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Annuities under random rates of interest—revisited

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Abstract

In the article we consider accumulated values of annuities-certain with yearly payments with independent random interest rates. We present the variance formulae of the final values of the annuities, which leads to a correction of Theorems 4.2, 4.3, 4.5 and 4.6 from Zaks [Insurance: Mathematics and Economics 28 (2001) 1].

MSC: IE50; IE51

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An annuity is defined as a sequence of payments of a limited duration which we denote by n (see, e.g. Gerber, 1997). The accumulated or final values of annuities are of our interest. Typically, for simplicity, it is assumed that underlying interest rate is fixed and the same for all years. However, the interest rate that will apply in future years is of course neither known nor constant. Thus, it seems reasonable to let interest rates vary in a random way over time, cf. e.g. Kellison (1991).

We assume that annual rates of interest are independent random variables with common mean and variance. We apply this assumption in order to compute, via recursive relationships, fundamental characteristics, namely mean and variance, of the accumulated values of annuities with payments varying in arithmetic and geometric progression (see, e.g. Kellison, 1991). Since these important varying annuities can be reduced to the cases considered by Zaks (2001), we discover several mistakes in main results of Zaks (2001). Of course, *errare humanum est*, but for the benefit of the readers we correct all of them here. Moreover, we note that the errors in main results of Zaks (2001) are 'independent', namely they are not merely a consequence of one incorrect result, e.g. formula (2.8).

We follow basic notation used in the theory of annuities, see, e.g. Gerber (1997). Suppose that j is the positive annual interest rate and fixed through the period of n years. We assume that $k \leq n$ throughout, unless otherwise specified. Let us now consider a standard increasing annuity-due. It corresponds to the case of an annuity-due with payments varying in arithmetic progression with $p = 1$ and $q = 1$. The accumulated value of such annuity with k annual payments of $1, 2, \ldots, k$, respectively, is

$$(I\ddot{s})_{\overline{k}|j} = \frac{\ddot{s}_{\overline{k}|j} - k}{d}. \tag{1}$$

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This can be expressed recursively as

$$(I\ddot{s})_{\overline{k}|j} = (1+j)^k + 2(1+j)^{k-1} + \cdots + k(1+j) = (1+j)(k + (I\ddot{s})_{\overline{k-1}|j}), \tag{2}$$

which corrects formula (2.8) from Zaks (2001).

Let us suppose that the annual rate of interest in the kth year is a random variable i_k. We assume that, for each k, we have $E(i_k) = j > 0$ and $\mathrm{Var}(i_k) = s^2$, and that $i_1, i_2, \ldots, i_n$ are independent random variables. We write $E(1+i_k) = 1+j = \mu$ and $E[(1+i_k)^2] = (1+j)^2 + s^2 = 1+f = m$, where $f = 2j + j^2 + s^2$. Obviously $\mathrm{Var}(1+i_k) = m - \mu^2$. Next we define r to be the solution of $1+r = (1+f)/(1+j)$. Using the relation between f and j we have $r = j + (s^2/(1+j))$.

For a k-year variable annuity-due with annual payments of $c_1, c_2, \ldots, c_k$, respectively, we denote their final value by C_k.

Fact 1. *If C_k denotes the final value of an increasing annuity-due with k annual payments of $1, 2, \ldots, k$, respectively, and if the annual rate of interest during the kth year is a random variable i_k such that $E(1+i_k) = 1+j$ and $\mathrm{Var}(1+i_k) = s^2$, and $i_1, i_2, \ldots, i_n$ are independent random variables, then*

(a) $E(C_k) = (I\ddot{s})_{\overline{k}|j}$,

(b) $M_{1k} = (I^2\ddot{s})_{\overline{k}|f}$,

(c) $$M_{2k} = \frac{(1+j)^{k+2}(I\ddot{s})_{\overline{k}|r} - (1+j)(I\ddot{s})_{\overline{k}|f} - j(1+j)(I^2\ddot{s})_{\overline{k}|f}}{j^2},$$

(d) $$m_k = \frac{2(1+j)^{k+2}(I\ddot{s})_{\overline{k}|r} - 2(1+j)(I\ddot{s})_{\overline{k}|f} - j(2+j)(I^2\ddot{s})_{\overline{k}|f}}{j^2}.$$

Part (b) corrects (4.6) from Zaks (2001) and (d) corrects Theorem 4.2 from Zaks (2001). These results can be summarized in the following corollary:

Fact 2. *Under the assumptions of* Fact 1 *we have*

$$\mathrm{Var}(C_k) = \frac{2(1+j)^{k+2}(I\ddot{s})_{\overline{k}|r} - 2(1+j)(I\ddot{s})_{\overline{k}|f} - j(2+j)(I^2\ddot{s})_{\overline{k}|f}}{j^2} - \frac{(I\ddot{s})_{\overline{2k}|j} - 2(1+kd)(I\ddot{s})_{\overline{k}|j} - k^2}{d^2}. \tag{3}$$

Fact 2 corrects Theorem 4.3 from Zaks (2001). Let us now consider the case of a decreasing annuity-due. It corresponds to the case of an annuity-due with payments varying in arithmetic progression with $p = n$ and $q = -1$.

