

**Levy-driven Ornstein-Uhlenbeck processes:
survey of results on first passage times**

Alexander Novikov

(University of Technology, Sydney and
Steklov Mathematical Institute, Moscow)

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Outline of topics

1. Introduction. Continuous and discrete-time O-U processes.
2. Some martingales associated with O-U.
3. Exponential boundedness of first passage times
4. Explicit cases for representations of Laplace transforms.
5. Asymptotic approximations.
6. Ruin probabilities.
7. Maximal inequalities.

1. Introduction. Continuous and discrete time versions.

Levy-driven **O-U** process X_t :

$$dX_t = -\beta X_t dt + dL_t, \quad t \in R^+ = [0, \infty) \quad (1)$$

where $\{L_t, t \geq 0\}$ is a Levy process,

$\beta > 0$ or $\beta < 0$.

Equation (1) is understood as the integral equation and its unique solution is

$$X_t = e^{-\beta t} \left(\int_0^t e^{\beta s} dL_s + X_0 \right).$$

Gaussian O-U processes (driven by Brownian Motion $L_t = W_t$) considered in many papers starting with Uhlenbeck and Ornstein (1930) (published in J.Phys.Rev.). Jacobsen (1996) found that, actually, Laplace discussed this type of processes in a context of limit theorems.

Generalized Levy driven O-U processes are defined as follows :

$$X_t = e^{-\xi t} \left(\int_0^t e^{\xi s} dL_s + X_0 \right)$$

where (ξ_t, L_t) is a bivariate Levy processes,

see Erickson & Maller (2004), Lindner & Maller (2005) and Carmona, Petit & Yor (2001).

Applications: econometrics (Barndorff-Nielsen & Shephard (2001)),
finance (interest rate and credit risk modelling)
(e.g. Leblanc & Scaillet (1998)),
insurance (ruin probabilities, $\beta < 0$) (Albrecher et al (2001)),
mechanical engineering (Grigoriu (1995)),
physics (Larralde (2004)),
neuron activity models (Girardo & Sacerdote (1997)),
dam theory (Kella & Stadge, W. (2001)),
computer science modelling (Louchard et al (1997)),
etc

Arato, Kolmogorov & Sinai (1959) considered two-dimensional Gaussian O-U process with a matrix

$$\beta = \begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix}, \quad \lambda > 0, \omega \geq 0$$

as a model for the displacement of earth magnetic pole. The Least-Square Estimators for parameters in this model are

$$\hat{\lambda}_t = \frac{\int_0^t (X_s^{(1)} dX_s^{(1)} + X_s^{(2)} dX_s^{(2)})}{\int_0^t ((X_s^{(1)})^2 + (X_s^{(2)})^2) ds},$$

$$\hat{\omega}_t = \frac{\int_0^t (X_s^{(1)} dX_s^{(2)} - X_s^{(2)} dX_s^{(1)})}{\int_0^t ((X_s^{(1)})^2 + (X_s^{(2)})^2) ds}.$$

The joint distribution of $(\hat{\lambda}_T, \hat{\omega}_T)$ for the Gaussian O-U and first-passage times of the form

$$\tau = \inf\{t \geq 0 : \int_0^t ((X_s^{(1)})^2 + (X_s^{(2)})^2) ds = C > 0\}$$

were studied in N.(1970), N.(1972), N.(1973), see also Arato (1982), Liptser & Shiryaev (2001).

About multidimensional Levy-driven processes see e.g. Masuda (2004), Maejima & Sato (2003), Sato et al (1996).

Any Gaussian stationary process with a rational spectral density (ARMA processes) can be considered as a component of a multidimensional O-U process (Arato (1982)).

Levy-driven ARMA processes have been studied by Brockwell (2004), Brockwell & Marquardt (2005).

The main problem to be discussed here is how to find a distribution of first passage times like

$$\tau_b = \inf\{t > 0 : X_t > b\}, \quad b > X_0 = x$$

or, equivalently, a distribution of $\max_{s \leq t} X_s$ having in mind the relation

$$P_x\{\tau_b < t\} = P_x\{\max_{s \leq t} X_s \geq b\}.$$

It is of interest also to find *approximations* for $P_x\{\tau_b < t\}$ and $E_x(\tau_b)$ when $b \rightarrow \infty$ or $\beta \rightarrow 0$.

The two-sided case:

$$\tau_H = \inf\{t : |X_t| > H\}$$

is also of interest for some applications (see EWMA procedure below).

First results about distributions of first passage time like τ_b were obtained by Darling & Siegert (1953).

Methods for study:

- 1) find a proper martingale e.g. in the form $e^{-\lambda t} f_\lambda(X_t)$ and use optional stopping theorem;
- 2) solve an integral (Dynkin) equation for Laplace transform $q_\lambda(x) = E_x e^{-\lambda \tau_b}$ (or solve directly an integral equation for a distribution function of τ_b , see e.g. Borovkov K. & N. (2001));
- 3) use extreme value theory ;
- 4) to develop technique of factorization identities (available so far only for Levy processes and random walks, see Bertoin (1996), Borovkov A. (1998));
- 5) develop numerical methods (see e.g. Cardoso & Waters (2003) for ruin problems).

Further we consider only one-dimensional O-U.

If $\beta > 0$ then under the assumption

$$E \log(|L_1| + 1) < \infty \quad (2)$$

then as $t \rightarrow \infty$

$$X_t \xrightarrow{d} X_\infty \stackrel{d}{=} \int_0^\infty e^{-\beta s} dL_s.$$

Condition (2) is necessary and sufficient for $P\{|\int_0^\infty e^{-\beta s} dL_s| < \infty\} = 1$, see Wolfe (1982). Vervaat (1979) got similar and more general results for the discrete-time O-U:

$$X_t = \rho X_{t-1} + \xi_t, \quad t = 1, 2, \dots, \quad X_0 = x \quad (3)$$

where ξ_t are iid r.v., $0 < \rho < 1$. In particular, his results imply that under condition

$$E \log(|\xi_1| + 1) < \infty \quad (4)$$

the following relation holds

$$X_t \xrightarrow{d} X_\infty \stackrel{d}{=} \sum_{t=0}^{\infty} \rho^t \xi_{t+1}$$

Let $\kappa_X(z) = \log(Ee^{izX})$ be a cumulant function of a r.v. X .

