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(*See also* **Catastrophe Derivatives**; **Diffusion Approximations**; **Derivative Pricing, Numerical Methods**; **Financial Economics**; **Inflation Impact on Aggregate Claims**; **Interest-rate Modeling**; **Lundberg Inequality for Ruin Probability**; **Market Models**; **Markov Chains and Markov Processes**; **Ornstein–Uhlenbeck Process**)

ECKHARD PLATEN

Simulation of Risk Processes

Introduction

The simulation of **risk processes** is a standard procedure for **insurance companies**. The generation of **aggregate claims** is vital for the calculation of the amount of loss that may occur. Simulation of risk processes also appears naturally in rating triggered step-up bonds, where the **interest rate** is bound to random changes of the companies' ratings.

Claims of random size $\{X_i\}$ arrive at random times T_i. The **number of claims** up to time t is described by the **stochastic process** $\{N_t\}$. The **risk process** $\{R_t\}_{t\geq 0}$ of an insurance company can be therefore represented in the form

$$R_t = u + c(t) - \sum_{i=1}^{N_t} X_i. \qquad (1)$$

This standard model for insurance risk [9, 10] involves the following:

- the claim arrival point process $\{N_t\}_{t\geq 0}$,
- an independent claim sequence $\{X_k\}_{k=1}^{\infty}$ of positive i.i.d. random variables with common mean μ,
- the nonnegative constant u representing the initial capital of the company,
- the premium function $c(t)$.

The company sells insurance policies and receives a **premium** according to $c(t)$. Claims up to time t are represented by the **aggregate claim** process $\{\sum_{i=1}^{N_t} X_i\}$. The **claim severities** are described by the random sequence $\{X_k\}$.

The simulation of the risk process or the aggregate claim process reduces therefore to modeling the point process $\{N_t\}$ and the claim size sequence $\{X_k\}$. Both processes are assumed independent; hence, can be simulated independently of each other. The modeling and computer generation of claim severities is covered in [14] and [12].

Sometimes a generalization of the risk process is considered, which admits gain in capital due to interest earned. Although risk processes with **compounding** assets have received some attention in the literature (see e.g. [7, 16]), we will not deal with them because generalization of the simulation schemes is straightforward.

The focus of this chapter is therefore on the efficient simulation of the claim arrival point process $\{N_t\}$. Typically, it is simulated via the arrival times $\{T_i\}$, that is, moments when the ith claim occurs, or the interarrival times (or waiting times) $W_i = T_i - T_{i-1}$, that is, the time periods between successive claims. The prominent scenarios for $\{N_t\}$, are given by the following:

- the homogeneous **Poisson process**,
- the nonhomogeneous Poisson process,
- the **mixed Poisson process**,
- the Cox process (or doubly stochastic Poisson process), and
- the **renewal process**.

In the section 'Claim Arrival Process', we present simulation algorithms of these five models. In the section 'Simulation of Risk Processes', we illustrate the application of selected scenarios for modeling the risk process. The analysis is conducted for the PCS (Property Claim Services [15]) dataset covering losses resulting from catastrophic events in the United States that occurred between 1990 and 1999.

Claim Arrival Process

Homogeneous Poisson Process

A continuous-time stochastic process $\{N_t : t \geq 0\}$ is a (homogeneous) **Poisson process** with intensity (or rate) $\lambda > 0$ if (i) $\{N_t\}$ is a **point process**, and (ii) the times between events are independent and identically distributed with an exponential(λ) distribution, that is, exponential with mean $1/\lambda$. Therefore, successive arrival times $T_1, T_2, \ldots, T_n$ of the Poisson process can be generated by the following algorithm:

Step 1: set $T_0 = 0$
Step 2: for $i = 1, 2, \ldots, n$ do
  Step 2a: generate an exponential random variable E with intensity λ
  Step 2b: set $T_i = T_{i-1} + E$

To generate an exponential random variable E with intensity λ, we can use the inverse transform method, which reduces to taking a random number U distributed uniformly on $(0, 1)$ and setting $E = F^{-1}(U)$, where $F^{-1}(x) = (-\log(1-x))/\lambda$ is the inverse of the exponential cumulative distribution function. In fact, we can just as well set $E = (-\log U)/\lambda$ since $1 - U$ has the same distribution as U.

