International Journal of Bifurcation and Chaos, Vol. 21, No. 2 (2011) 545–550

DOI: 10.1142/S0218127411028593

DUAL SKEW PRODUCTS, GENERICITY OF THE EXACTNESS PROPERTY AND FINANCE

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Received March 18, 2010

By introducing the concept of dual skew products and dual measures we obtain the class of skew products over Bernoulli shifts for which exactness is generic. We use the above to describe the stationary distributions of random walks determined by skew products as above. Finally, the application to binomial model for asset prices is presented.

Keywords: Homeomorphic extension of one-sided Bernoulli shift; invariant measure; exactness; random walk; binomial model.

1. Introduction

Let us consider two homeomorphims S_0, S_1 of the interval $I = [0, 1]$ onto itself. They determine the random walk on I as follows: x goes to $S_0(x)$ with probability p and x goes to $S_1(x)$ with probability q. Random walks on I may be realized as transformations of a larger space. Let Ω be the space $\{0, 1\}^N$, $N = \{0, 1, 2, \ldots\}$, with the (p, q)-measure μ_p on $(\Omega, \mathcal{B})$, where $\mathcal{B}$ is the Borel product σ-algebra and (p, q) is a prabability vector. Let σ be the one-sided shift on Ω. In the space $\Omega \times I$ we define the skew product

$$S(\omega, x) = (\sigma(\omega), S_{\omega(0)}(x)). \tag{1}$$

The distribution of the trajectory of the walk is characterized by S-invariant measures belonging to $M_p(S)$. Here, $M_p(S)$ denotes the set of S-invariant probability measures such that $m \,|\, \mathcal{B} \times \{\emptyset, I\} = \mu_p$ for $m \in M_p(S)$. Let us consider the inverse random walk i.e. x goes to $T_0(x)$ with probability p and x goes to $T_1(x)$ with probability q where $T_i = S_i^{-1}$ for $i = 0, 1$. Denote by $T(\omega, x)$ and $M_p(T)$ the skew product and the set of invariant measures respectively for this walk. The pair S, T will be called the dual pair. In this paper, we investigate the following question: What is the relationship between $M_p(S)$ and $M_p(T)$? We construct the map $* : M_p(T) \to M_p(S)$ which is one to one and onto. The construction of $*$ partially attaches to the map # for dual-fibre systems considered in Chap. 21 of [Schweiger, 1995]. Let Λ denote the Lebesgue measure on I. Now, we assume that $\mu_p \times \Lambda \in M_p(T)$. Such situation was first considered in [Kowalski, 1987]. In general $(T, \mu_p \times \Lambda)$ does not have any one-sided generator of finite entropy. We show that if the natural extension of $(T, \mu_p \times \Lambda)$ to the automorphism is K-automorphism then $(S, \mu_p \times \Lambda^*)$ has a one-sided generator of finite entropy and is exact. Moreover, $\mu_p \times \Lambda^*$ is nonatomic and nonabsolutely continuous. Next, we prove that there exist dense G_δ-sets in some compact metrizable spaces of homeomorphisms g for which

$$M_p(S_g) = \text{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}, \mu_p \times \Lambda^*\}$$

where $\mu_p \times \Lambda^*$ has the properties as previously. Here $S_{g_0} = g, S_{g_1} = \breve{g}$ (for the definition of $\breve{g}$

see (3)). The genericity of dynamical properties for deterministic homeomorphisms of Cantor set is studied in [Hochman, 2008]. The description of sets $M_p(S_g)$, $M_p(T_g)$ and ergodic properties of their elements allows us to get more exhaustive information about the distribution of the trajectory of the walk (see section below). The next section discusses the product structure of invariant sets and invariant measures of skew products with Bernoulli shift in the base. Among other things, the example of dissipative skew product with no product invariant set is given. Section 4 is assigned to the binomial model for asset prices. By using the results of Sec. 2, we show that the asset price may be changed in a chaotic way.

