# Empirical interpretation of the Pitman efficiency


Tadeusz Inglot

*Faculty of Pure and Applied Mathematics,*
*Wrocław University of Science and Technology*



**Abstract.** We study an empirical interpretation of the Pitman efficiency for testing uniformity in the two-parametric family of the beta distributions. For contamination models the efficiency aproximates empirical ratios of sample sizes very well.



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**1. Introduction.** Usually, the Pitman efficiency is determined for parametric problems with a real parameter (eg. Lehmann and Romano (2008)) or for parametric subproblems with a real parameter of some nonparametric problems. Such approaches do not allow to compare theoretical findings with their empirical counterparts for multidimensional sets of alternatives.

The aim of the present note is to study an empirical interpretation of the Pitman efficiency for a particular example of the two-dimensional family of the beta distributions. We start by recalling a definition of the Pitman efficiency we shall use in the sequel.

Let $\Gamma \subset \mathbb{R}^k,\ k \geqslant 1$, be a nonempty set, $\mathcal{P} = \{P_\gamma : \gamma \in \Gamma\}$ a family of distributions on a measurable space $(\mathcal{X}, \mathcal{A})$ and $X_1, ..., X_n$ a sample from a distribution $P \in \mathcal{P}$. Fix $\gamma_0 \in \Gamma$. We test the null hypothesis $H_0 : P = P_{\gamma_0}$ against $H_1 : P \neq P_{\gamma_0}$. Suppose we want to compare two upper-tailed tests given by statistics $T_n$, $V_n$. For $0 < \alpha < \beta < 1$ and an alternative $P$ let $N_T(\alpha, \beta, P)$ denote the minimal sample size such that for all $n \geqslant N_T(\alpha, \beta, P)$ the power of the test $T$ for the alternative $P$ at the significance level $\alpha$ and for the sample size $n$ is not smaller than $\beta$. Similarly we define $N_V(\alpha, \beta, P)$ for the test $V$. The relative efficiency of $T$ with respect to $V$ is defined to be

$$\mathcal{RE}_{TV}(\alpha, \beta, P) = \frac{N_V(\alpha, \beta, P)}{N_T(\alpha, \beta, P)}.$$

Let $\gamma(s) \in \Gamma,\ s \in [0, 1]$, be a continuous curve in $\Gamma$ such that $\gamma(0) = \gamma_0$. Assume that there exists a $\sigma$-finite measure $\lambda$ on $(\mathcal{X}, \mathcal{A})$ such that $P_{\gamma(s)} \ll \lambda$ for $s \in [0, 1]$ and $H(P_{\gamma(s)}, P_{\gamma_0}) \xrightarrow{s \to 0^+} 0$, where $H(P, Q)$ denotes the Hellinger distance between $P$ and $Q$. The family of distributions $\{P_{\gamma(s)}\} = \{P_{\gamma(s)} : s \in [0, 1]\}$ we shall call a path.

**Definition.** Given $0 < \alpha < \beta < 1$ and a path $\{P_{\gamma(s)}\}$. If there exists a limit

$$\lim_{s \to 0^+} \mathcal{RE}_{TV}(\alpha, \beta, P_{\gamma(s)}) = e^P_{TV}(\alpha, \beta, \{P_{\gamma(s)}\}) \in [0, \infty],$$

then we call it the Pitman efficiency of the test $T$ with respect to the test $V$ for the path $\{P_{\gamma(s)}\}$.

