Random walk as a model of motion in quantum harmonic oscillator

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Abstract

The one-dimensional quantum harmonic oscillator is considered. We complement the wave solution of the Schrödinger equation by presenting the motion of particles for n-quantum state at the moment of jump from $n-1$ to n-quantum state as orbits of stationary random walk. Consequently, we get two equlibrium positions of the oscillator in n-quantum state. The numerical solutions are given for n=2,3.

1 Introduction

One-dimensional quantum harmonic oscillator Ψ satisfies the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$$

where $H = \frac{1}{2}(P^2 + Q^2)$ is the Hamiltonian. Here $P = -i\frac{d}{dx}$ and Q is multiplication by x. Let us consider the n-quantum state solution

$$\Psi_n(x,t) = \frac{1}{\sqrt{n!2^n\sqrt{\pi}}}H_n(x)exp(-\frac{i}{2\hbar}(2n+1)t)exp(-\frac{x^2}{2}).$$

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for $n \geq 0$. Here H_n is nth Hermite polynomial i.e

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The quantum interpretation of

$$\Phi_n(x) = \int_{-\infty}^{x} |\Psi_n(y,t)|^2 \, dy = \frac{1}{n! 2^n \sqrt{\pi}} \int_{-\infty}^{x} H_n^2(y) exp(-y^2) dy$$

is the probability distribution of occurrence of a particle in $\mathbb{R}$. We consider a sudden change of n-quantum state Ψ_n to the state Ψ_{n+1} by analyzing only the probability distributions Φ_n and Φ_{n+1} . Therefore we don't apply theory of quantum transition.
In Theorem 2.1 [3] it has been observed that for $n \geq 1$ there is the unique partition on intervals $\{I_k : k = 1, ..., 2n\}$ of $\mathbb{R}$ such that $\mu_n(I_k) = \mu_{n-1}(I_k)$ where the measure μ_n has distribution Φ_n. The endpoints of intervals come from the equivalence

$$\Phi_n(x) = \Phi_{n-1}(x) \Leftrightarrow H_n(x) H_{n-1}(x) = 0.$$

They are singular points of density of measures μ_n and μ_{n-1} respectively for $n \geq 2$. It is convenient to consider the unit interval instead of $\mathbb{R}$, therefore we use the map $\Phi_{n-1} : \mathbb{R} \to (0, 1)$. Φ_{n-1} arise as the distribution function of the Lebesgue measure on the unit interval and Φ_n as $F^{(n)} = \Phi_n \circ \Phi_{n-1}^{-1}$. The last one has been considered in [3]. Here Φ_{n-1}^{-1} is the inverse function of Φ_{n-1}. The jump from $n-1$ to n-quantum state i.e. from Φ_{n-1} to Φ_n changes the structure of particles and their dynamics. In section 2 we describe the step skew product transformation which is determined by $F^{(n)}$ for $n \geq 2$ and is stationary one. This dynamical system has interpretation as a random walk on the interval on the basis of coin tossing with $p \in (\frac{1}{2}, \frac{1}{\sqrt[3]{2}}]$ which is the probability of head. Namely, $F^{(n)}$ determines the random walk on $[0, 1] - \Phi_{n-1}(I_1 \cup I_{2n})$ under assumption of symmetry i.e. the Lebesgue measure is stationary for symmetric coin (see Corollary 2.6). The walk is carried by Φ_{n-1}^{-1} on $\mathbb{R} - (I_1 \cup I_{2n})$ to random walk where x goes with probability p to $h_0(x)$ and probability $1 - p$ to $h_1(x)$ and μ_n is stationary measure.The self-homeomorphisms $h_0(x)$, $h_1(x)$ are determined in one way by $F^{(n)}$ and p and by assumption of stationarity of μ_{n-1} in the case of walking as above under tossing of symmetric coin.
According to the author, the possible physical interpretation of this walk is description of the oscillator in the nonequilibrium state at the moment of jump from $n-1$ to n-quantum state. Here the $n-1$-particle state changes

to n-particle state by adding one particle (see [1], chapter 1.5).
In section 3 we construct numerically the random walk for 2 and 3- quantum state (see Figures 2 and 4). Section 4 is devoted to the equilibrium states, where we additionally analyze the moment of jump from $n+1$ to n-quantum state.

2 n-quantum state

We will apply Theorem 2.5 i.e. Theorem 6.6 [3] for our purposes. At first we observe the following symmetry properties.

Lemma 2.1. *$F^{(n)}(1-u) = 1 - F^{(n)}(u)$ for $u \in (0,1)$.*

Proof. By $\Phi_n(-x) = 1 - \Phi_n(x)$ for $x \in \mathbb{R}$ we have $\Phi_n^{-1}(1-u) = -\Phi_n^{-1}(u)$ for $u \in (0,1)$. Therefore

$$F^{(n)}(1-u) = \Phi_n(\Phi_{n-1}^{-1}(1-u)) = \Phi_n(-\Phi_{n-1}^{-1}))$$

$$= 1 - \Phi_n(\Phi_{n-1}^{-1}(u) = 1 - F^{(n)}(u).$$

□

We will denote $F^{(n)}$ by F as n is fixed. We extend F to $[0,1]$ by putting $F(0) = 0$ and $F(1) = 1$. Let $p \in (0,1)$. Define

$$F_p(u,v) = pF(2u-v) + (1-p)F(v) - F(u) \text{ for } (u,v) \in [0,1] \times [0,1]$$

such that the right hand side of the equality make sense.