It can be checked that for any continuous nonrandom function $h(x)$ the cumulant function of the process

$$Y_t = \int_0^t h(s)dL_s$$

is

$$\kappa_{Y_t}(z) = \int_0^t \kappa_{L_1}(zh(s))ds.$$

(Wolfe (1982) referes these formulas to Lukacs (1969,1970)). For the case of O-U process defined in (1) we have $h(s) = e^{-\beta(t-s)}$ and it implies

$$\begin{aligned} \kappa_{X_t}(z) &= izxe^{-\beta t} + \int_0^t \kappa_{L_1}(ze^{-\beta(t-s)})ds = \\ &izxe^{-\beta t} + \frac{1}{\beta} \int_{ze^{-\beta t}}^z \frac{\kappa_{L_1}(u)}{u} du. \end{aligned}$$

If $\beta > 0$ then condition (2) guarantees a convergence of the last integral as $t \rightarrow \infty$ and so

$$\kappa_{X_\infty}(z) = \frac{1}{\beta} \int_0^z \frac{\kappa_{L_1}(u)}{u} du$$

(Wolfe (1982)). This result can be obtained by other ways, e.g. by use of stochastic calculus (see e.g. N.(2003)) or as a limit from a discrete-time model.

Similar formulas holds for the case when there exists a moment generating function (**mgf**) of L_1 . Denote

$$\Lambda_{L_1}(u) = \log(Ee^{uL_1}) < \infty, 0 \leq u < K, (K > 0).$$

Then under the condition

$$E \log(L_1^- + 1) < \infty$$

we have

$$\Lambda_{X_\infty}(u) = \frac{1}{\beta} \int_0^u \frac{\Lambda_{L_1}(z)}{z} dz, u < K.$$

For discrete-time O-U processes similar formulas can be found by direct calculations. Let e.g. for the model 3 there exist a cumulant function $\Lambda_{\xi_1}(u)$ for $0 \leq u < K$, ($K > 0$). Then under condition

$$E \log(\xi_1^- + 1) < \infty$$

we have

$$\Lambda_{X_\infty}(z) = \sum_{t=0}^{\infty} \Lambda_{\xi_1}(\rho^t z).$$

This also implies

$$\Lambda_{X_\infty}(z) = \Lambda_{X_\infty}(\rho z) + \Lambda_{\xi_1}(z). \quad (5)$$

Based on this relation we construct below a family of martingales as a function of X_t . It is worth to note that 5 is a consequence of the following relation:

$$X_\infty \stackrel{d}{=} \sum_{t=0}^{\infty} \rho^t \xi_{t+1} = \rho \sum_{t=1}^{\infty} \rho^{t-1} \xi_{t+1} + \xi_1 \stackrel{d}{=} \rho \bar{X}_\infty + \xi_1$$

where \bar{X}_∞ and ξ_1 are independent and $\bar{X}_\infty \stackrel{d}{=} X_\infty$.

O-U processes with special distributions.

Stable O-U processes. Let O-U processes be driven by a Levy process having one-sided (spectral negative) stable distribution with the mgf

$$Ee^{uL_1} = e^{um + CSgn(\alpha-1)u^\alpha}, 0 < \alpha \leq 2, u \geq 0, \alpha \neq 1$$

where m and $C > 0$ are some constants. Then

$$\Lambda_{X_\infty}(z) = \frac{1}{\beta} \left(mz + \frac{C Sgn(\alpha - 1)}{\alpha} z^\alpha \right), z \geq 0. \quad (6)$$

and so for the case $m = 0$ the r.v. X_∞ is also a stable (one-sided) r.v. A similar result holds for other stable distribution (see Wolfe (1982)).

Heyde & Leonenko (2005) have shown (besides other results) that if L_1 has a Student distribution then X_∞ has also a Student distribution and so there exists a stationary O-U process with a Student distribution. There are many other examples with explicit forms for distributions or cumulant functions like above. E.g. if L_t is a compound Poisson process with a double-exponential distribution for jumps then X_∞ has a Variance-Gamma distribution. See other examples e.g. in Eliazar & Klafter (2005).

For the discrete-time case one can obtain many similar results. E.g. if the r.v. ξ_t has a stable or Student distribution then X_∞ has the same type of distribution.

Note that if $\Lambda_{\xi_1}(u)$ has a Taylor expansion

$$\Lambda_{\xi_1}(u) = \sum_{k=1}^{\infty} \frac{s_k}{k!} u^k, \quad |u| < R$$

(here s_k are cummulants of ξ_1), then

$$\Lambda_{X_\infty}(u) = \sum_{k=1}^{\infty} \frac{s_k}{k!} \frac{u^k}{1 - \rho^k}, \quad |u| < R.$$

It is worth to mention that that for the **continuous-time case** the distribution of X_∞ always belongs to the class L (see Wolfe (1982), also Sato (1999)) and so it is *infinitely divisible, absolutely continuous and unimodal* (the last result proved by Yamazato (1978)).

2. Some martingales associated with O-U.

Discrete-time case.

Let $e^{-\lambda t} f_\lambda(X_t)$ be a martingale (here and further we consider only martingales with respect to a natural filtration $\mathcal{F}_t = \sigma\{X_s, s \leq t\}$).

Then

$$E(e^{-\lambda t} f_\lambda(X_t) | \mathcal{F}_{t-1}) = e^{-\lambda(t-1)} f_\lambda(X_{t-1}), t \geq 1$$

or, due to (3)

$$E(f_\lambda(\rho X_{t-1} + \xi_t) | \mathcal{F}_{t-1}) = e^\lambda f_\lambda(X_{t-1}).$$

As ξ_t does not depend on \mathcal{F}_{t-1} one could easily see that if we find $f(x)$ as a solution of the following integral equation

$$E f_\lambda(\rho x + \xi_t) = e^\lambda f_\lambda(x) \quad (7)$$

then we get a martingale property for the process $e^{-\lambda t} f_\lambda(X_t)$ and so for any bounded stopping time we have

$$E e^{-\lambda \tau} f_\lambda(X_\tau) = E f_\lambda(X_0).$$

Having this martingale property one could try to get some properties of τ .