Below we provide a version of the Pitman theorem in the form ready to apply in Section 2. We shall need the following assumption on an asymptotic behaviour of a statistic $W_n$ for a path $\{P_{\gamma(s)}\}$:

there exist scaling functions $\mu(s) \geqslant 0,\ \sigma(s) > 0,\ s \in [0,1]$, and a continuous distribution function $G(x)$ such that

$$\lim_{n\to\infty} P^n_{\gamma_0}\left(\frac{W_n - \sqrt{n}\mu(0)}{\sigma(0)} \leqslant x\right) = G(x) \tag{1}$$

for all $x \in \mathbb{R}$ and for any sequence $s_n \to 0,\ s_n > 0$, we have

$$\lim_{n\to\infty} P^n_{\gamma(s_n)}\left(\frac{W_n - \sqrt{n}\mu(s_n)}{\sigma(s_n)} \leqslant x\right) = G(x) \tag{2}$$

for all $x \in \mathbb{R}$.

Condition (2) is a little bit weaker than the uniform convergence in $s$ (cf. the condition (P1) in Serfling (1980) or the condition D in Noether (1955)). Rothe (1981) proposed three conditions instead of (2). One of them is a continuity of the power function with respect to $s$ at $s = 0$ for every fixed $n$. The proof of the theorem, given below, is a modification of well known ones (cf. Lehmann and Romano (2008), Nikitin (1995)). Therefore we omit it. Usually (1) and (2) are fulfilled with $G(x) = \Phi(x)$ the standard normal distribution function. But in Theorem below and in its proof this fact is unimportant.

**Theorem.** Suppose $T_n, V_n$ satisfy (1) and (2) for a path $\{P_{\gamma(s)}\}$ with the same distribution function $G(x)$ increasing on the set $\{x : 0 < G(x) < 1\}$, functions $\sigma_T(s), \sigma_V(s)$ are continuous at $s = 0$, while $\mu_T(s), \mu_V(s)$ have nonnegative derivatives at the point $s = 0$. Denote $c^P_T = (\mu'_T(0)/\sigma_T(0))^2$ and $c^P_V = (\mu'_V(0)/\sigma_V(0))^2$. If $\max\{c^P_T, c^P_V\} > 0$ then there exists the Pitman efficiency of $T$ with respect to $V$ for the path $\{P_{\gamma(s)}\}$, does not depend on $\alpha$ and $\beta$ and equals

$$e^P_{TV}(\{P_{\gamma(s)}\}) = e^P_{TV}(\alpha, \beta, \{P_{\gamma(s)}\}) = \left(\frac{\mu'_T(0)/\sigma_T(0)}{\mu'_V(0)/\sigma_V(0)}\right)^2 = \frac{c^P_T}{c^P_V}, \tag{3}$$

where $c/0$ is understood as $\infty$.

**2. Example and empirical interpretation.** In this section we study an empirical interpretation of the Pitman efficiency for testing uniformity in the family of the beta distributions. Set

$$\mathcal{P} = \{P_\gamma : P_\gamma = P_{(p,q,\varepsilon)} = (1-\varepsilon)P_{1\,1} + \varepsilon P_{pq},\ \varepsilon \in [0,1],\ p \geqslant q > 0,\ \tau(p,q) \geqslant 0\},$$

where $P_{pq}$ denotes the beta distribution on $[0,1]$ with parameters $p, q$ and $\tau(p,q) = 2p^2 - 2pq - q^2 + 2p - q$. Let $\gamma_0 = (1,1,0)$. Then $P_{\gamma_0} = P_{(1,1,0)} = P_{1\,1}$ is the uniform distribution. We test the simple null hypothesis $H_0 : P = P_{1\,1}$. Consider two (upper-tailed) tests given by the statistics $V_n = \sqrt{n}(\overline{X} - 1/2)$ and $T_n = (\sum_{i=1}^n (X_i^2 - 1/3))/\sqrt{n}$.