Since for the homogeneous Poisson process the expected value $\mathbb{E}N_t = \lambda t$, it is natural to define the premium function in this case as $c(t) = ct$, where $c = (1+\theta)\mu\lambda$, $\mu = \mathbb{E}X_k$ and $\theta > 0$ is the relative safety loading that 'guarantees' survival of the insurance company. With such a choice of the risk function, we obtain the classical form of the risk process [9, 10].

Nonhomogeneous Poisson Process

One can think of various generalizations of the homogeneous Poisson process in order to obtain a more reasonable description of reality. Note that the choice of such a process implies that the size of the portfolio cannot increase or decrease. In addition, there are situations, like in **automobile insurance**, where claim occurrence epochs are likely to depend on the time of the year or of the week [10]. For modeling such phenomena the nonhomogeneous Poisson process (NHPP) suits much better than the homogeneous one. The NHPP can be thought of as a Poisson process with a variable intensity defined by the deterministic intensity (rate) function $\lambda(t)$. Note that the increments of an NHPP do not have to be stationary. In the special case when $\lambda(t)$ takes the constant value λ, the NHPP reduces to the homogeneous Poisson process with intensity λ.

The simulation of the process in the nonhomogeneous case is slightly more complicated than in the homogeneous one. The first approach is based on the observation [10] that for an NHPP with rate function $\lambda(t)$ the increment $N_t - N_s$, $0 < s < t$, is distributed as a Poisson random variable with intensity $\tilde{\lambda} = \int_s^t \lambda(u)\,du$. Hence, the cumulative distribution function F_s of the waiting time W_s is given by

$$\begin{aligned} F_s(t) &= P(W_s \le t) = 1 - P(W_s > t) \\ &= 1 - P(N_{s+t} - N_s = 0) = \\ &= 1 - \exp\left(-\int_s^{s+t} \lambda(u)\,du\right) \\ &= 1 - \exp\left(-\int_0^t \lambda(s+v)\,dv\right). \end{aligned} \quad (2)$$

If the function $\lambda(t)$ is such that we can find a formula for the inverse F_s^{-1}, then for each s we can generate a random quantity X with the distribution F_s by using the inverse transform method. The algorithm, often called the 'integration method', can be summarized as follows:

Step 1: set $T_0 = 0$
Step 2: for $i = 1, 2, \ldots, n$ do
  Step 2a: generate a random variable U distributed uniformly on $(0, 1)$
  Step 2b: set $T_i = T_{i-1} + F_s^{-1}(U)$

The second approach, known as the **thinning** or 'rejection method', is based on the following observation [3, 14]. Suppose that there exists a constant $\bar{\lambda}$ such that $\lambda(t) \le \bar{\lambda}$ for all t. Let $T_1^*, T_2^*, T_3^*, \ldots$ be the successive arrival times of a homogeneous Poisson process with intensity $\bar{\lambda}$. If we accept the ith arrival time with probability $\lambda(T_i^*)/\bar{\lambda}$, independently of all other arrivals, then the sequence $T_1, T_2, \ldots$ of the accepted arrival times (in ascending order) forms a sequence of the arrival times of a nonhomogeneous Poisson process with rate function $\lambda(t)$. The resulting algorithm reads as follows:

Step 1: set $T_0 = 0$ and $T^* = 0$
Step 2: for $i = 1, 2, \ldots, n$ do
  Step 2a: generate an exponential random variable E with intensity $\bar{\lambda}$
  Step 2b: set $T^* = T^* + E$
  Step 2c: generate a random variable U distributed uniformly on $(0, 1)$
  Step 2d: if $U > \lambda(T^*)/\bar{\lambda}$, then return to step 2a ($\rightarrow$ reject the arrival time), else set $T_i = T^*$ ($\rightarrow$ accept the arrival time)

As mentioned in the previous section, the interarrival times of a homogeneous Poisson process have an exponential distribution. Therefore, steps 2a–2b generate the next arrival time of a homogeneous Poisson process with intensity $\bar{\lambda}$. Steps 2c–2d amount to rejecting (hence the name of the method) or accepting a particular arrival as part of the thinned process (hence the alternative name).