For the case of random walks ($\rho = 1$) under the assumption

$$\Lambda_{\xi_1}(u) < \infty, u < K$$

there is a solution of (7) of the form

$$f_\lambda(x) = e^{ux - \lambda t}, \lambda = \Lambda_{\xi_1}(u),$$

but it seems the case $0 < \rho < 1$ was firstly studied in N.(1990)

To have a martingale property for the process $g(X_t) - t$ the function $g(x)$ should satisfy another integral equation

$$Eg(\rho x + \xi_t) = g(x) + 1$$

and if $g(X_t) - t$ is a martingale then for any bounded stopping time

$$Eg(X_\tau) = Eg(X_0) + E(\tau).$$

Continuous-time case.

Integral equation for martingale functions $f_\lambda(x)$ and $g(x)$.

Consider the Levy process L_t in the following form

$$L_t = mt + \sigma W_t + Z_t$$

where m and σ are constants,

W_t is a standard Brownian Motion,

Z_t is a pure jump process

$$Z_t = \int_0^t \int x I\{|x| < 1\} [p(dx, ds) - \Pi(dx) ds] + \int_0^t \int x I\{|x| \geq 1\} p(dx, ds),$$

$\Pi(dx)$ is the Levy-Khinchin measure of L_t which is compensator for a Poisson random measure $p(dx, ds)$ (generated by L_t) and such that

$$\int \min(x^2, 1) \Pi(dx) < \infty.$$

Then under some assumptions

$$\frac{\sigma^2}{2} f''_{\lambda}(x) - \beta(x - m) f'_{\lambda}(x) - \lambda f_{\lambda}(x) + \int [f_{\lambda}(x + y) - f_{\lambda}(y)] \Pi(dy) = 0,$$

$$\frac{\sigma^2}{2} g''(x) - \beta(x - m) g'(x) + \int [g(x + y) - g(y)] \Pi(dy) = 1.$$

Martingales for discrete-time case, $0 < \rho < 1$.

We consider the equation

$$Ef_\lambda(\rho x + \xi_t) = e^\lambda f(x).$$

Assume that the mgf of ξ_t exists for all u :

$$Ee^{uL_1} = e^{\Lambda_{L_1}(u)} < \infty, 0 \leq u < \infty, (K = \infty)$$

and so the cumulant function $\Lambda_{X_\infty}(u) = \sum_{t=0}^{\infty} \Lambda_{\xi_1}(\rho^t u)$ is defined for all $u \geq 0$.

Set

$$f_\lambda(x) := \int_0^\infty e^{ux - \Lambda_{X_\infty}(u)} u^{\nu-1} du, \quad \nu > 0, \lambda = \nu \log\left(\frac{1}{\rho}\right).$$

This is a finite function for all x when the r.v. ξ_t has the one-sided stable distribution with the parameter $\alpha \in (1, 2]$ because for this case

$$\Lambda_{X_\infty}(u) = \frac{m}{1-\rho} u + \frac{C}{1-\rho^\alpha} u^\alpha, \quad u \geq 0, C > 0.$$

grows faster than a linear function.

For the case of uniformly distributed $\xi_t \sim Uniform(0,1)$ it can be shown that

$$\Lambda_{X_\infty}(u) \sim \frac{u}{1-\rho}, u \rightarrow \infty$$

and so $f_\lambda(x)$ is infinite for $x > \frac{u}{1-\rho}$.

Proposition. *Let $f_\lambda(x)$ be finite for all x . Then the process*

$$M(t) := e^{-\lambda t} f_\lambda(X_t), e^\lambda = \rho^{-\nu}, \nu > 0$$

is a martingale.

Proof. For $t \geq 1$ $E(M(t)|\mathcal{F}_{t-1}) =$

$$e^{-\lambda t} \int_0^\infty e^{u\rho X_{t-1} + \Lambda_{\xi_1}(u) - \Lambda_{X_\infty}(u)} u^{\nu-1} du =$$

(due to relation $\Lambda_{X_\infty}(u) = \Lambda_{X_\infty}(\rho u) + \Lambda_{\xi_1}(u)$)

$$= e^{-\lambda t} \int_0^\infty e^{u\rho X_{t-1} - \Lambda_{X_\infty}(\rho u)} (\rho u)^{\nu-1} d(\rho u) \rho^{-\nu} = M(t-1).$$

Set $G(x) := \frac{1}{\log(1/\rho)} \int_0^\infty \frac{e^{ux} - 1}{u} e^{-\Lambda_{X_\infty}(u)} du.$

Proposition. *Let $G(x)$ be finite for all x . Then the process*

$$m(t) := G(X_t) - t$$

is a martingale.

Proof. For $t \geq 1$ $\log(1/\rho) E(\Delta m(t) | \mathcal{F}_{t-1}) =$

$$= \int_0^\infty \frac{e^{u\rho X_{t-1} + \Lambda_{\xi_1}(u)} - 1}{u} e^{-\Lambda_{X_\infty}(u)} du - \int_0^\infty \frac{e^{uX_{t-1}} - 1}{u} e^{-\Lambda_{X_\infty}(u)} du - \log(1/\rho)$$

(setting $f(u, x) = e^{ux - \Lambda_{X_\infty}(u)}$)

$$= \int_0^\infty \frac{f(\rho u, X_{t-1}) - f(u, X_{t-1})}{u} du - \log(1/\rho) = 0.$$

The last relation holds due to the well-known formula (it is a so-called Frullani integral).

Martingales for continuous-time case, $\beta > 0$.

Assume that the mgf of L_1 exists for all u :

$$Ee^{uL_1} = e^{\Lambda_{L_1}(u)} < \infty, 0 \leq u < \infty, (K = \infty)$$

and so the cumulant function $\Lambda_{X_\infty}(u) = \int_0^u \frac{\Lambda_{L_1}(z)}{\beta z}$ is defined for all $u \geq 0$.

Set

$$f_\lambda(x) := \int_0^\infty e^{ux - \Lambda_{X_\infty}(u)} u^{\nu-1} du, \lambda = \nu\beta > 0.$$

This is a finite function for all x when e.g. the r.v. L_1 has the one-sided stable distribution with the parameter $1 < \alpha \leq 2$ but for the case $0 < \alpha < 1$ it is infinite for all $x > m$ (see (6)).