Recall that

$$E_{pq}X_1 = \frac{p}{p+q},\quad m_2 = E_{pq}X_1^2 = \frac{p(p+1)}{(p+q)(p+q+1)}, \tag{4}$$

$$m_4 = E_{pq}X_1^4 = \frac{p(p+1)(p+2)(p+3)}{(p+q)(p+q+1)(p+q+2)(p+q+3)}. \tag{5}$$

Hence, for a distribution $P_{pq}$ with $\tau(p,q) < 0$ we have $E_{pq}X_1^2 = 1/3 + \tau(p,q)/3(p+q)(p+q+1) < 1/3$ and for paths lying in the region $\tau(p,q) < 0$ the statistic $T_n$ does not satisfy (2), as $\mu_T(s) < 0$. Therefore we have considered the restriction $\tau(p,q) \geqslant 0$ for the set of parameters.

Fix $P_{pq} = P_{(p,q,1)} \in \mathcal{P},\ (p,q) \neq (1,1)$, and let $P_{\gamma(s)} = (1-s)P_{1\,1} + sP_{pq},\ s \in [0,1]$. So, we have $\gamma(s) = (p,q,s)$ and the path $\{P_{\gamma(s)}\}$ links by a linear segment (in the space of distributions) $P_{11}$ to $P_{pq}$. $\{P_{\gamma(s)}\}$ forms a contamination family determined by a single alternative. Here we shall call it a linear path. Lyapunov's theorem and (4) imply that $V_n$ satisfies (1) and (2) with $G(x) = \Phi(x)$, $\mu_V(s) = (1-s)/2 + sE_{pq}X_1 - 1/2 = s(p-q)/2(p+q)$ and $\sigma_V^2(s) = (1-s)/3 + sm_2 - (\mu_V(s) + 1/2)^2$. The assumptions of the above theorem are satisfied for this test and $\mu_V'(0) = (p-q)/2(p+q)$, $\sigma_V(0) = 1/\sqrt{12}$ and consequently $c_V^P = 3(p-q)^2/(p+q)^2$. Similarly from (4), (5) and Lyapunov's theorem it follows that $T_n$ satisfies (1) and (2) with $G(x) = \Phi(x)$, $\mu_T(s) = s\tau(p,q)/3(p+q)(p+q+1)$, $\sigma_T^2(s) = (1-s)/5 + sm_4 - (\mu_T(s) + 1/3)^2$. Hence the assumptions of Theorem are also satisfied for $T_n$ and $\mu_T'(0) = \tau(p,q)/3(p+q)(p+q+1)$, $\sigma_T^2(0) = 4/45$ and $c_T^P = 5\tau^2(p,q)/4(p+q)^2(p+q+1)^2$. By (3) it follows that for $p > q$ with $\tau(p,q) > 0$ the Pitman efficiency of $T$ with respect to $V$ for linear paths exists and equals

$$\mathcal{E}_{TV}^P = \mathcal{E}_{TV}^P(P_{pq}) = \frac{5(2p^2 - 2pq - q^2 + 2p - q)^2}{12(p-q)^2(p+q+1)^2} = \frac{5\tau^2(p,q)}{12(p-q)^2(p+q+1)^2}. \tag{6}$$

For $p = q < 1$ the efficiency is equal to $\infty$ while for $p, q$ with $\tau(p,q) = 0$ is equal to 0. Observe that for $p, q$ lying on the line given by the equation $p - 2q + 1 = 0$, $p > 1$, we have $\mathcal{E}_{TV}^P(P_{pq}) = 5/12$ while for $p, q$ on the line $2p - q - 1 = 0$, $1/2 < p < 1$, we have $\mathcal{E}_{TV}^P(P_{pq}) = 5/3$. For a path contained in one of these lines the efficiency $e_{TV}^P$ takes the same value 5/12 or 5/3, respectively.

Compare the theoretical formula (6) for linear paths with the empirical behaviour of both tests for several alternatives and the significance level 0.05. Results are shown in Table 1. For each case the parameter $s$ was chosen to get the power of the test $V$ close to 1/2 for a moderate sample size.