We finally note that since in the nonhomogeneous case the expected value $\mathbb{E}N_t = \int_0^t \lambda(s)\,ds$, it is natural to define the premium function as $c(t) = (1+\theta)\mu \int_0^t \lambda(s)\,ds$.

Mixed Poisson Process

The very high **volatility** of risk processes, for example, expressed in terms of the index of **dispersion** $\mathrm{Var}(N_t)/\mathbb{E}(N_t)$ being greater than 1 – a value obtained for the homogeneous and the nonhomogeneous cases, led to the introduction of the **mixed Poisson process** [2, 13]. In many situations, the portfolio of an insurance company is diversified in the sense that the risks associated with different groups of policy holders are significantly different. For example, in motor insurance, we might want to make a difference between male and female drivers or between drivers of different ages. We would then assume that the claims come from a heterogeneous group of clients, each one of them generating claims according to a Poisson distribution with the intensity varying from one group to another.

In the mixed Poisson process, the distribution of $\{N_t\}$ is given by a mixture of Poisson processes. This means that, conditioning on an extrinsic random variable Λ (called a structure variable), the process $\{N_t\}$ behaves like a homogeneous Poisson process. The process can be generated in the following way: first a realization of a nonnegative random variable Λ is generated and, conditioned upon its realization, $\{N_t\}$ as a homogeneous Poisson process with that realization as its intensity is constructed. Making the algorithm more formal we can write:

Step 1: generate a realization λ of the random intensity Λ
Step 2: set $T_0 = 0$
Step 3: for $i = 1, 2, \ldots, n$ do
  Step 3a: generate an exponential random variable E with intensity λ
  Step 3b: set $T_i = T_{i-1} + E$

Since for each t the claim numbers $\{N_t\}$ up to time t are Poisson with intensity Λt, in the mixed case, it is reasonable to consider the premium function of the form $c(t) = (1+\theta)\mu\Lambda t$.

Cox Process

The Cox process, or doubly stochastic Poisson process, provides flexibility by letting the intensity not only depend on time but also by allowing it to be a **stochastic process**. Cox processes seem to form a natural class for modeling risk and size fluctuations. Therefore, the doubly stochastic Poisson process can be viewed as a two-step randomization procedure. An intensity process $\{\Lambda(t)\}$ is used to generate another process $\{N_t\}$ by acting as its intensity; that is, $\{N_t\}$ is a Poisson process conditional on $\{\Lambda(t)\}$, which itself is a stochastic process. If $\{\Lambda(t)\}$ is deterministic, then $\{N_t\}$ is a nonhomogeneous Poisson process. If $\Lambda(t) = \Lambda$ for some positive random variable Λ, then $\{N_t\}$ is a mixed Poisson process.

This definition suggests that the Cox process can be generated in the following way: first a realization of a nonnegative stochastic process $\{\Lambda(t)\}$ is generated and, conditioned upon its realization, $\{N_t\}$ as a nonhomogeneous Poisson process with that realization as its intensity is constructed. Making the algorithm more formal we can write:

Step 1: generate a realization $\lambda(t)$ of the intensity process $\{\Lambda(t)\}$ for a sufficiently large time period
Step 2: set $\bar{\lambda} = \max\{\lambda(t)\}$
Step 3: set $T_0 = 0$ and $T^* = 0$
Step 4: for $i = 1, 2, \ldots, n$ do
  Step 4a: generate an exponential random variable E with intensity $\bar{\lambda}$
  Step 4b: set $T^* = T^* + E$
  Step 4c: generate a random variable U distributed uniformly on $(0, 1)$
  Step 4d: if $U > \lambda(T^*)/\bar{\lambda}$ then return to step 4a ($\rightarrow$ reject the arrival time) else set $T_i = T^*$ ($\rightarrow$ accept the arrival time)

In the doubly stochastic case, the **premium function** is a generalization of the former functions, in line with the generalization of the claim arrival process. Hence, it takes the form $c(t) = (1+\theta)\mu\int_0^t \Lambda(s)\,ds$.