Proposition. Let $f_\lambda(x)$ be finite for all x . Then the process

$$M(t) := e^{-\lambda t} f_\lambda(X_t), \nu > 0, \lambda = \beta\nu$$

is a martingale.

Proof. Heuristic arguments: put in (3)

$$\rho = 1 - \beta\delta, \delta \rightarrow 0, \xi_t^{(\delta)} = L_{t+\delta} - L_t, t = 0, \delta, 2\delta, \dots$$

and note $\Lambda_{\xi_t^{(\delta)}}(u) = \delta\Lambda_{L_1}(u)$. Then (5) has the form

$$\Lambda_{X_\infty^{(\delta)}}(u) = \Lambda_{X_\infty^{(\delta)}}((1 - \beta\delta)u) + \delta\Lambda_{L_1}(u)$$

which implies as the limit of $\Lambda_{X_\infty^{(\delta)}}(u)$ as $\delta \rightarrow 0$ is a solution of

$$\frac{d\Lambda_{X_\infty}(u)}{du} = \frac{\Lambda_{L_1}(u)}{\beta u}$$

and so $\Lambda_{X_\infty}(u) = \frac{1}{\beta} \int_0^u \frac{\Lambda_{L_1}(z)}{z} dz$, (as expected) etc.

A formal proof in N.(2003) is based on some calculations involving a stochastic exponent.

Set

$$G(x) := \frac{1}{\beta} \int_0^\infty \frac{e^{ux} - 1}{u} e^{-\Lambda_{X_\infty}(u)} du$$

Proposition. *Let $G(x)$ be finite for all x . Then the process*

$$m(t) = G(X_t) - t$$

is a martingale.

Proof.

**Martingales for discrete-time case, $0 < \rho < 1$,
the case of exponential distribution $\xi_t \sim \text{Exp}(1)$.**

Set

$$f_\lambda(x) := \sum_{k=0}^{\infty} \frac{C_k}{k!} x^k, \quad C_0 = \rho^\nu, \quad C_k = \rho(1 - \rho^{k-1+\nu})C_{k-1}, \quad \nu > 0;$$

$$g(x) := \sum_{k=0}^{\infty} \frac{C_k}{k!} x^k, \quad C_0 = 1, \quad C_k = \rho(1 - \rho^k)C_{k-1}.$$

Proposition. *The processes $e^{-\lambda t} f_\lambda(X_t)$ with $e^\lambda = \rho^{-\nu}$ and $g(X_t) - t$ are martingales.*

Proof.

3. Exponential boundedness of first passage times.

Continuous-time case, $\beta > 0$.

Theorem (N. (2003)) *Let $\sigma > 0$ or $\Pi((0, \infty)) > 0$. Besides,*

$$E(L_1^-)^\delta < \infty \text{ for some } \delta > 0.$$

Then there exists $\alpha > 0$ such that

$$Ee^{\alpha\tau_b} < \infty.$$

Proof. Set

$$H(\lambda, x) = \int_0^\infty (e^{ux - \Lambda_{X_\infty}(u)} - 1) u^{\nu-1} du, \nu \in (-\min(\delta, 1), 0), \lambda = \nu\beta$$

First step. Check a martingale property of the process

$$M_\lambda(t) = e^{-\lambda t} \int_0^\infty (e^{uX_t} - e^{ux}) e^{-\Lambda_{X_\infty}(u)} u^{\nu-1} du + H(\lambda, x)(e^{-\lambda t} - 1)$$

(which is, in fact, a kind of an analytical continuation of the martingale family $e^{-\lambda t} f_\lambda(X_t)$ from positive $\lambda > 0$ to negative ones).

Second step. Truncate jumps of a Levy process from above by some constant, say, A and note that for the O-U process $X_t^{(A)}$ driven by Levy-truncated process the stopping time $\tau_b^{(A)} \geq \tau_b$.

Third step. Use the optional stopping theorem and the fact that for the O-U process $X_t^{(A)}$

$$X_{\min(\tau_b^{(A)}, t)}^{(A)} \leq b + A$$

Discrete-time case, $0 < \rho < 1, x < b$.

Theorem. Let $P\{\xi_t > b(1 - \rho)\} > 0$ and

$$E(\xi_1^-)^\delta < \infty \text{ for some } \delta > 0.$$

Then there exists $\alpha > 0$ such that

$$Ee^{\alpha\tau_b} < \infty.$$

Wald's moment identity.

Reminder: the classic **Wald's moment identity for Levy processes:** if τ is a stopping time, $E|L_1| < \infty$, $E(\tau) < \infty$ then

$$E(L_\tau) = E(L_1)E(\tau).$$

Now we have another identity for the case of O-U processes? Set

$$K = \sup\{u > 0 : Ee^{uL_1} < \infty\}.$$

Theorem. Let $K = \infty$ and $\tau = \inf\{t \geq 0 : X_t \geq f(t)\}$ where $f(t)$ is a bounded from above continuous deterministic function.

Assume $\sigma > 0$ or $\Pi((0, \infty)) > 0$. Besides,

$$E(L_1^-)^\delta < \infty \text{ for some } \delta > 0.$$

Then

$$E(\tau) = \frac{1}{\beta} E \int_0^\infty (e^{uX_\tau} - e^{uX_0}) u^{-1} e^{-\Lambda_{X_\infty}(u)} du < \infty$$

4. Explicit representations for Laplace transform.

For continuous-time case with $\beta > 0$ explicit formulas are available for the following cases:

a. **Spectral-negative case:** $\Pi(dy) = 0, y > 0$

b. **Exponential jumps:** $\Pi(dy) = \mu\nu e^{-\nu y} dy, y > 0$

(e.g. $L_t = \sum_{k=1}^{N_t(\mu)} \xi_k, \xi_k \sim Exp(\nu)$).

c. **Uniformly distributed jumps**

$\Pi(dy) = \mu/C I\{0 < y < C\} dy, y > 0$

(e.g. $L_t = \sum_{k=1}^{N_t(\mu)} \xi_k, \xi_k \sim Uniform(0, C)$).

