(t)} = 1 \) and \( E_1(t) = -C_\gamma - \ln t - \sum_{k=1}^\infty (-1)^k \frac{t^k}{k}, \) the asymptotic behavior of \( \langle Y^2(t) \rangle \) is of the form

\[
\langle Y^2(t) \rangle \cong \begin{cases} (1 - C_\gamma) t - t \ln t & \text{for } t \ll 1, \\ C_\gamma + \ln t & \text{for } t \gg 1. \end{cases}
\]

For this transient anomalous process, the covariance function takes the form

\[
\langle Y(t)Y(s) \rangle = C_\gamma + e^{\min(t,s)} E_1(\min(t,s)) + \ln(\min(t,s)).
\]

The evolution of Eq. (14) is presented in Figure 1.

B. Inverse Gaussian subordinator

For the inverse Gaussian subordinator, the Laplace exponent is written as

\[
\Psi(u) = \delta\sqrt{\gamma^2 + 2u - \delta\gamma},
\]

where \( \gamma > 0, \delta > 0 \) are constant parameters.\(^{40}\) By rewriting \( 1/\Psi(u) \) as the following sum:

\[
\frac{1}{\delta\sqrt{\gamma^2 + 2u - \delta\gamma}} = \frac{1}{2\delta u} + \frac{1}{\delta\sqrt{\gamma^2 + 2u}} + \frac{\gamma^2}{2\delta\sqrt{2u}}
\]

it is not difficult to determine the memory function for the given diffusion. Using Eqs. (29.3.11) and (29.3.44) of the book in Ref. 39, the Laplace inverse of (18) gives

\[
K(t) = \frac{\gamma}{2\delta} + \frac{1}{\delta\sqrt{2\pi t}} \exp \left( -\frac{\gamma^2 t}{2} \right) + \frac{\gamma}{2\delta} \text{erf} \left( \frac{\sqrt{t}}{2} \right)
\]

where \( \text{erf} z = \frac{2}{\sqrt\pi} \int_0^z e^{-x^2} dx \) is the error function. Next, from the memory function we find the MSD dependence

\[
\langle Y^2(t) \rangle = \frac{\gamma}{2\delta} t + \frac{1}{\delta\gamma} \text{erf} \left( \frac{\sqrt{t}}{2} \right) + \frac{\gamma}{2\delta} \int_0^t \text{erf} \left( \frac{\sqrt{s}}{2} \right) ds
\]

For \( t \to 0 \) the value \( \langle Y^2(t) \rangle \) starts as \( \sqrt{\frac{2\gamma}{\delta}} \) due to the second term of (20), whereas for \( t \to \infty \) it accelerates to \( \frac{\gamma}{\delta} t^2 \) because of the first and third terms. The total behavior of (20) is shown in Figure 1.

VI. TRANSIENT SUBORDINATORS FOR COMPOUND RELAXATION LAWS

Relaxation process is defined as a decay of a macroscopic physical magnitude characteristic for an initially imposed state in the investigated system, and whatever transient relaxation is essentially a non-equilibrium phenomenon. From subordination point of view, relaxation problem should
be properly studied in the framework of the anomalous diffusion. The approach is based on analysis of diffusion front motions described in Ref. 41. Since the relaxation function $\phi(t)$ describes the temporal decay of a given excited mode $k$, it can be expressed through the Fourier transform of the diffusion process $X[S_r(t)]$ with respect to spatial coordinates $\phi(t) = \langle e^{ikX[S_r(t)]} \rangle$. (21)

Starting with Eq. (4), we can write the Laplace image of Eq. (21) as

$$\tilde{\phi}(u) = \frac{\Psi(u)}{u[\Xi(k) + \Psi(u)]},$$

where $\Xi(k)$ is the logarithm of the characteristic function of the process $X(t)$. In the frequency domain of measurements, it is more convenient to use the shape function, defined as the one-sided Fourier transform of the relaxation response $f(t) = -\frac{\phi(t)}{dt}$, namely,

$$\Phi^*(\omega) = \int_0^\infty e^{-i\omega t} \left(-\frac{d\phi(t)}{dt}\right) dt.'$$

Then, for the relaxation under the inverse subordinator $S_r(t)$, the above frequency-domain relaxation function is expressed in the following form:

$$\Phi^*(\omega) = \frac{1}{1 + \Upsilon(i\omega/\omega_p)},$$

where $\Upsilon(i\omega/\omega_p) = \Psi(i\omega)/\Xi(k)$, and $\omega_p > 0$ is the loss-peak frequency (inverse of the characteristic material relaxation time). As expected, for $\Psi(u) = u^\alpha$ ($\alpha$-stable subordinator) the function $\Phi^*(\omega)$, given by Eq. (23), tends to the well-known empirical Cole-Cole (CC) law. Similarly, if $\Psi(u) = (u + \delta)^\alpha - \delta^\alpha$ (tempered $\alpha$-stable subordinator), it provides transient diffusion (relaxation) that occupies an intermediate place between pure anomalous subdiffusion (CC law) and normal diffusion (Debye law).20