Renewal Process

Generalizing the **point process** we come to the position where we can make a variety of distributional assumptions on the sequence of waiting times $\{W_1, W_2, \ldots\}$. In some particular cases, it might be useful to assume that the sequence is generated by a **renewal process** of claim arrival epochs, that is, the random variables W_i are i.i.d. and nonnegative. Note that the homogeneous Poisson process is a renewal process with exponentially distributed interarrival times. This observation lets us write the following algorithm for the generation of the arrival times for a renewal process:

Step 1: set $T_0 = 0$
Step 2: for $i = 1, 2, \ldots, n$ do
  Step 2a: generate a random variable X with an assumed distribution function F
  Step 2b: set $T_i = T_{i-1} + X$

An important point in the previous generalizations of the Poisson process was the possibility to compensate risk and size fluctuations by the premiums. Thus, the premium rate had to be constantly adapted to the development of the total claims. For renewal claim arrival processes, a constant premium rate allows for a constant safety loading [8]. Let $\{N_t\}$ be a renewal process and assume that W_1 has finite mean $1/\lambda$. Then the premium function is defined in a natural way as $c(t) = (1+\theta)\mu\lambda t$, like in the homogeneous Poisson process case.

Simulation of Risk Processes

In this section, we will illustrate some of the models described earlier. We will conduct the analysis on the PCS [15] dataset covering losses resulting from catastrophic events in the United States of America that occurred between 1990 and 1999. The data includes market's loss amounts in USD adjusted for **inflation**. Only natural perils, which caused damages exceeding five million dollars, were taken into consideration. Two largest losses in this period were caused by **hurricane** Andrew (August 24, 1992) and the Northridge **earthquake** (January 17, 1994).

The **claim arrival process** was analyzed by Burnecki et al. [6]. They fitted exponential, log-normal, Pareto, Burr and gamma distributions to the waiting time data and tested the fit with the χ^2, Kolmogorov–Smirnov, Cramer–von Mises and Anderson–Darling test statistics; see [1, 5]. The χ^2 test favored the exponential distribution with $\lambda_w = 30.97$,