For discrete-time case with $0 < \rho < 1$ explicit formulas are available only for the case of double-exponential jumps.

a. Spectral-negative case. It was completely solved in N. (2003). Here is one particular case: if $E \log(1 + L_1^-) < \infty$ and $P\{\tau_b < \infty\} = 1$ then

$$E_x e^{-\lambda \tau_b} = \frac{f_\lambda(x)}{f_\lambda(b)}, \quad x \leq b$$

where

$$f_\lambda(x) := \int_0^\infty e^{ux - \Lambda_{X_\infty}(u)} u^{\nu-1} du, \quad \lambda = \beta\nu > 0$$

Firstly this result was proved by Hadjiev (1983) (in a slightly different form) under some assumptions, then in N. (1990) under more general assumptions.

Sufficient conditions for $P_x\{\tau_b < \infty\} = 1$:

$$\sigma > 0 \text{ or } m + \int |x|I\{-1 < x < 0\}\Pi(dx) > \beta b.$$

Remark. For one-sided stable L_t with the parameter $\alpha \in (0, 2)$

$$\int |x|I\{-1 < x < 0\}\Pi(dx) = \infty \text{ for } \alpha \geq 1,$$

$$\int |x|I\{-1 < x < 0\}\Pi(dx) < \infty \text{ for } 0 < \alpha < 1.$$

Remark. For the spectral negative Levy process with

$$Ee^{uL_1} = e^{Cu^\alpha}, 1 < \alpha \leq 2$$

the O-U process

$$X_t = e^{-\beta t} \left(x + \int_0^t e^{\beta s} dL_s \right) \stackrel{d}{=} e^{-\beta t} (x + L_{\theta(t)}), \quad \theta(t) = \int_0^t e^{\alpha\beta s} ds$$

and the result of the last Theorem is a consequence of the result from N. (1981) where the following first passage time

$$\tau = \inf\{t : L_t \geq a + b(1+t)^{1/\alpha}\}$$

was studied (see also for the case $\alpha = 2$ the papers of Breiman (1967), Shepp (1969) , N.(1971)).

Some other results and references about first passage-time densities of τ_b for Gaussian O-U processes ($\alpha = 2$) can be found e.g. in the survey paper Finch (2004) , PhD thesis of Patie (2005), Alili et al (2005).

For the special case $x < b = 0$ one can easily get with $\theta(t) = \int_0^t e^{\alpha\beta s} ds$

$$P\{\tau_b > T\} = P\{x + L_{\theta(t)} < 0, t \leq T\} = \\ P\{L_t < (-x), t \leq \theta^{(-1)}(T)\},$$

and so the distribution of τ_b can be expressed in terms of first-passage density (through a level) of L_t which is, actually, a density of stable subordinator with index $1/\alpha$.

Unfortunately, this type of results does not hold for $b > 0$ and that is a reason why an explicit formula for density of τ_b in Leblanc & Renault & Scaillet (2000) (for the case $\alpha = 2$) is not correct, see a clarification note of Going-Jaesche & Yor (2003).

b. Exponential jumps: $L_t = \sum_{k=1}^{N_t(\mu)} \xi_k, \xi_k \sim \text{Exp}(\nu)$.

$$E_x e^{-\lambda \tau_b} = \frac{\mu \Phi\left(\frac{\lambda}{\beta}, \frac{\mu + \lambda}{\beta} + 1; \nu x\right)}{(\mu + \lambda) \Phi\left(\frac{\lambda}{\beta}, \frac{\mu + \lambda}{\beta}; \nu b\right)}, \quad 0 \leq x < b$$

where $\Phi(a, b; x)$ represents Kummer's series

$$\Phi(a, b; x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k x^k}{(b)_k k!},$$

Another representation

$$E_x e^{-\lambda \tau_b} = \frac{\int_0^1 \exp\{\nu x u\} u^{\lambda/\mu - 1} (1 - u)^{\mu/\beta} du}{\int_0^1 \exp\{\nu b u\} u^{\lambda/\mu - 1} (1 - u)^{\mu/\beta - 1} du}.$$

The paper N. et al (2005) contains two different derivations of the last formula. Some related results one can find in Tsurui & Osaki (1976), Kella & Stadje (2001).

Uniformly distributed jumps.

$$L_t = \sum_{k=1}^{N_t(\mu)} \xi_k, \xi_k \sim \text{Uniform}(0, C), C \geq b.$$

Solving integral equations for the Laplace transform we got

$$E_x e^{-\lambda \tau_b} = 1 - \frac{\lambda (b/x)^{\frac{\mu+\lambda}{2\beta}} J_{\frac{\mu+\lambda}{\beta}} \left(2\sqrt{\frac{\mu}{C\beta} x} \right)}{\sqrt{\frac{\beta\mu}{C}} b J_{\frac{\mu+\lambda}{\beta}-1} \left(2\sqrt{\frac{\mu}{C\beta} b} \right)}, \quad 0 < x < b,$$

where $J_\nu(x)$ is the Bessel function of the first kind.

See the proof in N. et al (2005).

Explicit solutions for discrete time case, $0 < \rho < 1$.

$$f_\lambda(x) := \sum_{k=0}^{\infty} \frac{C_k}{k!} x^k, \quad C_0 = \rho^\nu, C_k = \rho(1 - \rho^{k-1+\nu})C_{k-1},$$

$$g(x) := \sum_{k=0}^{\infty} \frac{C_k}{k!} x^k, \quad C_0 = 1, C_k = \rho(1 - \rho^k)C_{k-1}.$$

Theorem. Let $x \geq 0$,

$$\begin{aligned} E_x(e^{-\lambda\tau_b}) &= \frac{e^{-\lambda} f_\lambda(x)}{f_\lambda(b/\rho)}, \quad \nu > 0, e^\lambda = \rho^{-\nu}; \\ E_x(\tau_b) &= g(b/\rho) - g(x) + 1, \quad 0 \leq x < b. \end{aligned}$$

Proof. Use the memory-less property of exponential distribution.

(Some details: $E_x g(\rho x + \xi_1) = g(x) + 1$, $X_{\tau_b} = b + \chi_b$, $\chi_b \stackrel{d}{=} \xi_1$,
 $E_x(\tau_b) = E_x g(X_{\tau_b}) - g(x) = g(b/\rho) + 1$.)

Larrald (2004) considered the case of a symmetric double-exponential distribution for ξ_t and found an explicit formula for $E_x(\tau_0)$, $x < 0$ without a use of martingale considerations.