**Table 1.** Empirical powers of the tests $V$ and $T$ (in %) and empirical efficiencies for selected alternatives. $\alpha = 0.05$, 100 000 MC.

| alternative $p$ | $q$ | $s$ | $n$ | test $V$ | $T$ | empirical efficiency | $\mathcal{E}_{TV}^P$ |
|---|---|---|---|---|---|---|---|
| 5 | 4 | 1 | 80 | 56 | 0 | | |
| | | | 30000 | 100 | 0 | $< 0.003$ | 0 |
| 4 | 3.15 | 0.9 | 100 | **63** | 03 | | |
| | | | 4000 | 100 | **63** | 0.025 | 0.026 |
| 6 | 4 | 0.5 | 100 | **54** | 15 | | |
| | | | 460 | 99 | **54** | 0.217 | 0.220 |
| 3 | 1 | 0.2 | 105 | **55** | 56 | | |
| | | | 100 | 54 | **55** | 1.05 | 1.067 |
| 0.6667 | 0.5 | 0.5 | 200 | **54** | 77 | | |
| | | | 100 | 36 | **54** | 2 | 2.074 |
| 0.55 | 0.5 | 0.9 | 640 | **58** | 100 | | |
| | | | 100 | 22 | **58** | 6.4 | 6.505 |
| 0.5 | 0.5 | 0.9 | 30000 | 09 | 100 | | |
| | | | 100 | 09 | 41 | $> 300$ | $\infty$ |

For the first alternative the power of $T$ is equal to 0.002 for both sample sizes and is denoted as 0 in Table 1. It may be seen a very good proximity of empirical and theoretical efficiencies for all values of $s$.

Now, take a path $\{P_{\gamma_1(s)}\}$ determined by the curve $\gamma_1(s) = \gamma_1(s;p,q) = (1 - s + ps, 1 - s - qs + 2qs^2, 1)$, where $p \geqslant q > 0$ are fixed with $\tau(p,q) > 0$ and $q < 3 + \sqrt{8}$. Here both parameters of the beta distribution are sent to 1 along a parabola without mixing with $P_{\gamma_0} = P_{11}$. The condition for $q$ guarantees that the curve is conatined in the set of parameters (i.e. $1 - s - qs + 2qs^2 > 0$ for $s \in [0,1]$).

Since densities of the beta distributions are continuous with respect to $(p, q)$ it follows by Lebesgue Bounded Convergence Theorem that the Hellinger distance $H(P_{\gamma_1(s)}, P_{11})$ tends to 0 when $s \to 0$ (parameters of the beta distibution are bounded away from 0 unifromly in $s$). From Lyapunov's theorem and (4) we get

$$\mu_V(s) = \frac{1-s+ps}{2-2s+ps-qs+2qs^2} - \frac{1}{2} = \frac{(p+q)s - 2qs^2}{2(2-2s+ps-qs+2qs^2)}, \quad s \in [0,1].$$

Thus $\mu_V'(0) = (p+q)/4$. As previously $\sigma_V^2(0) = 1/12$. So $c_V^P = 3(p+q)^2/4$. Analogously,

$$\begin{aligned}\mu_T(s) &= \frac{(1-s+ps)(2-s+ps)}{(2-2s+ps-qs+2qs^2)(3-2s+ps-qs+2qs^2)} - \frac{1}{3} \\ &= \frac{(4p+5q+1)s}{3(2-2s+ps-qs+2qs^2)(3-2s+ps-qs+2qs^2)} + O(s^2), \quad s \to 0^+.\end{aligned}$$

and hence $\mu_T'(0) = (4p+5q+1)/18$. Since $\sigma_T^2(0) = 4/45$ then $c_T^P = 5(4p+5q+1)^2/144$. Finally by (3) we obtain