Lemma 2.2. *If $F(1-u) = 1 - F(u)$ and g is an implicit function on $[a,b] \subset [0,1]$ i.e. $F_p(u, g(u)) = 0$ for $u \in [a,b]$ then $1 - g(1-u)$ is the implicit function on $[1-b, 1-a]$.*

Proof. By assumption we have

$$pF(2u - g(u)) + (1-p)F(g(u)) - F(u) = 0 \text{ for } u \in [a,b].$$

For $u \in [1-b, 1-a]$ we get

$$pF(2u - (1 - g(1-u)) + (1-p)F(1 - g(1-u)) - F(u) =$$

$$pF(1 - (2(1-u) - g(1-u)) + (1-p)F(1 - g(1-u)) - F(u) =$$

$$-[pF((2(1-u) - g(1-u)) + (1-p)F(g(1-u)) - F(1-u)] = 0$$

as $1 - u \in [a,b]$. □

From now on we will assume that $n \geq 2$. Let a_k be the left endpoint of interval I_{k+1} and $b_k = \Phi_{n-1}(a_k)$ be the left endpoint of interval $J_{k+1} = \Phi_{n-1}(I_{k+1})$ for $k = 1, ..., 2n-1$. The sequence a_k is symmetric with respect to 0 and the sequence b_k is the set of fix points of F and it is symmetric with respect to $\frac{1}{2}$. Denote by $int(J_{k+1})$ the interval (b_k, b_{k+1}).

Lemma 2.3. *If $H_{n-1}(a_k) = 0$ for some $1 \leq k \leq 2n-1$ then $F'(b_k^+) = \infty$, $F'(b_{k+1}^-) = 0$ and $F''(u) < 0$ for $u \in int(J_{k+1})$. Similarly, if $H_n(a_k) = 0$ for some $1 \leq k \leq 2n-1$ then $F'(b_k^+) = 0$ and $F'(b_{k+1}^-) = \infty$, and $F''(u) > 0$ for $u \in int(J_{k+1})$.*

Proof. By using the definitions of $H_n(x)$, $\Phi_n(x)$ and $F(x)$ we get

$$\frac{d}{du}F(u) = F'(u) = \frac{1}{2n}\Big(\frac{H_n(x)}{H_{n-1}(x)}\Big)^2|_{x=\Phi_{n-1}^{-1}(u)} =$$

$$\frac{1}{2n}\Big(\frac{(exp(-x^2)^{(n)}}{(exp(-x^2)^{(n-1)}}\Big)^2|_{x=\Phi_{n-1}^{-1}(u)}$$

and

$$F''(u) = \frac{\sqrt{\pi}2^{n-1}(n-1)!}{n}exp(x^2)\frac{H_n(x)}{H_{n-1}^5(x)}(H_n^2(x) - H_{n+1}(x)H_{n-1}(x))|_{x=\Phi_{n-1}^{-1}(u)}.$$

Moreover,

$$\Big(\frac{H_n}{H_{n-1}}\Big)'(x) = \frac{H_n^2(x) - H_{n+1}(x)H_{n-1}(x)}{H_{n-1}^2(x)} > 0$$

by Turan's inequality. We only show the first part of the lemma. The roots of H_n and H_{n-1} lie alternately. Therefore, if $H_{n-1}(a_k) = 0$ then $H_n(a_{k+1}) = 0$. Hence $F'(b_k^+) = \infty$ and $F'(b_{k+1}^-) = 0$ and

$$sgn(F''(u)) = sgn\Big(\frac{H_n}{H_{n-1}} \circ \Phi_{n-1}^{-1}(u)\Big) < 0.$$

□

Let us denote $F' = F'|J_k$ for some k. Then $F'^{-1} : (0, \infty) \to J_k$. We define

$$\varphi_p(u) = 2u - F'^{-1}\Big(\frac{1}{2p}F'(u)\Big) \text{ for } u \in int(J_k), k = 1, ..., 2n-1,$$

as in [3] and extend φ_p on $\cup_{k=1}^{2n-1} J_k$ to the continuous function by putting $\varphi_p(b_k) = b_k$ for $k = 1, ..., 2n-1$.