2. Dual Measures

Let $\overline{\sigma}$ be the two-sided (p, q)-Bernoulli shift on the space $\overline{\Omega} = \{0,1\}^Z, Z = \{0, \pm 1, \pm 2, \ldots\}$, with the (p,q)-measure $\overline{\mu}_p$ on $(\overline{\Omega}, \overline{\mathcal{B}})$, where $\overline{\mathcal{B}}$ is the Borel product σ-algebra. We define the transformations

$$\overline{S}(\overline{\omega}, x) = (\overline{\sigma}(\overline{\omega}), S_{\overline{\omega}(0)}(x)),$$

$$\overline{T}(\overline{\omega}, x) = (\overline{\sigma}(\overline{\omega}), T_{\overline{\omega}(0)}(x)),$$

which are homeomorphisms on $\overline{\Omega} \times I$.

Theorem 1. *$\overline{T}$ and $\overline{S}^{-1}$ are topologically conjugate.*

Proof. Let $\check{T}(\overline{\omega}, x) = (\overline{\sigma}^{-1}(\overline{\omega}), T_{\overline{\omega}(0)}(x))$. We designate homeomorphism $\Phi : \overline{\Omega} \times I \to \overline{\Omega} \times I$ by formula $\Phi(\overline{\omega}, x) = (\phi(\overline{\omega}), x)$ where $\phi(\overline{\omega})(i) = \overline{\omega}(-i)$. By $\overline{\sigma}(\phi(\overline{\omega})) = \phi(\overline{\sigma}^{-1}\overline{\omega})$ we get $\Phi\check{T} = \overline{T}\Phi$. Next, let $\Psi(\overline{\omega}, x) = (\overline{\sigma}^{-1}(\overline{\omega}), x)$. Then $\Psi\overline{S}^{-1} = \check{T}\Psi$. Therefore, $\Phi\Psi$ makes the topological conjugation between $\overline{T}$ and $\overline{S}^{-1}$. ■

Let $\mu \in M_p(T)$ and $\overline{\mu}$ be the $\overline{T}$-invariant measure such that $(\overline{T}, \overline{\mu})$ is the natural extension to automorphism of (T, μ). The measure $\overline{\mu}$ satisfies the equalities

$$\begin{aligned}&\overline{\mu}({}_{-n}[i_1, \ldots, i_n] \times B)\\ &\qquad = \mu({}_0[i_1, \ldots, i_n] \times S_{i_1} \circ \cdots \circ S_{i_n} B),\end{aligned}$$

where ${}_{-n}[i_1, \ldots, i_n] = \{\overline{\omega} : \overline{\omega}(-n) = i_1, \ldots, \overline{\omega}(i_{-1}) = i_n\}$ denotes the cylinder set in $\overline{\Omega}$, ${}_0[i_1, \ldots, i_n]$ — the cylinder set in Ω respectively and B is a Borel set.

Definition 1. We determine the measure μ^* by the formula

$$\mu^*(A) = \overline{\mu}(\Phi\Psi(\pi^{-1}(A)) \quad \text{for } A \in \mathcal{B}.$$

Here $\pi(\overline{\omega}, x) = (\omega, x)$ for $(\overline{\omega}, x) \in \overline{\Omega} \times I$.

By the definition $\mu^* \in M_p(S)$ and

$$\begin{aligned}&\mu^*({}_0[i_0, \ldots, i_{n-1}] \times B)\\ &\qquad = \mu({}_0[i_{n-1}, \ldots, i_0] \times S_{i_{n-1}} \circ \cdots \circ S_{i_0} B). \quad (2)\end{aligned}$$

Therefore

$$\mu^{**} = \mu$$

Property 1. *The transformation $*$ maps $M_p(T)$ on $M_p(S)$ in one-one manner.*

Definition 2. $(\overline{T}, \overline{\mu})$ is said to be K-automorphism if there exists sub σ-algebra $\overline{\mathcal{D}}$ of $\overline{\mathcal{B}}$ such that

$$\bigvee_{n=-\infty}^{\infty} \overline{T}^n(\overline{\mathcal{D}}) = \overline{\mathcal{B}} \quad \text{and} \quad \bigwedge_{n=-\infty}^{\infty} \overline{T}^n(\overline{\mathcal{D}}) = \overline{\mathcal{R}}.$$

where $\overline{\mathcal{R}}$ is trivial in the sense that it contains only sets of measure 0 or 1.