Under- and overshooting subordination scenario for fractional two-power-law relaxation responses was proposed in Ref. 44. This compound subordination results in contracting and stretching of the operational time $S_r(t)$ characteristic for the ML or CC relaxation mechanism

$$Z^U_{\alpha,\gamma}(t) \leq S_r(t) \leq Z^O_{\alpha,\gamma}(t),$$

where $Z^U_{\alpha,\gamma}(t)$ and $Z^O_{\alpha,\gamma}(t)$ denote the compound operational times (see Figure 2(a)) defined as under- and overshooting transient subordinators, respectively, and $0 < \alpha < 1, \gamma < 1$. Thus, the transient anomalous diffusion process $B(Z^U_{\alpha,\gamma}(t))$ is subdiffusive since its MSD has the form $2D^\gamma/\Gamma(1 + \alpha)$ and the transient anomalous diffusion process $B(Z^O_{\alpha,\gamma}(t))$ is superdiffusive since its MSD diverges. It should be noticed here that the pure anomalous subdiffusion process $B(S_r(t))$ takes an intermediate place between the under- and overshooting anomalous transient diffusion (Figure 2(b)). The
involves different modifications stretching or contracting of dissimilarity in the waiting-jump and jump-waiting schemes transient anomalous diffusion. Moreover, it indicates that the clustered complex system in which a given mode undergoes permittivity spectra of dipole moments in materials, in some of them (e.g., tric spectroscopy. This method, being sensitive to reorientation relaxation data in a wide range of frequency is the dielectric permittivity. Ref. 44). (GML) empirical relaxation patterns (for more details, see Negami (HN) and less typical Generalized Mittag-Leffler different relaxation scenarios covering the typical Havriliak-tended within the transient subordination approach into two scenario. As a consequence, the empirical CC scheme is ex-

\[ \varepsilon^*(\omega) = \varepsilon_\infty + \sum_{k=1}^{n} \Delta\varepsilon_k \left[ 1 + (i\omega\tau_k)^{\alpha_k} \right]^{\beta_k}, \tag{24} \]

based on a superposition of \( n \) HN equations, or its variants, Cole-Davidson (for \( \alpha_k = 1 \)), CC (for \( \beta_k = 1 \)), or Debye functions (for \( \alpha_k = 1 \) and \( \beta_k = 1 \)). In the above formula, \( \Delta\varepsilon_k = \varepsilon_{ok} - \varepsilon_\infty \), where \( \varepsilon_{ok} \) denotes the static permittivity and \( \varepsilon_\infty \) – the infinite-frequency limit.\(^3\) The power law exponents \( \alpha_k \) and \( \beta_k \) take values from the range (0, 1) and 0 < \( \alpha_k\beta_k < 1 \), \( \tau_k \) is the characteristic time constant. The dielectric response given in Eq. (24) suggests a transition between different types of relaxation in the investigated frequency range (usually of few decades).

A prominent example of such a behavior is the dynamics of hydrogen-bonded liquids such as aqueous mixtures of alcoholic systems.\(^4\) Investigations of the dielectric permittivity of water,\(^4\) alcohols, alcohol-water, and alcohol-alcohol mixtures (see, e.g., Refs. 49–53) or electrolyte solutions\(^5\) have led to a better understanding of the unique characteristics of water as a bulk liquid and as a solvent embedding solutes. It has been found that the dipole relaxation experiments yield a good insight to the wetting properties of water and to the physical basis of the hydrophobic effect as a general phenomenon of the interplay among hydrophobic and hydrophilic compounds.\(^55\), \(^56\) It is well-known that studies of dielectric relaxation in biological systems, especially at interfaces, help elucidate the biological role of water. However, despite the broad attention (so that it is difficult to afford a fair list of key references in a regular research paper) the theoretical understanding of the relaxation behavior in the above mentioned systems is still insufficiently developed.

We hope that the proposed mathematical model of the transient anomalous diffusion underlying the compound dielectric spectra may introduce a progress in understanding of the (stochastic in nature) glass-forming-liquid dynamics. In the case of the above mentioned hydrogen-bonded liquids, it may help to elucidate the role of system’s operational time in superposition of different relaxation responses. The superposed compound dielectric spectra, observed for such systems in a broad frequency range, from one side are clearly related with a cooperative-ensemble, single alcoholic-chain, and a free OH-group stochastic motion, and, from the other side, with the operational time of the system. Hence, clarification of the relationship among the system’s operational time, responsible for different responses at low, intermediate, and high frequency ranges, and the different levels of the structural motion is of special interest. A details will be presented elsewhere. A knowledge on the microdynamics of such systems is still one of the most important problems in condensed-matter physics and biophysics as well.

VII. CONCLUSIONS

A main feature of the transient anomalous diffusion is the nonlinear behavior of the MSD of a walking particle without any evident scaling. We have presented its stochastic representation as the subordinated stochastic process \( X[S_n(t)] \) whose PDF obeys Eq. (10) characterized by memory effects. The memory function of this equation has a direct connection with properties of the inverse subordinator \( S_n(t) \). By specific examples we have demonstrated a wide variety of MSD evolution of the transient anomalous diffusion in time. Moreover,
the relaxation function (23) obtained with the help of this approach generalizes the CC law, very popular in the dielectric studies.43 The experience in the stochastic background investigations of the CC law allows one to extend the analysis on a new type of relaxation phenomena described by other empirical dependences in frequency.44 Using a compound subordinators, the approach has good perspectives in development of the theoretical description of the compound dielectric spectra of glass-forming liquids.

ACKNOWLEDGMENTS

A.S. is grateful to the Institute of Physics and the Hugo Steinhaus Center for pleasant hospitality during his visit in Wrocław University of Technology. The research of A.W. was partially supported by NCN Maestro Grant No. 2012/06/A/ST1/00258.

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