Denote

$$\varphi_p'(x) = \varphi_p' \circ \Phi_{n-1}(x) \text{ and } F'(x) = F' \circ \Phi_{n-1}(x), F''(x) = F'' \circ \Phi_{n-1}(x).$$

So

$$\varphi_p'(x) = 2 - \frac{1}{2p}\frac{F''(x)}{F''(F'^{-1}(\frac{1}{2p}F'(x)))} \text{ for } x \in int(I_k), k = 1, ..., 2n-1.$$

Theorem 2.4. *If $H_n(a_k) = 0$ then*

$$\lim_{x\to a_k} \varphi_p'(x) = 2 - \frac{1}{\sqrt{2p}}$$

and if $H_{n-1}(a_k) = 0$ then

$$\lim_{x\to a_k} \varphi_p'(x) = 2 - (2p)^{\frac{3}{2}}$$

for $k = 1, ..., 2n-1$.

Proof. Let us assume that $H_n(a_k) = 0$. Then

$$F'(x) \approx c_n(x-a_k)^2 \text{ and } F''(x) \approx d_n(x-a_k) \text{ for } x \to a_k.$$

Hence

$$F'^{-1}(x) \approx a_k \pm \sqrt{\frac{x}{c_n}} \text{ for } x \to a_k^{\pm}.$$

Here c_n, d_n are the nonzero constants. Therefore

$$F'^{-1}\Big(\frac{1}{2p}F'(x)\Big) \approx a_k \pm \frac{1}{\sqrt{2p}}|x-a_k| \text{ for } x \to a_k^{\pm}.$$

By definition of $\varphi_p'(x)$ we have

$$\varphi_p'(a_k) = 2 - \frac{1}{2p}\lim_{x\to a_k^{\pm}} \frac{x-a_k}{\pm\frac{1}{\sqrt{2p}}|x-a_k|} = 2 - \frac{1}{\sqrt{2p}}.$$

For case $H_{n-1}(a_k) = 0$ we have

$$F'(x) \approx \frac{c_{n-1}}{(x-a_k)^2} \text{ for } x \to a_k,\ c_{n-1} \neq 0.$$

Then we proceed as before. □

Theorem 2.5. ([3]). *Let $F \in C^2(0,1) \cap C[0,1]$, $F(0) = 0$, $F(1) = 1$, $F'(0^+) = \infty$, $F'(1^-) = 0$ and $F''(x) < 0$ for $x \in (0,1)$. Moreover, let $p \in (\frac{1}{2}, 1)$ be such that φ_p and φ_{1-p} are increasing functions. Then there exists exactly one implicit function $y = g(x)$ such that $g(x) < x$ for $x \in (0,1)$. The function g is a homeomorphism of I and $g \in C^1(0,1)$. Moreover $2x - g(x)$ is a homeomorphism too.*

Corollary 2.6. *By Lemma 2.3 assumptions of Theorem 2.5 are satisfied for F on invariant intervals J_{k+1} for k such that $H_{n-1}(a_k) = 0$ and $\varphi'_p(u) > 0$ and $\varphi'_{1-p}(u) > 0$ for $u \in int(J_{k+1})$. Moreover, Theorem 2.4 implies that $p \in (\frac{1}{2}, \frac{1}{\sqrt[3]{2}}]$.*

Let us denote by g_p the self-homeomorphism given by Corollary 2.6 for some $1 \leq k \leq 2n-1$.

Theorem 2.7. *If $\frac{1}{2} < p < q \leq \frac{1}{\sqrt[3]{2}}$ then $g_q(u) < g_p(u)$ for every $u \in int(J_{k+1})$.*

Proof. If $p < q$ then $F_p(u,v) < F_q(u,v)$ for $0 < v < u$. Therefore

$$0 = F_q(u, g_q(u)) = F_p(u, g_p(u)) < F_q(u, g_p(u)).$$

By Lemma 6.3 [3]

$$\frac{\partial F_q(u,v)}{\partial v} > 0 \text{ for } 0 < v < g_q(u).$$

Hence $g_q(u) < g_p(u)$ for $u \in int(J_{k+1})$. □

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Let Ω be the space $\{0,1\}^{\mathbb{N}}$, $\mathbb{N} = \{0,1,2,...\}$, with the $(p, 1-p)$-Bernoulli measure μ_p on $(\Omega, \mathcal{B})$, where $\mathcal{B}$ is the Borel product σ-algebra. We denote by $\mathcal{A}$ the Borel σ-algebra of subsets of $\mathbb{R}$ and by Λ the Lebesgue measure. Let σ be the one-sided shift on Ω i.e. $\sigma(\omega)(i) = \omega(i+1)$. Let us assume that conditions of Corollary 2.6 hold for some $p \in (\frac{1}{2}, \frac{1}{\sqrt[3]{2}}]$ and for some $1 \leq k \leq 2n-1$. By Theorem 2.5 we get the step skew product transformation in the space $\Omega \times J_{k+1}$ as follows

$$S_p(\omega, u) = \begin{cases} (\sigma(\omega), (2u - g(u))^{-1}) & \text{for } \omega_0 = 0 \text{ and} \\ (\sigma(\omega), g^{-1}(u)) & \text{for } \omega_0 = 1. \end{cases}$$