5. Asymptotic approximations.

Continuous-time case. Assume the global Cramer condition holds.

Recall under some conditions

$$E_x(\tau_b) = \frac{1}{\beta} E \int_0^\infty (e^{uX_{\tau_b}} - e^{ux}) u^{-1} e^{-\Lambda_{X_\infty}(u)} du.$$

As $X_{\tau_b} \geq b$ we get a lower bound:

$$E_x(\tau_b) \geq \frac{1}{\beta} \int_0^\infty (e^{ub} - e^{ux}) u^{-1} e^{-\Lambda_{X_\infty}(u)} du$$

Set $\Delta_b(\beta) = X_{\tau_b} - b$. If $\beta \rightarrow 0$ and $b \rightarrow \infty$ then one may expect that

$$\Delta_b(\beta) \xrightarrow{d} \Delta_\infty(0)$$

and so

$$E_x(\tau_b) = \frac{1}{\beta} \int_0^\infty (e^{ub} E e^{u\Delta_\infty(0)} - e^{ux}) u^{-1} e^{-\Lambda_{X_\infty}(u)} du (1 + o(1))$$

Special cases

1. $Y_t = \sum_{k=1}^{N_t(\lambda)} \xi_k, \quad \xi_k \sim N(0, 1).$

Then

$$E\Delta_{\infty}(0) = -\frac{\zeta(1/2)}{\sqrt{2\pi}} = .5826..$$

$$E\Delta_{\infty}^2(0) = 3.5366$$

2. $Y_t = \sum_{k=1}^{N_t(\lambda)} \xi_k, \quad \xi_k > 0.$

Then

$$E\Delta_{\infty}(0) = \frac{E\xi_k^2}{2E\xi_k}, \quad E\Delta_{\infty}^2(0) = \frac{E\xi_k^3}{3E\xi_k}, \dots$$

Numerical results (N. et al (2005))

$$dX_t = -\beta X_t dt + dL_t, \quad X_0 = 0, \quad L_t = \sum_{k=1}^{N_t(\mu)} \xi_k, \quad \lambda = 10.$$

Case 1. $\xi_k \sim N(0, 1)$, $\beta = 0.1, 0.01$

Case 2. $\xi_k \sim Uniform(0, 1)$, $\beta = 0.1, 0.01$

Normal approximation.

Assume $Var(L_1) < \infty$.

Theorem (N. & Ergashev (1993)). *Let*

$$\beta \rightarrow 0, \quad \beta b = \text{const} < E(L_1) = m.$$

Then

$$\frac{\tau_b - E_x(\tau_b)}{\sqrt{Var(\tau_b)}} \rightarrow N(0, 1).$$

Besides,

$$E_x(\tau_b) = \frac{1}{\beta} \log\left(\frac{m}{m - \beta b}\right)(1 + o(1)),$$

$$\beta Var(\tau_b) = \frac{Var(L_1)}{2\beta} \left(\frac{1}{(m - \beta b)^2} - \frac{1}{m^2} \right) (1 + o(1)).$$

Exponential approximation.

Theorem (Borovkov & N. (2003)). Let L_t be a Compound Poisson subordinator, $X_0 = x \geq 0$ and

$$E \ln(1 + L_1) < \infty.$$

Then

$$P_x(\beta b p(b) \tau_b > s) \rightarrow e^{-s} \quad \text{as } b \rightarrow \infty$$

and

$$E_x(\tau_b) \sim \frac{1}{\beta b p(b)} \quad \text{as } b \rightarrow \infty.$$

Proof: based on "renewal" type arguments.

Theorem. Let L_t be a Levy process with a bounded positive jumps, $\sigma > 0$,

$$E \ln(1 + L_1^-) < \infty.$$

Then

$$P_x\left(\frac{\tau_b}{E_x(\tau_b)} > s\right) \rightarrow e^{-s} \quad \text{as } b \rightarrow \infty$$

Proof: based on martingale "identities" and the Wald identity

$$E_x(\tau_b) = \frac{1}{\beta} E \int_0^\infty (e^{uX_{\tau_b}} - e^{ux}) u^{-1} e^{-\Lambda_{X_\infty}(u)} du < \infty$$

Numerical illustration.

Application for Exponential Moving Average Algorithm

(EWMA)

This part is based on results from Sukparungsee & Novikov (2006).

Let $e_t \sim N(\mu, 1)$ be observed independent r.v.'s.

Need to check *sequentially* hypothesis:

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu \neq \mu_0$$

Interpretation: the process is *in control* or *out of control*.

Let τ be an alarm time.

Typical constraints. The average run length (ARL) should be large:

$$\text{ARL} = E(\tau|H_0) \geq T \rightarrow \infty$$

but the the average delay (AD) should should be minimal:

$$\text{AD} = E(\tau|H_1) \rightarrow \min.$$

EWMA had **two parameters** (λ, H) :

$$X_t = \rho X_{t-1} + (1 - \rho)(e_t - \mu_0), t = 1, 2, \dots, X_0 = 0,$$

with the alarm time

$$\tau = \tau_H = \inf\{t : |X_t| > H\}.$$

Problem: find ARL and AD.

Existing software for finding optimal parameters (λ, H) are based on discretization of X_t or on Monte-Carlo simulation.

Further assume $\mu_0 = 0$.

Martingale approach.

To approximate **ARL** we write the "exact" formula

$$ARL = E(\tau_H|H_0) = \frac{1}{\log(1/\rho)} \int_0^\infty (E \cosh(uX_{\tau_H}) - 1) u^{-1} e^{-\frac{(1-\rho)}{2(1+\rho)}u^2} du,$$

We suggest the following approximation

$$E \cosh(uX_{\tau_H}) \approx \cosh(u(H + C))$$

with some constant C .

For the average delay we suggest the following approximating formula

$$AD = E(\tau_H|H_1) \approx \frac{1}{\log(1/\rho)} \int_0^\infty (E \exp(uX_{\tau_H}) - 1) u^{-1} e^{-\mu_1 u - \frac{(1-\rho)}{2(1+\rho)}u^2} du.$$

For the case $\mu_1 > 0$ We suggest the following approximation:

$$E \exp(uX_{\tau_H}) \approx \exp(uH + uC)$$

with some constant C .