Definition 3. (T, μ) is called exact if

$$\bigcap_{n=0}^{\infty} T^{-n}(\mathcal{B}) = \mathcal{R}.$$

Theorem 2. *Let $\mu_p \times \Lambda \in M_p(T)$. If $(\overline{T}, \overline{\mu_p \times \Lambda})$ is K-automorphism then $(S, \mu_p \times \Lambda^*)$ is exact.*

The proof will be continued by auxiliary lemmas and properties.

Property 2. *The Frobenius–Perron operator (F-P) for S with respect to $\mu_p \times \Lambda^*$ is given by*

$$Pf(\omega, x) = pf(0\omega, T_0(x)) + qf(1\omega, T_1(x))$$

for $f \in L^1(\mu_p \times \Lambda^*)$. Here $i\omega = \sigma^{-1}(\omega) \cap {}_0[i]$.

Proof. By (2) it is easy to check the equality

$$\begin{aligned}&\mu_p \times \Lambda^*(S^{-1}({}_0[i_0, \ldots, i_{n-1}] \times B) \cap {}_0[j_0, \ldots, j_{n-1}] \times C)\\ &\qquad = \int 1_{{}_0[i_0,\ldots,i_{n-1}]\times B}(\omega, x) P 1_{{}_0[j_0,\ldots,j_{n-1}]\times C}(\omega, x) d\mu_p \times \Lambda^*\end{aligned}$$

for any blocks ${}_0[i_0, \ldots, i_{n-1}]$, ${}_0[j_0, \ldots, j_{n-1}]$, $n \in N$ and Borel sets B, C. ■

Lemma 1. $h(S, \mu_p \times \Lambda^*) = h(T, \mu_p \times \Lambda) = h(\sigma, \mu_p)$.

Proof. We have immediately

$$h(\sigma, \mu_p) \leq h(T, \mu_p \times \Lambda) = h(S, \mu_p \times \Lambda^*).$$

Therefore it is enough to show that $h(T, \mu_p \times \Lambda) \leq h(\sigma, \mu_p)$. Let us consider the sequence of partitions $\alpha \times \beta_k, k = 2, 3, \ldots$. Here $\alpha = \{A_0, A_1\}$ where $A_i = \{\omega : \omega(0) = i\}$ for $i = 0, 1$ and $\beta_k = \{[0, 1/k), [1/k, 2/k), \ldots, [(k-1)/k, 1]\}$, $k = 2, 3, \ldots$. It is sufficient to show that $h(T, \alpha \times \beta_k) \leq h(\sigma, \mu_p)$ for every k because

$$\bigvee_{i=0}^{\infty} T^{-i}(\alpha \times \beta_k) \to \epsilon \quad \text{as } k \to \infty$$

(see Theorem 5.10 in [Parry, 1969]). Here ϵ denotes the point partition. Let us denote

$$(\alpha \times \beta_k)_n = \bigvee_{i=0}^{n-1} T^{-i}(\alpha \times \beta_k).$$

By strict monotonicity and continuity of T_i, $i = 0, 1$

$$(\alpha \times \beta_k)_n = \{A \times B_A : A \in \alpha_n,\, B_A \in \beta_A\} \quad \text{where}$$

$$\alpha_n = \bigvee_{i=0}^{n-1} \sigma^{-i}\alpha$$

and β_A is the partition of I on intervals. By induction on n we will prove that $\mathrm{card}(\beta_A) \leq nk$ for every $A \in \alpha_n, n = 1, 2, \ldots$. For $n = 1$, $(\alpha \times \beta_k)_1 = \alpha \times \beta_k$. Assume the inequality holds for n; we will argue this for $n + 1$. We have

$$\begin{aligned}(\alpha \times \beta_k)_{n+1} &= T^{-1}(\alpha \times \beta_k)_n \vee (\alpha \times \beta_k) \\ &= \{A \times B_A : A = \sigma^{-1}(\overline{A}) \cap A_i, \overline{A} \in \alpha_n, \\ &\qquad B_A \in \beta_A = S_i\beta_{\overline{A}} \vee \beta_k \text{ for some } i \in \{0, 1\}\}\end{aligned}$$