Here $g(u)$ and $2u - g(u)$ are the increasing self-homeomorphisms of J_{k+1}. The skew product as above preserves the measure $\mu_p \times \mu_F$ on $\Omega \times J_{k+1}$ where μ_F has the distribution F. The formula $(2u - g(u))^{-1}$ for the first homeomorphism is equivalent to S_p invariance of the measure $\mu_{\frac{1}{2}} \times \Lambda$. Moreover, its natural extension to automorphism is Bernoulli one (see Corollary 5.2 [4]) and

$$\int |\mu_p(S_p^j(\cdot, u) \in A) - \frac{\mu_F(A)}{\mu_F(J_{k+1})}| d\mu_F(u) \to 0$$

as $j \to \infty$ for any measurable set $A \subset J_{k+1}$ by [3].

Remark 2.8. By Lemma 2.2 we get the similar skew product with the same properties for symmetric interval $\hat{J}_{k+1} = [1 - b_{k+1}, 1 - b_k]$. We put above $h(u) = 1 - g(1 - u)$ instead of $g(u)$. Such skew products are called random dynamical systems or random walks.

The interval symmetric to I_{k+1} i.e. $\hat{I}_{k+1}$ is $[-a_{k+1}, -a_k] = \Phi_{n-1}^{-1}(\hat{J}_{k+1})$ by $\Phi_{n-1}^{-1}(1-u) = -\Phi_{n-1}^{-1}(u)$ for $u \in (0,1)$. Let

$$H_g(x) = \Phi_{n-1}^{-1}(g(\Phi_{n-1}(x)) \text{ for } x \in I_{k+1}.$$

Lemma 2.9.

$$H_{1-g(1-u)}(x) = -H_g(-x) \text{ and } H_{2u-(1-g(1-u))}(x) = -H_{2u-g(u)}(-x)$$

for $x \in \hat{I}_{k+1}$.

Proof. We only show the first equality .

$$H_{1-g(1-u)}(x) = \Phi_{n-1}^{-1}(1 - g(1 - \Phi_{n-1}(x)) = -\Phi_{n-1}^{-1}(g(1 - \Phi_{n-1}(x)) =$$
$$-\Phi_{n-1}^{-1}(g(\Phi_{n-1}(-x)) = -H_g(-x) \text{ for } x \in \hat{I}_{k+1}.$$

□

We define the skew product transformation $\hat{S}_p(\omega, x)$ in the space $\Omega \times I_{k+1}$ by putting H_g and $H_{2u-g(u)}$ instead of g and $2u - g(u)$ in the definition of $S_p(\omega, u)$. This skew product preserves the measure $\mu_p \times \mu_n$ on $\Omega \times I_{k+1}$ where μ_n has distribution Φ_n. Moreover, its natural extension to automorphism is Bernoulli one and

$$\int |\mu_p(\hat{S}_p^j(\cdot, x) \in A) - \frac{\mu_n(A)}{\mu_n(I_{k+1})}| d\mu_n(x) \to 0$$

as $j \to \infty$ for any measurable set $A \subset I_{k+1}$. We also have by [3] that

$$\frac{1}{m} \sum_{j=0}^{m-1} 1_{\Omega \times J}(\hat{S}_p^j(\omega, x)) \to \frac{\mu_n(J)}{\mu_n(I_{k+1})}$$

as $m \to \infty$ for μ_p-almost every $\omega \in \Omega$, every $x \in int(I_{k+1})$ and any interval $J \subset I_{k+1}$. By Lemma 2.9 we get the similar skew product with the same properties for symmetric interval $\hat{I}_{k+1}$.

Let us denote by $\hat{X}_m(\omega, x)$ the random walk on I_{k+1} defined as

$$\hat{X}_0(\omega, x) = x \text{ and } \hat{X}_m(\omega, x) = \pi(\hat{S}_p^m(\omega, x))$$

for $m = 1, 2, \ldots$. Here $\pi(\omega, x) = x$ for $x \in \mathbb{R}$.

Proposition 2.10. $\hat{X}_m(\omega, x)$ *is the Markov process with measure* $\mu_p \times \mu_n$ *where* μ_n *is the stationary one.*

For $p = \frac{1}{2}$ *the Markov measure is* $\mu_{0,5} \times \mu_{n-1}$ *with* μ_{n-1} *as the stationary measure.*

3 2 and 3-quantum states

We use the following formula

$$\Phi_n(x) = \Phi_0(x) - \frac{1}{n!2^n\sqrt{\pi}}e^{-x^2}\sum_{l=0}^{n-1}\binom{n}{l}^2 2^l l! H_{2(n-l)-1}.$$