7. Ruin probabilities.

Continuous-time ruin model (Segerdahl (1942), Harrison (1977)):

$$dX_t = \delta X_t dt - dS_t, \quad X_0 = x > 0$$

where S_t is a Levy process (usually, a compound Poisson subordinator).

Discrete-time ruin model (Gerber (1981)). Let

$$X_t = \rho X_{t-1} - \eta_t, \quad t = 1, 2, \dots, \quad \rho > 1, \quad X_0 = x > 0,$$

where η_t are iid for $t = 0, 1, \dots$

Set

$$\tau_L = \inf\{t \geq 0 : X_t < L\}, \quad L < x.$$

Problem: find ruin probability

$$\psi(x, T, L) = P_x\{\tau_L < T\}, \quad T \leq \infty.$$

Some exact solutions for $\psi(x, \infty, L)$ have been obtained e.g. in Segerdahl (1942), Harrison (1977); a method for finding $\psi(x, T, L)$ numerically is developed in Cardoso & Waters (2003).

To find explicit solutions (which was not available so far as for the case $T < \infty$) we use again a martingale approach which we illustrate for the case of discrete-time ruin model. Set

$$Z_t(\rho, \eta) = \sum_{k=1}^t \rho^{-k} \eta_k,$$

Then

$$X_t = x\rho^t - \rho^t Z_t(\rho, \eta).$$

Further we always assume that

$$E(\log(|\eta_k| + 1)) < \infty.$$

It is known that under this condition and the assumption $\rho > 1$ there exists a finite limit

$$Z(\rho, \eta) = \lim_{t \rightarrow \infty} \sum_{k=1}^t \rho^{-k} \eta_k \quad (a.s.).$$

In particular, if $\eta_k \sim \text{Exp}(\mu)$ then for $u > 0$

$$Ee^{-uZ(\rho,\eta)} = \frac{1}{\prod_{k=1}^{\infty} (1 + \frac{u}{\mu}\rho^{-k})} = \frac{1 + \frac{u}{\mu}}{(-\frac{u}{\mu}; 1/\rho)}.$$

where we use standard notations from theory of q-Series:

$$(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k), \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (n \geq 1)$$

for all a and $|q| < 1$, $(a; q)_0 = 1$.

By the uniqueness of Laplace transform one can check that the tail distribution of $Z(\beta, \eta)$ is

$$P(Z(\rho, \eta) > x) = \sum_{k=1}^{\infty} C_k e^{-x\mu\rho^k}, \quad C_k = (1; 1/\rho)_{\infty} (1; \rho)_{n-1}$$

The construction of martingales will be based on the following property of $Z(\rho, \eta)$:

$$Z(\rho, \eta) + \eta_0 \stackrel{d}{=} \rho Z(\rho, \eta), \quad (\rho > 1). \quad (*)$$

Let $Z(\rho, \eta)$ as above and set for $v \geq 0$

$$G(x, v) = EI\{Z(\rho, \eta) > x\}(Z(\rho, \eta) - x)^v = E(Z(\rho, \eta) - x)^{+v}.$$

In particular,

$$G(x, 0) = P\{Z(\rho, \eta) > x\}.$$

Proposition. If $E|\eta_t|^v < \infty$ then the process

$$G_v(X_t, v)\rho^{-vt}$$

is a martingale with respect to the natural filtration.

Proof. Set

$$\hat{Z} \stackrel{d}{=} Z(\rho, \eta)$$

and further assume that r.v. \hat{Z} does not depend on the sequence $\{\eta_t\}$. Note that by Fubini theorem

$$E(G_v(X_t, v)) = E((\hat{Z} - X_t)^+)^v < \infty.$$

We have

$$\begin{aligned} E(G(X_t, v)\rho^{-vt}|\mathcal{F}_{t-1}) &= E((\hat{Z} - X_t)^{+v}|\mathcal{F}_{t-1})\rho^{-vt} = \\ &E((\hat{Z} - \rho X_{t-1} + \eta_t)^{+v}|\mathcal{F}_{t-1})\rho^{-vt} = \\ E((\rho\hat{Z} - \rho X_{t-1}))^{+v}|\mathcal{F}_{t-1})\rho^{-vt} &= G(X_{t-1}, v)\rho^{-v(t-1)}. \end{aligned}$$

QED

Proposition. *Let $E|\eta_t|^v < \infty$. Then*

$$EI(\tau_L < \infty)\rho^{-v\tau_L}G(X_t, v) = G(x, v)$$

and, in particular,

$$P(\tau_L < \infty) = \frac{G(x, v)}{E[G(X_{\tau_L}, v)|\tau_L < \infty]}.$$

The case of $\eta_k \sim \text{Exp}(\mu)$. It can be shown that on the set $\{\tau_L < \infty\}$

$$X_{\tau_L} = L - \Delta, \quad \Delta \stackrel{d}{=} \eta_k$$

X_{τ_L} and τ_L are independent r.v.'s. It implies

$$\begin{aligned} E[G(X_{\tau_L}, v) | \tau_L < \infty] &= E(Z(\rho, \eta) - L + \Delta)^{+v} = \\ E(\rho Z(\rho, \eta) - L)^{+v} &= \rho^v E(Z(\rho, \eta) - L/\rho)^{+v} = \rho^v G_-(L/\rho, v) \end{aligned}$$

In particular, we have

$$\begin{aligned} P(\tau_L < \infty) &= \frac{G(x, 0)}{G(L/\rho, 0)}, \\ E[I(\tau_L < \infty) \rho^{-v\tau_L}] &= \rho^{-v} \frac{G(x, v)}{G(L/\rho, v)}, \quad \text{where} \end{aligned}$$

$$\begin{aligned} G(x, 0) &= P(Z(\rho, \eta) > x) = \sum_{k=1}^{\infty} C_k e^{-x\mu\rho^k}, \\ C_k &= (1; 1/\rho)_{\infty} (1; \rho)_{n-1}. \end{aligned}$$

Continuous-time ruin model

$$dX_t = \delta X_t dt - dS_t, \quad \delta > 0, \quad X_0 = x > 0$$

where S_t is a Levy process

Let

$$Z(\delta, L) = \int_0^\infty e^{-\delta s} dL_s.$$

Let $\hat{Z}(\delta, L) \stackrel{d}{=} Z(\delta, L)$, $\hat{Z}(\delta, L)$ and L_t are independent. Then we have the following analog of relation (*)

$$e^{-\delta t} \hat{Z}(\delta, L) + \int_s^t e^{-\delta y} dL_y \stackrel{d}{=} e^{-\delta s} Z(\delta, L).$$

Proposition. Let $E(|L_1|)^v < \infty$, $G_v(x) = E(\hat{Z}(\delta, L) - x)^{+v}$. Then

$$G_v(X_t) e^{-\delta vt}$$

is a martingale with respect to the natural filtration.