Since $S_i\beta_{\overline{A}}$ and β_k are partitions on intervals we get

$$\mathrm{card}(\beta_A) \leq \mathrm{card}(\beta_{\overline{A}}) + k \leq (n + 1)k.$$

Consequently

$$\begin{aligned}h(T, \alpha \times \beta_k) &= \lim_{n\to\infty} \frac{1}{n} H((\alpha \times \beta_k)_n) \\ &\leq \lim_{n\to\infty} \frac{1}{n}(H(\alpha_n) + \ln[(n + 1)k]) = h(\sigma, \mu_p)\end{aligned}$$

■

Lemma 2. *$(S, \mu_p \times \Lambda^*)$ has a one-sided generator of finite entropy.*

Proof. Let S' denote the Jacobian of S with respect to $\mu_p \times \Lambda^*$. By Property 2 we see that

$$S'(\omega, x) = \begin{cases} p^{-1} & \text{for } \omega(0) = 0 \\ q^{-1} & \text{for } \omega(0) = 1. \end{cases}$$

Therefore, by Lemma 1

$$\int \ln S' d\mu_p \times \Lambda^* = h(\sigma, \mu_p) = h(S, \mu_p \times \Lambda^*).$$

This implies that $(S, \mu_p \times \Lambda^*)$ has a one-sided generator of finite entropy. The last conclusion follows as consequence of Remark 8.10, p. 97 and Lemma 10.5, p. 110 from [Parry, 1969]. ■

Remark 1. $(S, \mu_p \times \Lambda^*)$ has a 1-sided 3-element generator by Theorem 1 [Kowalski, 1988].

Remark 2. $(T, \mu_p \times \Lambda)$ has non one-sided generator of finite entropy if S_i is not Λ preserving for some $i \in \{0, 1\}$ (see Theorem 2 [Kowalski, 1987]).

Proof of Theorem 2. If $(\overline{T}, \overline{\mu_p \times \Lambda})$ is K-automorphism then $(\overline{S}, \overline{\mu_p \times \Lambda^*})$ has the same property. Therefore $(S, \mu_p \times \Lambda^*)$ is exact as an endomorphism with a 1-sided generator of finite entropy. For detailed motivations see Theorem 6.17, p. 74 of [Parry, 1969]. ■

In order to apply Theorem 2 let us consider the set $G_{a,b}$ of homeomorphisms $g : I \to I$ included in [Kowalski & Liardet, 2000] as follows:

(i) $g(0) = 0$, $g(1) = 1$;
(ii) $g(x) \leq x$ for any $x \in I$;
(iii) for all $(x, y) \in I^2$:

$$x \neq y \Rightarrow a \leq \frac{g(y) - g(x)}{y - x} \leq \frac{1}{b}.$$

Here $a, b \in (0, 1)$ and $p < b$.

Let $\breve{g}$ be the homeomorphism defined by the equality

$$pg + q\breve{g} = Id. \tag{3}$$

For every $g \in G_{a,b}$ we consider similarly as in (1) the skew product

$$S_g(\omega, x) = (\sigma(\omega), g_{\omega(0)}(x))$$

where $g_0 = g, g_1 = \breve{g}$. Let T_g denote the dual skew product of S_g. It follows from (3) that $\mu_p \times \Lambda \in M_p(T_g)$. The main theorem of [Kowalski & Liardet, 2000] i.e. Theorem 1 states that the set of g in $G_{a,b}$

such that $(\overline{T}_g, \overline{\mu_p \times \Lambda})$ is K-automorphism containing a dense G_δ-set. Moreover, for g as above