Here

$$\Phi_0(x) = \frac{1}{2}(1 + erf(x)), \text{ for } x \geq 0.$$

We extend the above to $\mathbb{R}$ by putting

$$\Phi_n(x) = 1 - \Phi_n(-x) \text{ for } x < 0 \text{ and } n \geq 0.$$

For erf we use the rational approximation (see [2]) as follows

$$erf(x) \approx 1 - \frac{1}{(1 + a_1x + \cdots + a_6x^6)^{16}}$$

where $a_1 = 0.0705230784, a_2 = 0.0422820123, a_3 = 0.0092705272, a_4 = 0.0001520143, a_5 = 0.0002765672, a_6 = 0.0000430638$. The maximum error is $3 \cdot 10^{-7}$. Let us consider the case $n = 2$. The partition of $\mathbb{R}$ on four intervals is given by zeros of $H_1(x) = 2x$ and $H_2(x) = 4x^2 - 2$ i.e. $\{-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\}$. For our aims consider intervals I_2 and I_3 i.e. $I_2 = \hat{I}_3$, $I_3 = [0, \frac{\sqrt{2}}{2}]$ and $J_3 = [\frac{1}{2}, \Phi_1(\frac{\sqrt{2}}{2})]$, $\hat{J}_3 = [1 - \Phi_1(\frac{\sqrt{2}}{2}), \frac{1}{2}]$. Just restrict our considerations to J_3. Denote $\Phi_1(\frac{\sqrt{2}}{2})$ by a and put $F(u) = F^{(2)}(u) = \Phi_2(\Phi_1^{-1}(u))$. Here

$$F'(u) = (\frac{1}{2x} - x)^2|_{x=\Phi_1^{-1}(u)} \text{ for } u \in J_3,$$

$$F'^{-1}(x) = \Phi_1(\frac{1}{2}(\sqrt{x+2} - \sqrt{x})) \text{ for } x \in [0, \infty)$$

and

$$\varphi_p(u) = 2u - F'^{-1}(\frac{1}{2p}F'(u)).$$

Since we have

$$\varphi_p'(x) = 2 - (8p)^{-\frac{3}{2}}h^2(x)x^{-2}e^{(x^2 - h^2(x)(8p)^{-1})}(1 + \frac{1}{2x^2})\cdot$$

$$(1 - \frac{\frac{1}{2x} - x}{\sqrt{(\frac{1}{2x} - x)^2 + 4p}}),$$

where

$$h(x) = \frac{4p}{\sqrt{(\frac{1}{2x} - x)^2 + 4p} + (\frac{1}{2x} - x)}.$$

Here $x \in I_3$. The numerical calculation shows that $\varphi_p' > 0$ on $(0, \frac{\sqrt{2}}{2}]$ for $p = \frac{1}{\sqrt[3]{2}}$ and $p = 1 - \frac{1}{\sqrt[3]{2}}$. In this situation we use Lemmas 6.1, 6.3, 6.4 from [3] and Theorem 2.5 to construct numerically $g(u)$ as follows.
For $u_0 = \frac{1}{2} + h$, where $h = \frac{a-\frac{1}{2}}{1000}$, we find the first $k_0 \in \mathbb{N}$ such that for $v_0 = \frac{1}{2} + k_0 \frac{h}{1000}$ we have $F(u_0, v_0) > 0$. Here

$$F(u,v) = \frac{1}{\sqrt[3]{2}} F(2u - v) + (1 - \frac{1}{\sqrt[3]{2}}) F(v) - F(u).$$

Next, we put $u_{n+1} = u_n + h$, $v_{n+1} = v_n + k_{n+1} \frac{h}{1000}$, where $k_{n+1} \in \mathbb{N}$ is the smallest number such that $F(u_{n+1}, v_{n+1}) > 0$, $n = 0, ..., 998$.
The solution $v = g(u)$ satisfies $v_n - \frac{h}{1000} < g(u_n) < v_n$ for $n = 0, ..., 998$ by Lemma 6.4 ([3]). In Figure 1 we present the point graphs of (u_n, v_n) and $(u_n, 2u_n - v_n)$ which approximate $g(u)$ and $2u - g(u)$ respectively on the interval $[1 - a, a]$. The quantum justification for $g(\frac{1}{2}) = \frac{1}{2}$ is that the value of the density of the distribution function in $\frac{1}{2}$ is $F'(\frac{1}{2}) = \infty$. Similarly, $g(a) = a$ because $F'(a) = 0$.

Let us denote

$$h_0(x) = H^{-1}_{2u-g(u)}(x) = \Phi_1^{-1}((2u - g(u))^{-1} \circ \Phi_1(x)) \text{ for } x \in I_2 \cup I_3$$

and

$$h_1(x) = H_g^{-1}(x) = \Phi_1^{-1}(g^{-1}(\Phi_1(x)) \text{ for } x \in I_2 \cup I_3.$$