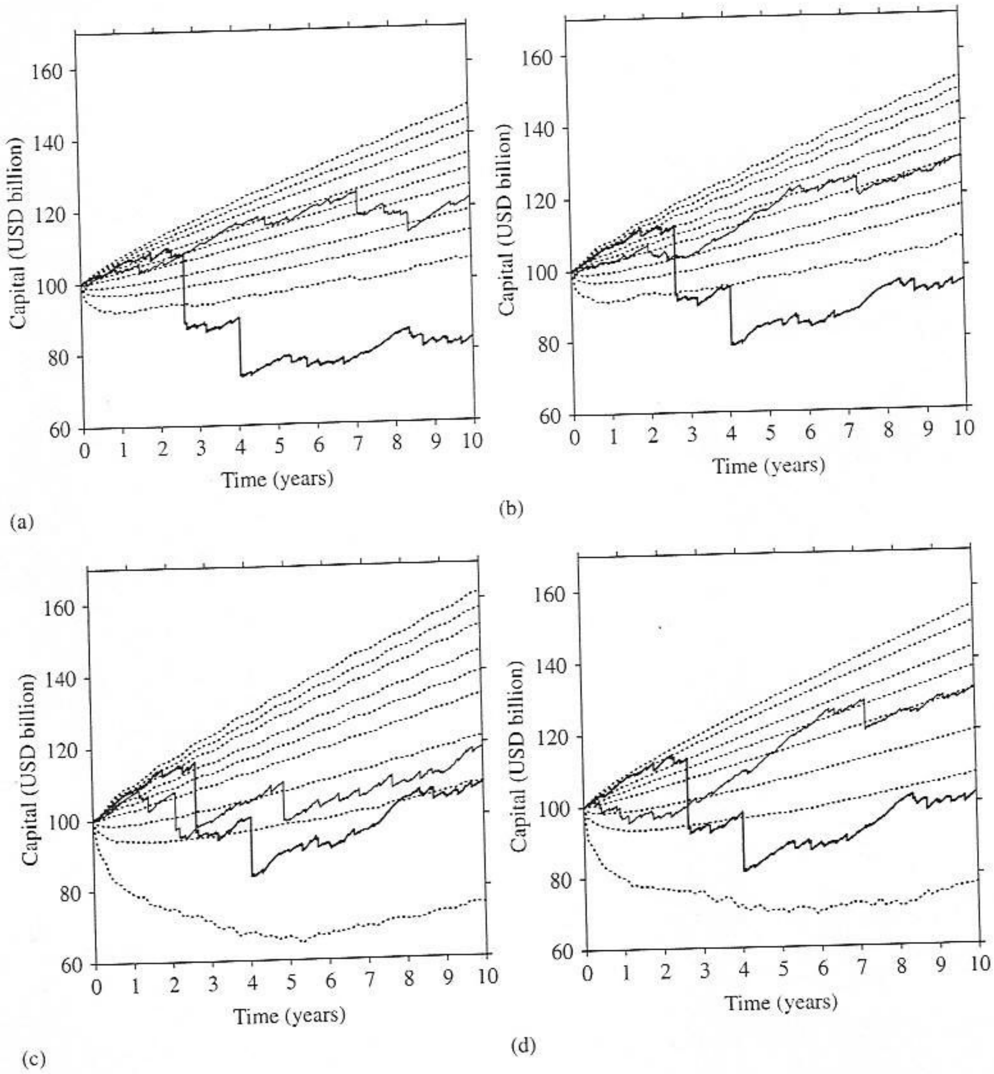

Figure 1 Simulation results for a homogeneous Poisson process with log-normal claim sizes (a), a nonhomogeneous Poisson process with log-normal claim sizes (b), a nonhomogeneous Poisson process with Pareto claim sizes (c), and a renewal process with Pareto claim sizes and log-normal waiting times (d). Figures were created with the Insurance library of XploRe [17]

justifying application of the homogeneous Poisson process. However, other tests suggested that the distribution is rather log-normal with $\mu_w = -3.88$ and $\sigma_w = 0.86$ leading to a renewal process. Since none of the analyzed distributions was a unanimous winner, Burnecki et al. [6] suggested to fit the rate function $\lambda(t) = 35.32 + 2.32(2\pi)\sin[2\pi(t - 0.20)]$ and treat the claim arrival process as a nonhomogeneous Poisson process.

The claim severity distribution was studied by Burnecki and Kukla [4]. They fitted **log-normal**, **Pareto**, **Burr** and gamma distributions and tested the fit with various nonparametric tests. The log-normal distribution with $\mu_s = 18.44$ and $\sigma_s = 1.13$ passed all tests and yielded the smallest errors. The Pareto distribution with $\alpha_s = 2.39$ and $\lambda_s = 3.03\ 10^8$ came in second.

The simulation results are presented in Figure 1. We consider a hypothetical scenario where the insurance company insures losses resulting from catastrophic events in the United States. The company's initial capital is assumed to be $u =$ USD 100 billion and the relative safety loading used is $\theta = 0.5$. We choose four models of the risk process whose application is most justified by the statistical results described above: a homogeneous Poisson process with log-normal claim sizes, a nonhomogeneous Poisson process with log-normal claim sizes, a nonhomogeneous Poisson process with Pareto claim sizes, and a renewal process with Pareto claim sizes and log-normal waiting times. It is important to note that the choice of the model has influence on both – the ruin probability and the **reinsurance** of the company.

In all subplots of Figure 1, the thick solid line is the 'real' risk process, that is, a trajectory constructed from the historical arrival times and values of the losses. The thin solid line is a sample trajectory. The dotted lines are the sample 0.001, 0.01, 0.05, 0.25, 0.50, 0.75, 0.95, 0.99, 0.999-quantile lines based on 50 000 trajectories of the risk process. Recall that the function $\hat{x}_p(t)$ is called a sample p-quantile line if for each $t \in [t_0, T]$, $\hat{x}_p(t)$ is the sample p-quantile, that is, if it satisfies $F_n(x_p-) \le p \le F_n(x_p)$, where F_n is the **empirical distribution function**. Quantile lines are a very helpful tool in the analysis of stochastic processes. For example, they can provide a simple justification of the **stationarity** (or the lack of it) of a process; see [11]. In Figure 1, they visualize the evolution of the density of the risk process. Clearly, if claim severities are Pareto distributed then **extreme events** are more probable than in the log-normal case, for which the historical trajectory falls even outside the 0.001-quantile line. This suggests that Pareto-distributed claim sizes are more adequate for modeling the 'real' risk process.