$$M_p(T_g) = \mathrm{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}, \mu_p \times \Lambda\}$$

by Theorem 1 [Kowalski, 2003] and

$$M_p(S_g) = \mathrm{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}, \mu_p \times \Lambda^*\}$$

by Property 1. Therefore, we get directly by Theorem 2,

Theorem 3. *Assume that $a, b \in (0,1)$ and $p < b$. Then the set of g in $G_{a,b}$, such that*

$$M_p(S_g) = \mathrm{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}, \mu_p \times \Lambda^*\}$$

and $\mu_p \times \Lambda^*$ is exact contains a dense G_δ-set with respect to the uniform topology.

Remark 3. If $(S_g, \mu_p \times \Lambda^*)$ is exact then the measure $\mu_p \times \Lambda^*$ is nonabsolutely continuous.

Proof. By (2) and definition of S_g, $\mu_p \times \Lambda^*$ is not a product measure and therefore nonabsolutely continuous. ■

Theorem 3 gives a positive answer to the open problem proposed in [Kowalski & Liardet, 2000]. Now, it is the correct time for the analysis on T_g and S_g as iterated random functions. The process determined by random walk T can be written as $X_0 = x_0, X_1 = T_{\omega(0)}(x_0), X_2 = T_{\omega(1)} \circ T_{\omega(0)}(x_0), \ldots$ Inductively

$$X_{n+1} = T_{\omega(n)}(X_n).$$

We are interested in the situation where there is a stationary probability distribution $\mathcal{P}$ on I with

$$\mu_p\{X_n \in A\} \to \mathcal{P}(A) \quad \text{as } n \to \infty.$$

Let g belong to the generic set given by Theorem 3. For $T = T_g$ we obtain

$$\mu_p\{X_n \in A\} = P^n 1_A(x_0).$$

Therefore

$$\mu_p\{X_n \in A\} \to \Lambda(A) \quad \text{as } n \to \infty$$

in $L^1(\mu_p \times \Lambda^*)$ convergence by exactness of $(S_g, \mu_p \times \Lambda^*)$. We also get

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu_p\{X_k \in J\} \to \Lambda(J) \quad \text{as } n \to \infty$$

for every $x_0 \in (0,1)$ and any interval $J \subset I$, by $\mu_p \times \Lambda \in M_p(T_g)$ and Theorem 2 in [Kowalski & Liardet, 2000]. For $T = S_g$

$$\mu_p\{X_n \in A\} = P^{*n} 1_A(x_0)$$

where $P^* f(x) = pf(g_0(x)) + qf(g_1(x))$. By $P^* f \geq f$ for any f positive, strictly convex and increasing function on I (see proof of Lemma 3) and by uniqueness of absolutely continuous invariant measure for T_g we obtain

$$\lim_{n\to\infty} P^{*n} f(x) = (f(1) - f(0))x + f(0)$$

for every $x \in [0,1]$ and any continuous function f. Therefore, we get

$$\mu_p\{X_n \in J\} \to (1 - x_0)\delta_{\{o\}}(J) + x_0 \delta_{\{1\}}(J) \quad \text{as } n \to \infty$$

for every $x_0 \in [0,1]$ and any interval $J \subset I$. The idea of dual pairs allows us to construct new examples of iterated random functions. For related discussion and other examples see [Diaconis & Freedman, 1999].

In the further considerations, we introduce additional assumptions about smoothness of S_i, $i = 0, 1$. Namely, we assume that S is described by

$$S_i = (1 - \varepsilon_i)x + \varepsilon_i g(x), \quad i = 0, 1 \tag{4}$$

such that $g \in C^2[0,1]$, $g(0) = 0$, $g(1) = 1, (1 - \sup g')^{-1} < \varepsilon_0, \varepsilon_1 < (1 - \inf g')^{-1}$. Furthermore, we suppose that there exists exactly one point x_0 for which $g'(x_0) = 1$ and $g'(x) < 1$ for $x < x_0$. The operator F-P for T with respect to $\mu_p \times \Lambda$ is given by the equality

$$P_T f(\omega, x) = pS_0'(x) f(0\omega, S_0(x)) + qS_1'(x) f(1\omega, S_1(\omega)).$$