In Figure 2 we present the approximate graphs of self-homeomorphisms h_0 and h_1.
Now, we consider the case $n = 3$. Here $F(u) = F^{(3)}(u) = \Phi_3(\Phi_2^{-1}(u))$. The partition of $\mathbb{R}$ on six intervals is given by zeros of $H_2(x) = 4x^2 - 2$ and $H_3(x) = 4x(2x^2 - 3)$ i.e. $\{-\frac{\sqrt{6}}{2}, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}\}$. For our aims consider intervals I_2, I_3, I_4, I_5 i.e. $I_4 = [0, \frac{\sqrt{2}}{2}]$, $I_5 = [\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}]$ and $I_2 = \hat{I}_5$, $I_3 = \hat{I}_4$. Let $b = \Phi_2(\frac{\sqrt{2}}{2})$ and $c = \Phi_2(\frac{\sqrt{6}}{2})$. Then $J_4 = [\frac{1}{2}, b]$, $J_5 = [b, c]$, $J_2 = \hat{J}_5$ and $J_3 = \hat{J}_4$. The numerical calculation shows that $\varphi_p'(x) > 0$ on $[0, \frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}]$ for $p = 0,785$ and $p = 0,215$. Moreover $\varphi_p'(x) > 0$ on $(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}]$ for $p = \frac{1}{\sqrt[3]{2}}$ and $p = 1 - \frac{1}{\sqrt[3]{2}}$. We numerically construct $g(u)$ on J_3 and J_5 in the similar way as in the case $n = 2$. In Figure 3 we present the point graphs of (u_n, v_n) and $(u_n, 2u_n - v_n)$ which approximate $g(u)$ and $2u - g(u)$ respectively on the interval $[1 - c, c]$. Similarly, in Figure 4 we get the approximate graphs of self-homeomorphisms h_0 and h_1 on interval $[-\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}]$.

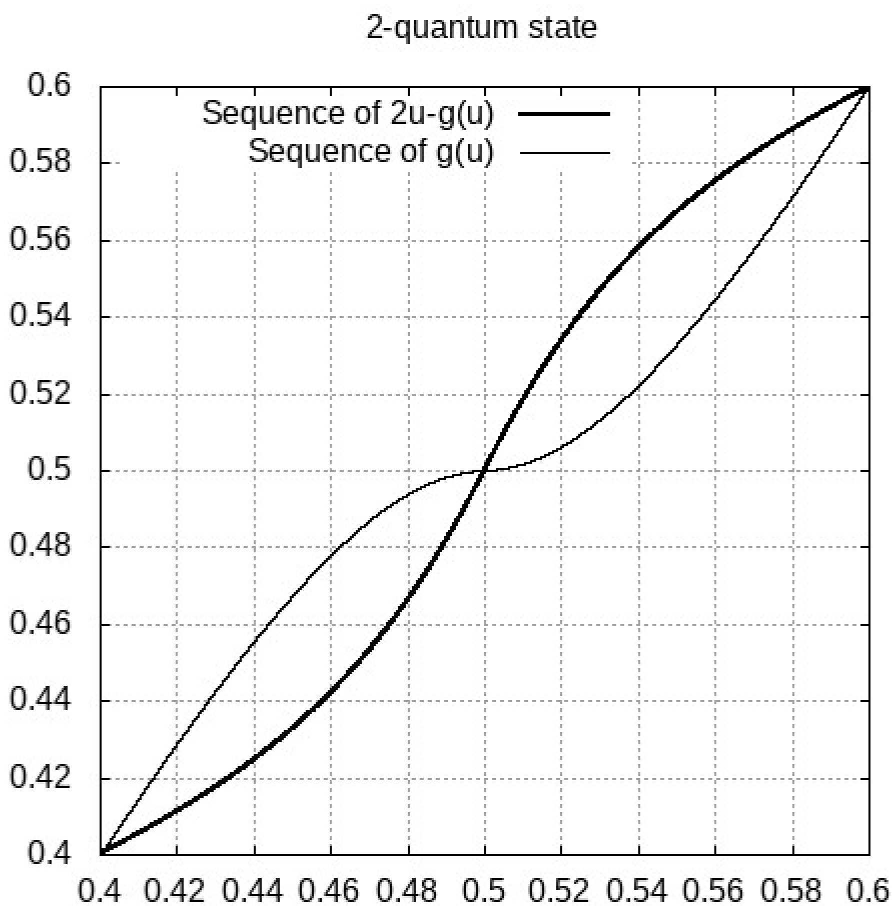

Figure 1: The approximate plot of implicit function $v = g(u)$ and $v = 2u - g(u)$ on interval $[1 - \Phi_1(\frac{\sqrt{2}}{2}), \Phi_1(\frac{\sqrt{2}}{2})]$