$$e_{TV}^P(\{P_{\gamma_1(s)}\}) = \frac{5(4p+5q+1)^2}{108(p+q)^2}.$$

For example, take $p = 6, q = 4$. Then $e_{TV}^P(\{P_{\gamma_1(s;6,4)}\}) = 15/16 = 0.9375$. Choose several points on the considered path taking $s = 1, 0.5, 0.2, 0.1, 0.05, 0.02$. In Table 2 we show empirical powers and empirical efficiencies for selected distributions on the path $\{P_{\gamma_1(s;6,4)}\}$ and values of $\mathcal{E}_{TV}^P$ for linear paths corresponding to each $P_{p_1(s)q_1(s)}$ with $p = 6$, $q = 4$ calculated from (6), where $p_1(s) = 1 - s + ps$, $q_1(s) = 1 - s - qs + 2qs^2$. Since both tests are very sensitive for considered alternatives we mix them with $P_{11}$ in the form $(1-\varepsilon)P_{11} + \varepsilon P_{p_1(s)\,q_1(s)}$ with $\varepsilon$ such that sample sizes were not too small.

**Table 2.** Empirical powers and empirical efficiencies of $T$ with respect to $V$ for selected distributions on the path $\{P_{\gamma_1(s;6,4)}\}$. $\alpha = 0.05$, 100 000 MC.

| alternative | | | | | test | | empirical | |
|---|---|---|---|---|---|---|---|---|
| $s$ | $p_1(s)$ | $q_1(s)$ | $\varepsilon$ | $n$ | $V$ | $T$ | efficiency | $\mathcal{E}_{TV}^P$ |
| 0.5 | 3.5 | 0.5 | 0.1 | 200 | **57** | 67 | | |
| | | | | 148 | 47 | **57** | 1.35 | 1.375 |
| 0.2 | 2 | 0.32 | 0.1 | 180 | **52** | 62 | | |
| | | | | 133 | 43 | **52** | 1.35 | 1.420 |
| 0.1 | 1.5 | 0.58 | 0.2 | 150 | **59** | 65 | | |
| | | | | 128 | 54 | **59** | 1.17 | 1.217 |
| 0.05 | 1.25 | 0.77 | 0.2 | 200 | **54** | 55 | | |
| | | | | 189 | 52 | **54** | 1.06 | 1.083 |
| 0.02 | 1.10 | 0.9032 | 0.2 | 200 | **52** | 50 | | |
| | | | | 207 | 53 | **52** | 0.97 | 0.996 |

It follows from Table 2 that empirical efficiencies are very close to $\mathcal{E}_{TV}^P$ (similarly as it was seen in Table 1) but for distributions, on the considered path, relatively far from $P_{11}$ the Pitman efficiency $e_{TV}^P(\{P_{\gamma_1(s)}\}) = 0.9375$ does not reflect empirical behaviuor of the tests. Only for alternatives very close to $P_{11}$, $e_{TV}^P(\{P_{\gamma_1(s)}\})$ has a good empirical interpretation. But in this case both efficiencies are close each other and still empirical efficiencies are closer to $\mathcal{E}_{TV}^P$ than to $e_{TV}^P$.

Elementary calculations give

$$\mathcal{E}_{TV}^P(P_{p_1(s)\,q_1(s)}) = \frac{5}{12}\left(\frac{4p+5q+1+o(s)}{3(p+q)+o(s)}\right)^2.$$

Hence $\lim_{s\to 0^+} \mathcal{E}^P_{TV}(P_{p_1(s)\,q_1(s)}) = e^P_{TV}(\{P_{\gamma_1(s)}\})$ which is nicely seen in the last column of Table 2.