See other examples and numerical procedures in Sidorowicz (2006).

7. Maximal inequalities.

For the Gaussian O-U process X_t with $\beta > 0$ driven by a standard Wiener process $L_t = W_t$, $X_0 = 0$, Graverson & Peskir (2000) proved the following inequality: for any stopping time τ

$$C_1 E \sqrt{\log(1 + \beta\tau)} \leq \sqrt{\beta} E(\max_{t \leq \tau} |X_t|) \leq C_2 E \sqrt{\log(1 + \beta\tau)} \quad (8)$$

where

$$C_1 \geq 1/3, \quad C_2 \leq 3.3795.$$

We derive below an analog of (8) for the case of O-U driven by a stable Levy process with

$$E e^{uL_1} = e^{u^\alpha}, \quad 1 < \alpha \leq 2, \quad u \geq 0, \quad X_0 = x \geq 0.$$

Theorem (N. (2003)). *For all stopping times τ and all $p > 0$*

$$c_p E_x [(\log(1 + \beta\tau))^{p(1 - \frac{1}{\alpha})}] \leq E_x [(\max_{t \leq \tau} X_t)^p] \leq a_p + C_p E_x [(\log(1 + \beta\tau))^{p(1 - \frac{1}{\alpha})}]$$

where positive constants c_p, a_p and C_p does not depend on τ .

Remark. In the case of $\alpha = 2$ an application of this result to both $\max_{t \leq \tau} X_t$ and $\max_{t \leq \tau} (-X_t)$ leads to the inequality of type 8 but without specification of constants.

Proof. For the proof we used the method of Graverson & Peskir (2000) which is based on a Wald's type identity with $E_x(\tau_b)$ (see above). Also, we used the following simple consequence of the Lenglart's domination principle (see e.g. Revuz & Yor (1999)).

Proposition. *Let Q_t be a non-negative right-continuous process and let A_t be an increasing continuous process, $A_0 = 0$. Assume that for all bounded stopping times τ*

$$E(Q_\tau) \leq E(A_\tau) \tag{9}$$

Then for all $p > 0$ and for all bounded stopping times τ there exist constant c_p and C_p such that

$$E([\log (1 + \sup_{t \leq \tau} Q_t)]^p) \leq c_p + C_p E([\log (1 + A_\tau)]^p)$$

where positive constants c_p and C_p do not depend on τ .

Proof. (Sketchy) By Lenglart's principle inequality (9) implies that for any increasing continuous function $H(x)$ satisfying $H(0) = 0$,

$$E(\sup_{t \leq \tau} H(Q_t)) \leq E(\tilde{H}(A_\tau))$$

where

$$\tilde{H}(x) = \int_x^\infty \frac{x}{s} dH(s) + 2H(x)$$

Consider

$$H(x) = (\log(1+x))^p, x \geq 0$$

This implies that and that there exist c_p and C_p such that

$$\tilde{H}(x) \leq c_p + C_p H(x).$$

Proof of Theorem. Denote

$$X_t^* = \max_{s \leq t} X_s$$

By the Wald identity for any bounded stopping time τ

$$\beta E(\tau) \leq E \int_0^\infty (e^{uX_\tau^*} - e^{ux}) u^{-1} e^{-Cu^\alpha/(\alpha\beta)} du = E(G(X_\tau^*) - G(x))$$

where we denote

$$G(y) = \int_0^\infty (e^{uy} - 1) u^{-1} e^{-Cu^\alpha/(\alpha\beta)} du$$

So, we have got inequality (9) with $Q_t = \beta t$ and $A_t = G(X_t^*) - G(x)$ where A_t is continuous process (since X_t does not have positive jumps).

Note that using a standard technique of asymptotic analysis one can easily obtain the following asymptotic relation

$$G(y) = \exp(C_\alpha y^{\alpha/(\alpha-1)}(1 + o(1))), \quad y \rightarrow \infty$$

Applying Lemma 2 we have

$$E[(\log(1 + \beta_\tau))^{p(1-1/\alpha)}] \leq c_p + C_p E[(\log(1 + G(X_\tau^*) - G(x)))^{p(1-1/\alpha)}]$$

So, to get the lower bound in (50) we only need the following estimate

$$\log(1 + G(y)) \leq C_1 + C_2 y^{\alpha/(\alpha-1)}, \quad y > 0$$

which easily follows.

To get the upper bound we note that the Wald identity implies that for any bounded stopping time τ

$$E(G(X_\tau^+)) \leq G(x) + \beta E(\tau) + E \int_0^\infty (1 - e^{-uX_\tau^-}) u^{-1} e^{-u^\alpha/(\alpha\beta)} du$$

Since

$$E \int_0^\infty (1 - e^{-uX_\tau^-}) u^{-1} e^{-u^\alpha/(\alpha^2\beta)} du \leq E(X_\tau^-) \int_0^\infty e^{-u^\alpha/(\alpha^2\beta)} du$$

and

$$E(X_\tau^-) \leq |x|/\beta + CE(\tau)$$

we have the following estimate

$$E(G(X_\tau^+)) \leq c + CE(\tau)$$

Then one can recognize that we got inequality (9) with $Q_t = G(X_t^+)$ and $A_t = c + Ct$ where A_t is continuous process (since X_t does not have positive jumps). Applying Lemma we have

$$E[(\log(1 + G(X_\tau^*)))^{p(1-1/\alpha)}] \leq c_p + C_p E[(\log(1 + \beta\tau))^{p(1-1/\alpha)}]$$

So, to get the upper bound in Theorem we only need the following estimate

$$C + \log(1 + G(y)) \geq by^{\alpha/(\alpha-1)}, y > 0$$

which is obvious.

Open problems:
distribution of overshoot,
accurate approximations for distribution of $\max X_s$.

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