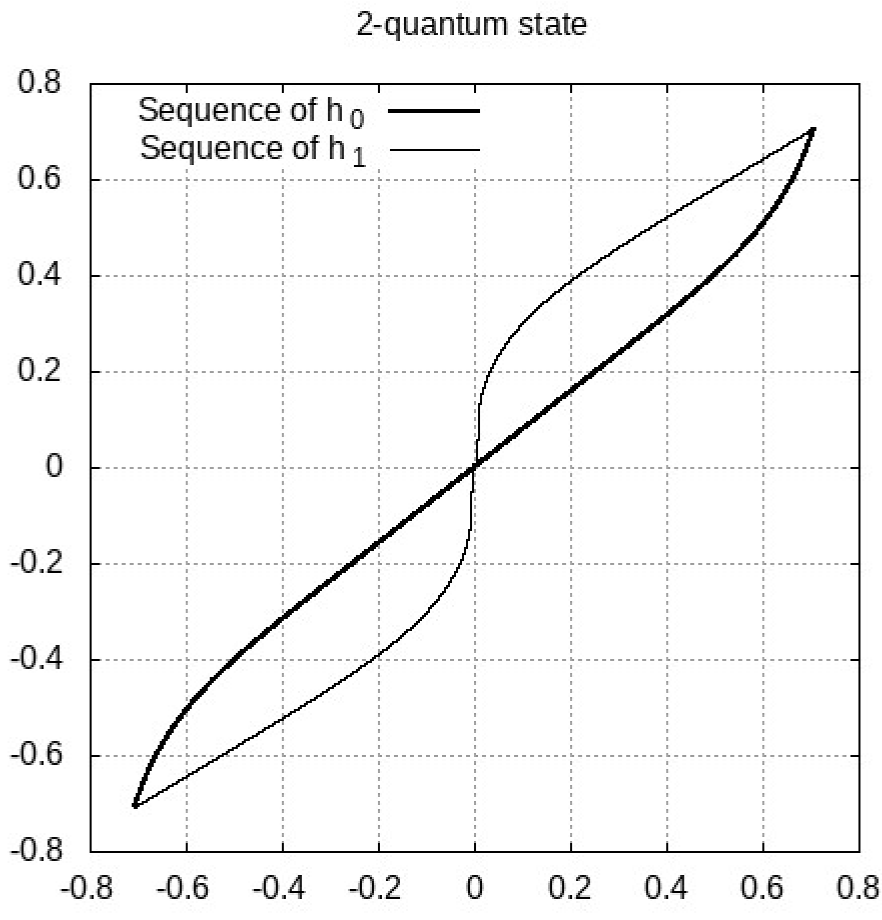

Figure 2: The approximate plot of self-homeomorphisms $y = h_0(x)$ and $y = h_1(x)$ for $p = \frac{1}{\sqrt[3]{2}}$ on interval $[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$.

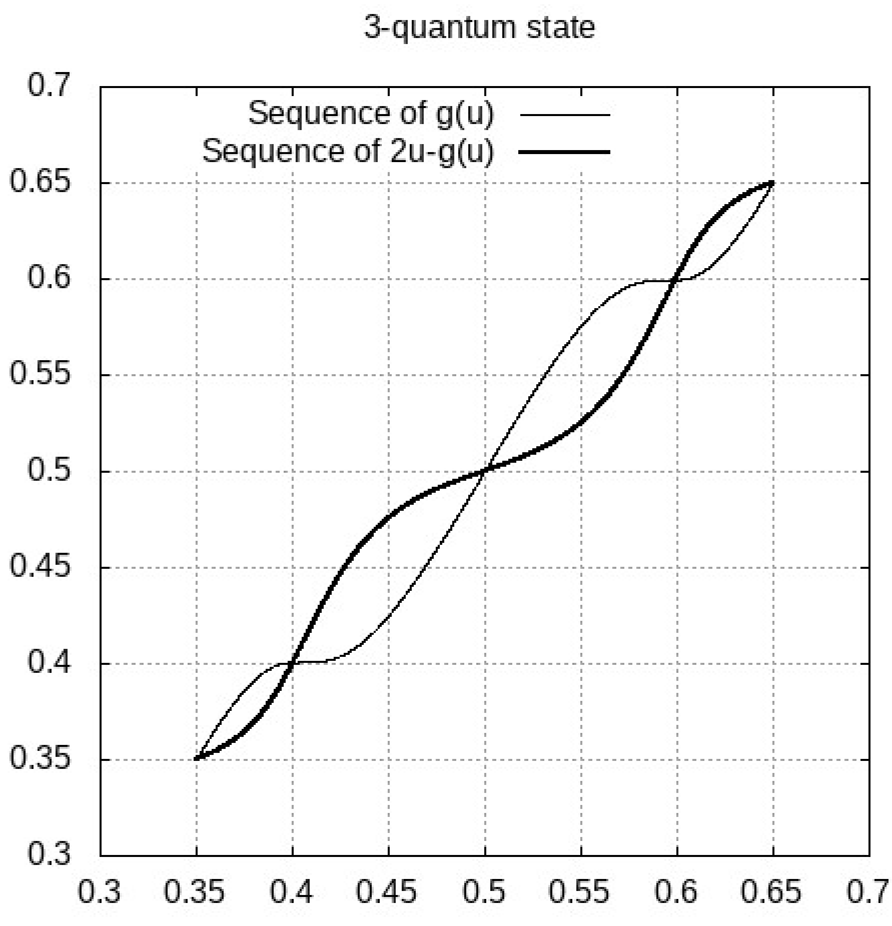

Figure 3: The approximate plot of implicit function $v = g(u)$ and $v = 2u - g(u)$ on interval $[1 - \Phi_2(\frac{\sqrt{6}}{2}), \Phi_2(\frac{\sqrt{6}}{2})]$