Since for $\{P_{\gamma_1(s)}\}$ we have had to mix distributions lying on the path with $P_{1\,1}$ we consider two other paths determined by the curves $\gamma_2(s) = (1+2s+s^2, 1+s+s^2, 1)$ and $\gamma_3(s) = (1-s/2+s^2/2, 1-2s/3, 1)$. They join $P_{1\,1}$ with $P_{4\,3}$ and $P_{1\,1}$ with $P_{1\,1/3}$, respectively. Similarly as above, using (4) and (5) after simple calculations we get for $\gamma_2(s)$: $\mu'_V(0) = 1/4$, $\mu'_T(0) = 1/6$ and $c^P_V = 3/4$, $c^P_T = 5/16$, $e^P_{TV}(\{P_{\gamma_2(s)}\}) = 5/12 \approx 0.4167$ and for $\gamma_3(s)$: $\mu'_V(0) = 1/24$, $\mu'_T(0) = 2/27$ and $c^P_V = 1/48$, $c^P_T = 5/81$, $e^P_{TV}(\{P_{\gamma_3(s)}\}) = 80/27 \approx 2.963$.

On both paths choose 5 distributions taking $s = 1, 0.5, 0.2, 0.1, 0.05$. In Table 3 we present empirical powers and empirical efficiencies for selected didtributions on the path $\{P_{\gamma_2(s)}\}$. In this case mixing with $P_{1\,1}$ is unncessary to keep empirical powers close to 1/2 for moderate sample sizes. This fact is marked in the table in the column denoted by $\varepsilon$.

**Table 3.** Empirical powers and empirical efficiencies of $T$ with respect to $V$ for selected distributions on the path $\{P_{\gamma_2(s)}\}$. $\alpha = 0.05$, 100 000 MC.

| alternative | | | | | test | | empirical | |
|---|---|---|---|---|---|---|---|---|
| $s$ | $p_2(s)$ | $q_2(s)$ | $\varepsilon$ | $n$ | $V$ | $T$ | efficiency | $\mathcal{E}^P_{TV}$ |
| 1 | 4 | 3 | 1 | 50 | **57** | 05 | | |
| | | | | 497 | 100 | **57** | 0.101 | 0.104 |
| 0.5 | 2.25 | 1.75 | 1 | 70 | **59** | 18 | | |
| | | | | 290 | 100 | **59** | 0.241 | 0.250 |
| 0.2 | 1.44 | 1.24 | 1 | 200 | **58** | 27 | | |
| | | | | 582 | 95 | **58** | 0.344 | 0.354 |
| 0.1 | 1.21 | 1.11 | 1 | 600 | **58** | 29 | | |
| | | | | 1600 | 92 | **58** | 0.375 | 0.387 |
| 0.05 | 1.1025 | 1.0525 | 1 | 2000 | **56** | 30 | | |
| | | | | 5015 | 89 | **56** | 0.399 | 0.402 |

In the next table we present the results for the path $\{P_{\gamma_3(s)}\}$. For two last cases we have taken 10000 MC runs while for the rest cases 100 000 MC runs.

**Table 4.** Empirical powers and empirical efficiencies of $T$ with respect to $V$ for selected distributions on the path $\{P_{\gamma_3(s)}\}$. $\alpha = 0.05$.

| alternative | | | | | test | | empirical | |
|---|---|---|---|---|---|---|---|---|
| $s$ | $p_3(s)$ | $q_3(s)$ | $\varepsilon$ | $n$ | $V$ | $T$ | efficiency | $\mathcal{E}^P_{TV}$ |
| 1 | 1 | 0.3333 | 0.2 | 100 | **53** | 64 | | |
| | | | | 73 | 44 | **54** | 1.37 | 1.437 |
| 0.5 | 0.875 | 0.6667 | 1 | 70 | **62** | 74 | | |
| | | | | 49 | 50 | **62** | 1.43 | 1.496 |
| 0.2 | 0.92 | 0.8667 | 1 | 1000 | **50** | 72 | | |
| | | | | 535 | 33 | **50** | 1.87 | 1.936 |
| 0.1 | 0.955 | 0.9333 | 1 | 7000 | **51** | 79 | | |
| | | | | 3120 | 30 | **51** | 2.24 | 2.295 |
| 0.05 | 0.97625 | 0.9667 | 1 | 35000 | **48** | 81 | | |
| | | | | 13650 | 26 | **48** | 2.56 | 2.58 |

Results shown in Tables 3 and 4 confirm observations made for $\{P_{\gamma_1(s)}\}$. It is seen that Pitman efficiency depends on the actual path and its values can considerable differ (in our examples: 0.9375, 0.4167, 2.963). Its empirical interpretation is correct only for alternatives lying on a given path and corresponding to very small $s$. Contrary to it, efficiency

$\mathcal{E}_{TV}^P$ has good empirical interpretation at any point of a path.