Let us observe, that

$$\int f P_T(g) d\Lambda = \int P(f) g d\Lambda$$

for $f \in L^\infty(\Lambda)$ and $g \in L^1(\Lambda)$. The above equality is justified by naming S and T as a dual pair. We can also generalize Remark 3 as follows

Lemma 3. *Let S be given by (4). Then $M_p(S)$ has no invariant absolutely continuous measure for any $p \in (0,1)$.*

Proof. Assume on the contrary that $\mu \ll \mu_p \times \Lambda$ is S-invariant and finite. Then $\mu = \mu_p \times \Lambda_G$ (see Theorem 3.1 of [Morita, 1988]), where

$$G = \frac{d\Lambda_G}{d\Lambda}.$$

Moreover, $P_S G = G$ and $P^* Id \geq Id$ or $P^* Id \leq Id$. Here $P^* f(x) = pf(S_0(x)) + qf(S_1(x))$. The last inequalities hold by Lemma 3 in [Kowalski, 2003]. Let us assume that $P^* Id \geq Id$. For F positive, strictly convex and increasing function on I we get

$$\int GFd\Lambda = \int P_S GFd\Lambda = \int GP^* Fd\Lambda$$
$$> \int GF(P^* Id)d\Lambda \geq \int GFd\Lambda$$

which is impossible. ■

3. Invariant Sets for Dissipative Extensions

We start with a more general situation i.e. S_i are only nonsingular maps of I ($\Lambda(B) = 0 \Rightarrow \Lambda(S_i^{-1}(B)) = 0$) for $i = 0, 1$. It is known by Theorem 1 in [Kowalski, 2009] that if S is conservative and a set E of positive $\mu_p \times \Lambda$ measure is S invariant i.e. $S(E) \subset E$, then $E = \Omega \times B$ for some Borel set $B \subset I$. We show by the example that in the totally dissipative case, invariant sets cannot be product sets. Let S be given by

$$S_0 = \frac{1}{2}x, \quad S_1 = \frac{1}{2}x + \frac{1}{2}.$$

So S is baker's transformation.

Theorem 4. *Let $p = 1/6$. Then $(S, \mu_p \times \Lambda)$ is totally dissipative.*

Proof. Let P_S denote F-P operator for S. Therefore,

$$P_S f(x) = \left[\frac{1}{3} 1_{S_0 I}(x) + \frac{5}{3} 1_{S_1 I}(x)\right] f(T(x)),$$
$$\text{for } f \in L^1(\Lambda).$$

Here $T(x) = 2x \bmod 1$. For $f \equiv 1$ we get

$$P_S^n 1 = \sum_{\mathbf{i} \in \{0,1\}^n} \frac{5^{\sum_{k=1}^n \mathbf{i}(k)}}{3^n} 1_{I_{\mathbf{i}}}(x),$$

where $\{I_{\mathbf{i}}\}_{\mathbf{i} \in \{0,1\}^n}$ is the partition of I on dyadic intervals of rank n. The limit average of attendance of 1 in dyadic extension is $1/2$ for a.e. x. Therefore, for a.e. x, there exists n_0 such that for any $n > n_0$ if $x \in I_{\mathbf{i}}$, where $\mathbf{i} \in \{0,1\}^n$, then

$$\sum_{i=1}^n \mathbf{i}(k) \leq \frac{2}{3}n.$$

Hence,

$$\sum_{n=0}^\infty P_S^n 1(x) = \sum_{n=0}^{n_0} P_S^n 1(x) + \sum_{n=n_0+1}^\infty P_S^n 1(x)$$
$$\leq \sum_{n=0}^{n_0} P_S^n 1(x) + \sum_{n=n_0+1}^\infty \left(\frac{5^{\frac{2}{3}}}{3}\right)^n < \infty$$

for a.e. x. Therefore,

$$\left\{x : \sum_{n=0}^\infty P_S^n 1(x) < \infty\right\}$$

has measure 1. By Proposition 1.3.1 [Aaronson, 1997] S is totally dissipative. ■

Conclusion 1. *S is not ergodic.*

Theorem 5. *If $A \in \mathcal{B}, \mathcal{S}^{-\infty}\mathcal{A} = \mathcal{A}$ and $0 < \mu_p \times \Lambda(A) < 1$ then A is not a product set.*