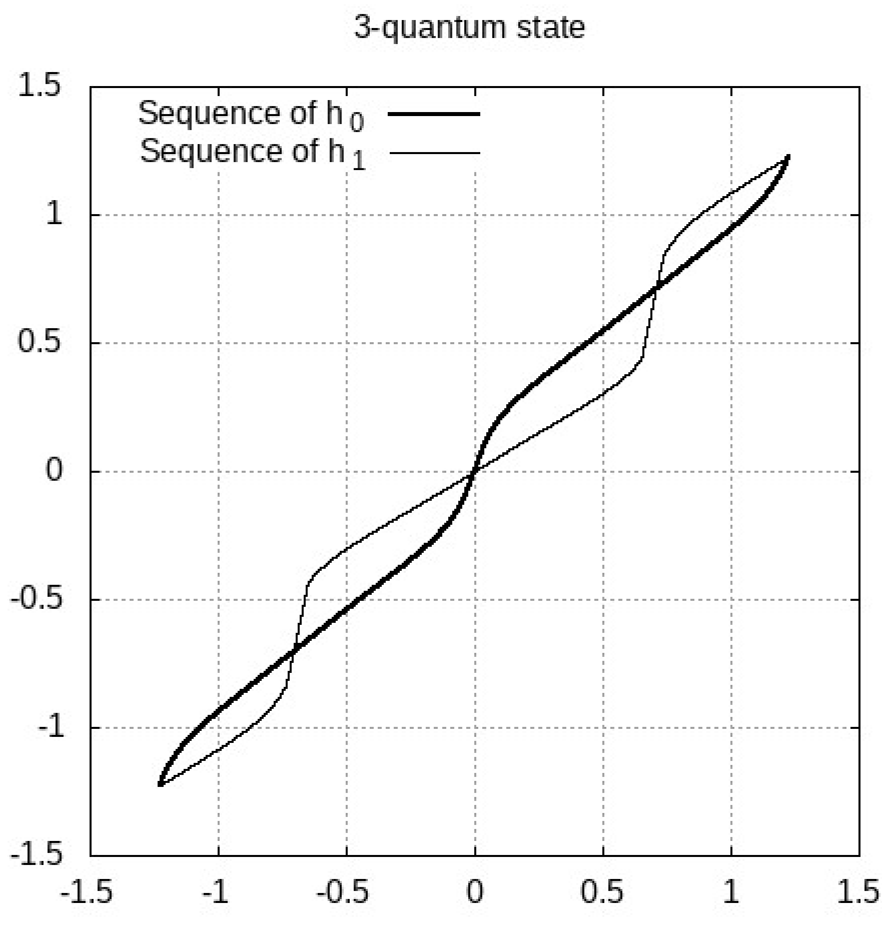

Figure 4: The approximate plot of self-homeomorphisms $y = h_0(x)$ and $y = h_1(x)$ for $p = 0,785$ on interval $[-\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}]$

4 Equilibrium position of n-quantum state

We consider a random walk description of the oscillator in the equilibrium position of n-quantum state. Therefore , let us consider additionally the passage from $n+1$ to n-quantum state for $n \geq 1$. In this case we consider the function $G^{(n)} = \Phi_n \circ \Phi_{n+1}^{-1} = (F^{(n+1)})^{-1}$. The fixed points for $G^{(n)}$ are the same as for $F^{(n+1)}$ and we can also use Theorem 2.5 for $G^{(n)}$. For example we take $p = 0,675$ for $n = 1$. Therefore we obtain the random walk of the same complexity as in the case $F^{(n+1)}$. But now the $n+1$-particle state changes to n-particle one by annihilation of one particle (see [1], chapter 1.5). Here the oscillator starts from the above walk but with $p = \frac{1}{2}$ (i.e. under tossing of symmetric coin) which is its an equilibrium $n+1$-state. Similarly we obtain another equilibrium $n+1$-state as the starting walk for the passage from $n+1$ to $n+2$-quantum state. The above walks have the same stationary measure μ_{n+1} and have Bernoulli property on their invariant components. By Proposition 2.10 we see that the above random walks as stochastic processes have the same basic properties as diffusion processes in stochastic quantum physics.

Remark 4.1. The description of an equilibrium position in 1-quantum state i.e for $n = 0$ is more complicated. We get one equilibrium state by passage from 1 to 2-quantum state (see Figure 2 with $p = \frac{1}{2}$). But in the case of passage from 1 to 0-state the function $G^{(0)}$ fulfills neither the assumptions of Theorem 2.5 nor of Hypothesis 6.11 [3]. The description of equilibrium position in 1-quantum state by diffusion process can be found in [5] p. 109. This process has two invariant ergodic components which are the same as ones given by Theorem 4.1 [3] for $n = 0$.

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