Consider a path $\{P_{\gamma_4(s)}\}$ determined by the curve $\gamma_4(s) = (1+s, 1+0.5s-1.18s^2, 1)$. Both curves $\gamma_3(s)$ and $\gamma_4(s)$ are tangent at $s = 0$ to the line $p - 2q + 1 = 0$. Similar calculations, as previously, give $e_{TV}^P(\{P_{\gamma_4(s)}\}) = 5/12$. The paths $\{P_{\gamma_1(s)}\}$ for $p = 6$, $q = 4$ and $\{P_{\gamma_4(s)}\}$ intersect at $P_{2\,0.32}$. So, one can assign to this distribution numbers 5/12 or 15/16 as efficiencies (theoretical). However, they have nothing to do with the empirical efficiency equal to 1.35 (cf. Table 2). In Table 5 we present empirical efficiencies for selected distributions on $\{P_{\gamma_4(s)}\}$. Also in this case empirical efficiencies are better approximated by $\mathcal{E}_{TV}^P$ than by $e_{TV}^P \approx 0.4167$.

The path $\{P_{\gamma_4(s)}\}$ intersects $\{P_{\gamma_5(s)}\}$ with $\gamma_5(s) = (1+s, 1, 1)$ at $P_{84/59\,1}$. We have $e_{TV}^P(\{P_{\gamma_5(s)}\}) = 20/27$ and $e_{TV}^P(\{P_{\gamma_4(s)}\}) = 5/12$ and both numbers could be assigned to $P_{84/59\,1}$ as its efficiency. But, the empirical efficiency is equal to 0.800 and is better approximated by $\mathcal{E}_{TV}^P(P_{84/59\,1}) \approx 0.835$.

**Table 5.** Empirical powers and empirical efficiencies of $T$ with respect to $V$ for selected distributions on the path $\{P_{\gamma_4(s)}\}$. $\alpha = 0.05$, 100 000 MC.

| alternative | | | | | test | | empirical | |
|---|---|---|---|---|---|---|---|---|
| $s$ | $p_4(s)$ | $q_4(s)$ | $\varepsilon$ | $n$ | $V$ | $T$ | efficiency | $\mathcal{E}_{TV}^P$ |
| 0.5 | 1.5 | 0.955 | 0.5 | 100 | **61** | 56 | | |
| | | | | 116 | 67 | **61** | 0.862 | 0.900 |
| 0.2 | 1.2 | 1.0528 | 1 | 250 | **56** | 40 | | |
| | | | | 405 | 74 | **56** | 0.617 | 0.637 |
| 0.1 | 1.1 | 1.0382 | 1 | 1300 | **56** | 36 | | |
| | | | | 2480 | 81 | **56** | 0.524 | 0.536 |

The true (unknown) distribution $P \neq P_{1\,1}$ of the samle at hand (under $H_1$) lies on many paths. So, one cannot assign it a single number interpreted as its efficiency meant as a ratio of sample sizes guaranteing the same power of both tests. Even for alternatives close to $P_{1\,1}$ (in the sense of Hellinger distance) the Pitman efficiency can significantly differ from the empirical efficiency as it depends on a shape of a considered path and a value of the parameter $s$ corresponding to a selected alternative. This is the case when a path runs close to $P_{1\,1}$ for large values of $s$. For example, for $\gamma_6(s) = (1-s+11s^2/10, 1-2s+2s^2, 1)$ we obtain $e_{TV}^P(\{P_{\gamma_6(s)}\}) = 5/3$, but for $s = 1$ we have $P_{\gamma_6(1)} = P_{1.1\,1}$ with $H(P_{1.1\,1}, P_{1\,1}) = 0.048$ and its empirical efficiency 0.735 is close to $\mathcal{E}_{TV}^P(P_{1.1\,1}) \approx 0.765$ and has nothing to do with 5/3.