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(*See also* **Coupling**; **Diffusion Processes**; **Dirichlet Processes**; **Discrete Multivariate Distributions**; **Hedging and Risk Management**; **Hidden Markov Models**; **Lévy Processes**; **Long Range Dependence**;

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KRZYSZTOF BURNECKI, WOLFGANG HÄRDLE & RAFAŁ WERON

Simulation of Stochastic Processes

Many problems in insurance and finance require **simulation** of the sample path of a stochastic process $\{X_t\}$. This may be the case even when only a single value X_T is required, but the distribution of X_T is inaccessible, say, the reserve at time T in a **compound Poisson risk process** with the interest rate described by a Cox–Ingersoll–Ross process. Other problems may require a functional depending on the whole sample path, like the average $\int_0^T X_t \, dt$ arising in Asian options.

Of course, only a finite segment of a sample path can be generated and only at a finite set of time points, not the whole continuum of values. Even so, the generation on a computer may be nontrivial and usually depends heavily on the specific structure of the process whereas there are few general methods applicable to a broad spectrum of stochastic processes.

The Markov Case

We refer to **Markov chains and Markov processes** for background. The simulation of a Markov chain $\{X_n\}_{n=0,1,\ldots}$ (with discrete state space E) being specified by the transition probabilities p_{ij} and the initial probabilities μ_i is straightforward: one just starts by selecting $X_0 \in E$ w.p. μ_i for i. If i is selected for X_0, one then selects j for X_1 w.p. p_{ij}; k is then selected for X_2 w.p. p_{jk} and so on. Note, however, that in practice one may choose to base the simulation on a model description rather than the transition probabilities: for example, if a **bonus system** in **car insurance** works the way that the bonus class X_n in year n is calculated as $\varphi(X_{n-1}, Y_{n-1})$ where Y_{n-1} is the number of accidents in year $n-1$, one would simulate using this recursion, not the transition probabilities.

For a discrete Markov process $\{X_t\}_{t\geq 0}$ in continuous time, one would usually just use the fact that the sequence $\{Z_n\}_{n=0,1,\ldots}$ of different states visited forms a Markov chain, and that the holding times are exponential with a parameter depending on the current state. The exponential distribution of interevent times is also the most natural way to simulate a **Poisson process** (or compound Poisson process as arising in the **classical risk model**).

Diffusions and SDEs

Standard **Brownian motion** $\{B_t\}_{t\geq 0}$ is usually just simulated as a discrete skeleton: if the time horizon is $[0, T]$, one selects a large integer N, generates N i.i.d. r.v.'s. $V_1, \ldots, V_N$ with a $N(0, h)$ distribution where $h = T/N$ and let $B_{kh} = V_1 + \cdots + V_k$, $k = 0, \ldots, N$.

For a stochastic differential equation

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + a(t, X_t)\,\mathrm{d}B_t, \quad X_0 = x_0, \tag{1}$$

the most straightforward way is the *Euler scheme*, which generates an approximation $\{\widehat{X}_{kh}\}$ to $\{X_{kh}\}$ as

$$\widehat{X}_0 = x_0, \quad \widehat{X}_{(k+1)h} = b(kh, \widehat{X}_{kh})h + a(kh, \widehat{X}_{kh})V_{k+1}, \quad k = 0, \ldots, N-1, \tag{2}$$

where as above the V_i are i.i.d. $N(0, h)$ r.v.'s.

The obvious intuition behind the Euler scheme is

$$\int_{kh}^{(k+1)h} a(t, X_t)\,\mathrm{d}t \approx a(kh, X_{kh})h, \tag{3}$$

$$\int_{kh}^{(k+1)h} b(t, X_t)\,\mathrm{d}B_t \approx b(kh, X_{kh})(B_{(k+1)h} - B_{kh}). \tag{4}$$

Here by the **box calculus** one expects the error from (4) to be larger than that from (3) since $B_{(k+1)h} - B_{kh}$ if of order $h^{1/2}$, not h. A more refined approximation than (2) produces the *Milstein scheme* $X_0 = x_0$,

$$\widehat{X}_{(k+1)h} = bh + aV_{k+1} + bb_x(V_{k+1}^2 - h), \tag{5}$$