Fact 3. *If C_k denotes the final value of a decreasing annuity-due with k annual payments of $n, n-1, \ldots, n-k+1$, respectively, and if the annual rate of interest during the kth year is a random variable i_k such that $E(1+i_k) = 1+j$ and $\mathrm{Var}(1+i_k) = s^2$, and $i_1, i_2, \ldots, i_n$ are independent random variables, then*

$$\mathrm{Var}(C_k) = \frac{\ell}{d^2}\left[\frac{(n-1/j)^2(1+j)^{2k}\ddot{s}_{\overline{k}|\ell}}{1+\ell} - \frac{2(n-1/j)^2(1+j)^k\ddot{s}_{\overline{k}|r}}{1+r} + \frac{(n-1/j)^2\ddot{s}_{\overline{k}|f}}{1+f} + \frac{2(n-1/j)(1+j)^k(I\ddot{s})_{\overline{k}|r}}{1+r} - \frac{2(n-1/j)(I\ddot{s})_{\overline{k}|f}}{1+f} + \frac{(I^2\ddot{s})_{\overline{k}|f}}{1+f}\right], \tag{4}$$

where $\ell = (s/(1+j))^2$.

Fact 3 corrects Theorem 4.5 from Zaks (2001).

We now study the case of an annuity-due with k annual payments of $1, 1+u, (1+u)^2, \ldots, (1+u)^{k-1}$, respectively. It corresponds to an annuity-due with payments varying in geometric progression with $p = 1$ and $q = 1 + u$. We assume that $1 + u = (1 + j)(1 + t)$, $1 + f = (1 + u)^2(1 + h)$ and $1 + f = (1 + j)^2(1 + t)(1 + w)$.

Fact 4. *If C_k denotes the final value of an annuity-due with k annual payments of $1, 1+u, (1+u)^2, \ldots, (1+u)^{k-1}$, respectively, and if the annual rate of interest during the kth year is a random variable i_k such that $E(1+i_k) = 1+j$ and $\mathrm{Var}(1 + i_k) = s^2$, and $i_1, i_2, \ldots, i_n$ are independent random variables, then*

$$\mathrm{Var}(C_k) = \frac{(1+u)^{2k}(2+t)\ddot{s}_{\overline{k}|h} - 2(1+j)^{2k}(1+t)^k\ddot{s}_{\overline{k}|w}}{t} - \frac{(1+j)^{2k}(\ddot{s}_{\overline{2k}|t} - 2\ddot{s}_{\overline{k}|t})}{t(1+t)}. \tag{5}$$

Fact 4 corrects Theorem 4.6 from Zaks (2001). For details see Burnecki et al. (2003).

Finally, we note that one can approximate mean and variance of the final values of general annuities using numerical approach (cf. Kellison, 1991). In order to apply it let us assume that random variables i_k have common normal distribution with parameters $\mu = 0.08$ and $\sigma = 0.02$. This yields that $j = 0.08$ and $s^2 = 0.0004$. Moreover, we set $n = 10$ and the number of simulations $m = 100\,000$. We focus on the variance results. We plot $\mathrm{Var}(C_k)$ as a function of k for the preceding annuity using the analytical and numerical outcomes.

Fig. 1 depicts the comparison for an annuity with payments varying in geometric progression, see Fact 4, where we set $u = 0.1$. We can observe in the left panel that our analytical (see Fact 4) and numerical results agree while the corresponding Theorem 4.6 from Zaks (2001) yields outcome which is essentially smaller (right panel). Fig. 1

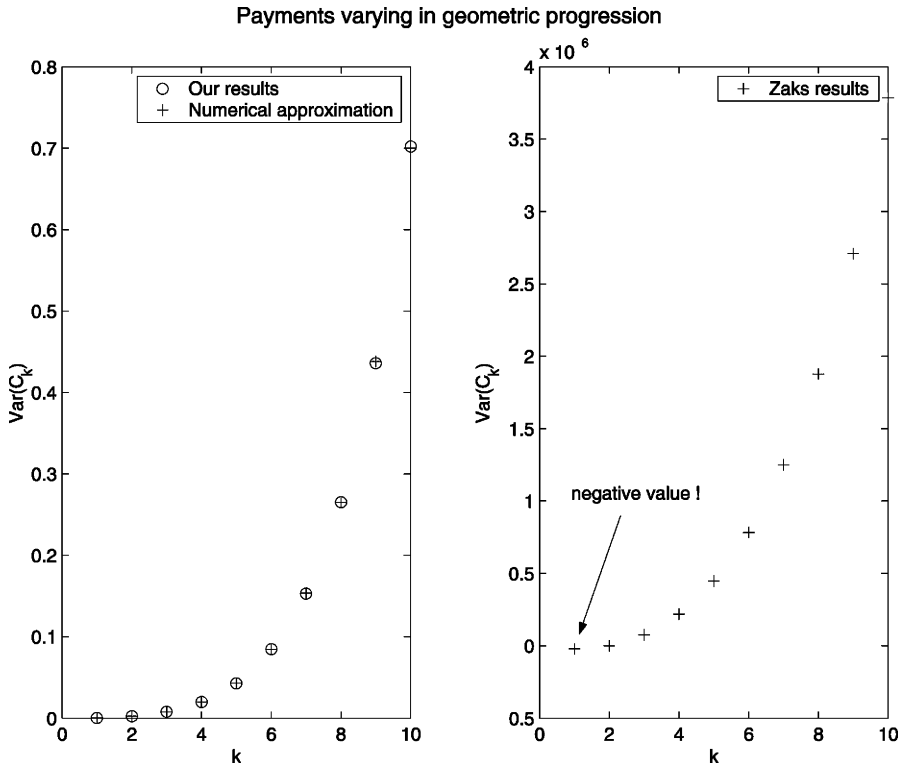

Fig. 1. Comparison of the analytical (+) and numerical (○) results on variance of the final value of an annuity-due with payments varying in geometric progression with $p = 1$ and $q = 1 + u$.

shows that the variance of the final values of annuities in Zaks (2001) can be 100 000 (!!!) times smaller than the correct number. Moreover, the variance is negative for $k = 1$.

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