Finally, consider a path determined by $\gamma_7(s) = (2+s, 1+s, s)$. For any $s_n \to 0$ we have the sequence of alternatives of the form $(1-s_n)P_{1\,1} + s_n P_{2+s_n\,1+s_n}$. So, we have a contamination sequence determined by a sequence of different alternatives. The sequence of the densities of $P_{2+s_n\,1+s_n}$ is uniformly bounded and uniformly bounded away from 0 and converges to the density of $P_{2\,1}$. The Pitman efficiency for this path equals to $e_{TV}^P(\{P_{\gamma_7(s)}\}) = 15/16$ and coincides with $\mathcal{E}_{TV}^P(P_{2\,1})$ for the limiting distribution $P_{2\,1}$. However, again empirical efficiencies are for each $n$ close to $\mathcal{E}_{TV}^P(P_{2+s_n\,1+s_n})$ but not to $e_{TV}^P(\{P_{\gamma_7(s)}\})$ which can be seen in Table 6. It is easy to see that $\mathcal{E}_{TV}^P(P_{2+s_n\,1+s_n}) \to 15/16 = e_{TV}^P(\{P_{\gamma_7(s)}\})$ as $n \to \infty$ and from Table 6 we see that empirical efficiencies also approach to 15/16 when $s \to 0$.

**Conclussion.** The notion of Pitman efficiency for arbitrary paths meets some difficulties with an empirical interpretation. For linear paths it approximates empirical relative efficiency very well.

**Table 6.** Empirical powers and empirical efficiencies of $T$ with respect to $V$ for selected distributions on the path $\{P_{\gamma_7(s)}\}$. $\alpha = 0.05$, 100 000 MC.

| alternative | | | | | test | | empirical | |
|---|---|---|---|---|---|---|---|---|
| $s$ | $p_7(s)$ | $q_7(s)$ | $\varepsilon$ | $n$ | $V$ | $T$ | efficiency | $\mathcal{E}^P_{TV}$ |
| 1 | 3 | 2 | 1 | 30 | 64 | 29 | | |
| | | | | 76 | 98 | 64 | 0.39 | 0.417 |
| 0.5 | 2.5 | 1.5 | 0.5 | 80 | 63 | 45 | | |
| | | | | 127 | 81 | 63 | 0.63 | 0.651 |
| 0.2 | 2.2 | 1.2 | 0.2 | 300 | 55 | 47 | | |
| | | | | 380 | 63 | 55 | 0.79 | 0.817 |
| 0.1 | 2.1 | 1.1 | 0.1 | 1000 | 54 | 48 | | |
| | | | | 1205 | 60 | 54 | 0.83 | 0.876 |
| 0.05 | 2.05 | 1.05 | 0.05 | 4000 | 55 | 50 | | |
| | | | | 4440 | 59 | 55 | 0.90 | 0.907 |

## References

Lehmann, E. L. and Romano, J. P. (2008), *Testing Statistical Hypotheses*, Springer Texts in Statistics, Springer, New York, 3rd ed.

Nikitin, Y. (1995), *Asymptotic Efficiency of Nonparametric Tests*, Cambridge University Press, Cambridge.

Noether, G. E. (1955), On a theorem of Pitman, *Ann. Math. Statist.*, **26** 64-68.

Rothe, G. (1981), Some properties of the asymptotic relative Pitman efficiency, *Ann. Statist.*, **9** 663-669.

Serfling, R., J. (1980), *Approximation Theorems of Mathematical Statistics*, Wiley, New York.