Shapley Value in cooperative differential games with random duration

Ekaterina Shevkoplyas
 St.Petersburg State University, Faculty of Applied Mathematics,
 University pr., 35,
 Petrodvorets, 198 904 St.Petersburg
 Russia
 e-mail: shevka@mail.ru

Abstract

We consider n -person differential game with standard dynamic equation:

$$\dot{x} = g(x, u_1, \dots, u_n), \quad x \in R^m, u_i \in U \subseteq \text{comp } R^l, \quad x(t_0) = x_0. \quad (1)$$

But we suppose that the game has a random duration because it seems to be more realistic to expect the end of the game at the random time instant T [3]. Let T be a random variable with distribution probability function $F(t)$. So, if we consider cooperative form of the game, the players would seek to maximize the total expected payoff. We get non- standard problem of dynamic programming:

$$\max_{\{u_i\}} \sum_{i=1}^n \int_{t_0}^{\infty} \int_{t_0}^t h_i(x(\tau)) d\tau dF(t), \quad (2)$$

where $h_i(x(\tau))$ is an instantaneous payoff of the player i .

The Hamilton-Jacobi-Bellman equation, appropriate for the problem with random duration had been derived by Shevkoplyas E. in 2005 [4] under condition of the density probability function $f(t) = F'(t)$ existency:

$$\frac{f(\vartheta)}{1 - F(\vartheta)} W(x, \vartheta) = \frac{\partial W(x, \vartheta)}{\partial \vartheta} + \max_u \left[H(x(\vartheta), u(\vartheta)) + \frac{\partial W(x, \vartheta)}{\partial x} g(x, u) \right], \quad (3)$$

where $\frac{f(\vartheta)}{1 - F(\vartheta)}$ is Hazard rate, $W(x, \vartheta)$ - Bellman function, $H(x(\vartheta), u(\vartheta)) = \sum_{i=1}^n h_i(x(\vartheta), u(\vartheta))$.

The Hazard rate for exponential distribution is constant ρ . Then the HJB equation (3) looks like HJB equation for the problem with prescribed duration under payoffs discounting with discount rate ρ [1].

The classical way of characteristic function values calculating is to use maxmin solution proposed by Neumann and Morgenshtern. A new concept of the characteristic function construction was introduced by L. Petrosjan and G. Zaccour in [2] for environmental and other problems where players out of coalition wouldn't form an anticoalition to harm their efforts. Following Petrosjan and Zaccour we suppose that the left- over players stick to their feedback Nash strategies.

So, we have the following definition of the characteristic function:

$$V(S, x^*(\vartheta)) = \begin{cases} 0, & S = \emptyset; \\ W_i(x^*(\vartheta), \vartheta), & i = 1, \dots, n; \quad \{i\} \in I; \\ W_K(x^*(\vartheta), \vartheta), & K \subseteq I, \end{cases} \quad (4)$$

where $W_i(x^*(\vartheta), \vartheta), W_K(x^*(\vartheta), \vartheta)$ are the results of the corresponding Hamilton-Jacobi-Bellman equations (3).

The characteristic function (4) is not superadditive in general. So, we need to prove its superadditivity before calculating Shapley Value.

We have 2 illustrations of our theory. We calculate Shapley Value for the model of pollution cost reduction [2] and for the model of nonrenewable resource extraction [1].

We research time-consistency [3] and agreeability [5] of the Shapley Value in this games.

References

- [1] *E.J. Dockner, S. Jorgensen, N. van Long, G. Sorger. Differential Games in Economics and Management Science.* Cambridge University Press, 2000.
- [2] *Petrosjan L. A., Zaccour G.* Time-consistent Shapley Value Allocation of Pollution Cost Reduction. // *Journal of Economic Dynamics and Control*, Vol. 27, 2003, pp. 381-398.
- [3] *Petrosjan L. A., Shevkoplyas E. V.* Cooperative Solutions for Games with Random Duration. *Game Theory and Applications*, Volume IX. Nova Science Publishers, 2003, pp.125-139.
- [4] *Shevkoplyas E. V.* On the Construction of the Characteristic Function in Cooperative Differential Games with Random Duration. *International Seminar "Control Theory and Theory of Generalized Solutions of Hamilton-Jacobi Equations" (CGS'2005)*, Ekaterinburg, Russia. Ext.abstracts. Vol.1, pp. 262-270. (in Russian)
- [5] *S. Jorgensen, G. Martin-Herran, G. Zaccour.* Agreeability and Time-consistency in Linear-State Differential Games. // *Journal of optimization theory and applications*, Vol.119, no. 1, 2003, pp.49-63.