Proof. Suppose on the contrary that $A = D \times B$. $SA = A$ implies $\sigma(D) = D$ and hence $D = \Omega$. $S^{-1}A = A$ implies $TB = B$ and therefore $B = I$. Finally, we get $A = \Omega \times I$ which is impossible. ■

We can also obtain invariant absolutely continuous measure which is not a product measure. By Theorem 4.1 [Friedman, 1970] there exists σ-finite invariant measure $\mu \approx \mu_p \times \Lambda$. The measure $\mu_A = \mu \,|\, A$ is S-invariant absolutely continuous but not a product measure.

4. Binomial Model for Asset Prices

Let us consider the classical one-asset binomial model. For details see [Bahsoun *et al.*, 2007, Section 5]. At each period of time there are two possibilities: the security price may go up by a factor $u(x)$ or it may go down by a factor $d(x)$. The factors u and d are functions of the prices, $u : (0, 1) \to (1, \infty)$ and $d : (0, 1) \to (0, 1)$. Given the functions $u(x), d(x)$ and the probabilities $p = p_u, q = p_d = 1 - p_u$, we can construct the random map S which consists of the transformations S_u, S_d

$$S_u(x) = u(x)x \quad \text{and} \quad S_d(x) = d(x)x.$$

Here $S_u : [0,1] \to [0,1], S_d : [0,1] \to [0,1]$ are continuous maps,

$$\bigwedge_{x\in[0,1]} S_u(x) \geq x \quad \text{and} \quad S_d(x) \leq x.$$

We will assume that S_u and S_d are homeomorphisms so S is given by (1). The subscript u of S_u illustrates that transformation S_u comprises the law which moves the price up and the subscript d of S_d denotes that transformation S_d is the law which moves the price down. The process determined by the random walk S and starting from $x_0 \in (0,1)$ can be written as X_n. We now give examples to illustrate the structure of the above model.

Example 1. Let

$$S_0(x) = S_u(x) = \frac{x}{\lambda_0 x + 1 - \lambda_0}, \quad \lambda_0 \in (0,1),$$

$$S_1(x) = S_d(x) = \frac{x}{\lambda_1 x + 1 - \lambda_1}, \quad \lambda_1 < 0.$$

Such S has been studied in [Kowalski, 2009, Section 3]. It appears that if

$$p < \frac{\ln(1-\lambda_0)}{\ln\left(\dfrac{1-\lambda_0}{1-\lambda_1}\right)}$$

then $X_n(\omega) \to 0$ for a.e. ω, so the security price goes to 0. If

$$p > \frac{\ln(1-\lambda_0)}{\ln\left(\dfrac{1-\lambda_0}{1-\lambda_1}\right)}$$

then $X_n(\omega) \to 1$ for a.e. ω, so the security price goes to 1. In the case

$$p = \frac{\ln(1-\lambda_0)}{\ln\left(\dfrac{1-\lambda_0}{1-\lambda_1}\right)}$$

S is conservative. If additionally p is irrational then S is ergodic but not exact. So for this p we have chaotic behavior of the security price.

Let us consider the general case. If p is such that

$$pS_u + qS_d = I_d \quad \text{or} \quad pS_u^{-1} + qS_d^{-1} = I_d$$

then the trajectory of X_n is extremely chaotic for typical S_u. This is the consequence of Theorem 3. The behavior of X_n for other p needs individual analysis. For example, if S is given by (4) then the description of X_n can be found in [Kowalski, 2009, Section 3].

Acknowledgment

This research was supported partially by grant MENII N N201 394537, Poland.

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