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STABILITY OF SMOOTH EXTENSIONS OF BERNOULLI SHIFTS

Abstract. Let S_i, $i = 0, 1$, be homeomorphisms of $I = [0, 1]$ such that $S_i^{-1}(x) = (1 - \epsilon_i)x + \epsilon_i g(x)$, $i = 0, 1$, for some reals $\epsilon_0 < 0$ and $\epsilon_1 > 0$. Here g is a $C^1(0, 1)$ homeomorphism and $g(x) < x$ for $x \in (0, 1)$. Let $(\Omega, \mathcal{B}, \mu_p, \sigma)$ be the one-sided Bernoulli shift where $\Omega = \{0, 1\}^{\mathbb{N}}$ and μ_p is the (p, q) measure for some $p \in I$. In the space $\Omega \times I$ we define the skew product $S(\omega, x) = (\sigma(\omega), S_{\omega(0)}(x))$. For some class of distribution functions $F \in C^2(0, 1)$ of probability measures and all $\epsilon_0 < 0$, $\epsilon_1 > 0$, and $p \in (\epsilon_1/(\epsilon_1 - \epsilon_0), 1)$, we give sufficient conditions for existence of exactly one pair of homeomorphisms as above such that $\mu_p \times \mu_F$ is S-invariant. Here μ_F is the measure determined by F. For example, as a consequence of the above, we show that if $S_0^{-1}(x) = 1.307x - 0.307x^2$ and $S_1^{-1}(x) = 0.26x + 0.74x^2$, then for every $p \in [0.706781, \sqrt{2}/2)$, S possesses ergodic invariant measure $\mu_p \times \mu_{G_p}$ which is a kind of Sinai–Ruelle–Bowen measure. We apply the above results to the quantum harmonic oscillator and a binomial model for asset prices.

1. Introduction. Let us consider two increasing homeomorphisms S_0, S_1 of the interval $I = [0, 1]$ into itself such that $S_0(x) < x$ and $S_1(x) > x$ for $x \in (0, 1)$ and let (p, q) be a probability vector. They determine a random walk on I as follows: x goes to $S_0(x)$ with probability p and to $S_1(x)$ with probability q.

Random walks on I may be realized as transformations of a larger space. Let Ω be the space $\{0, 1\}^{\mathbb{N}}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, with the (p, q)-Bernoulli measure μ_p on $(\Omega, \mathcal{B})$, where $\mathcal{B}$ is the Borel product σ-algebra. We denote by $\mathcal{A}$

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the Borel σ-algebra of subsets of I, and by Λ the Lebesgue measure on I. Let σ be the one-sided shift on Ω. In the space $\Omega \times I$ we define the skew product

$$S(\omega, x) = (\sigma(\omega), S_{\omega(0)}(x)) \tag{1}$$

and call it the *random walk* (S, p). Let us assume for the moment that S_0 and S_1 commute and $\mu_p \times \nu$ is an S-invariant measure for every $p \in (0,1)$, where ν is σ-finite and equivalent to Λ. Then by [1, Corollary 8.15] there exists exactly one $p_0 \in (0,1)$ such that $(S, \mu_{p_0} \times \nu)$ is conservative and $(S, \mu_p \times \nu)$ is totally dissipative for $p \neq p_0$ (for example see [8, Theorem 4]).

Definition 1.1. We say that a property P of random walk (S, p_0) is *physically essential* if there exists an interval $J \subset I$ with $\Lambda(J) > 0$ and $p_0 \in J$ such that the random walk (S, p) has the property P for every $p \in J$.

We see that conservativity is not physically essential for commuting S_0, S_1. Now, let S_i, $i = 0, 1$ be as in the abstract. At the beginning of Section 3, without loss of generality we assume that $\epsilon_0 = -1$ and $\epsilon_1 = 1$. Then $\mu_{0.5} \times \Lambda$ is S-invariant. We show that the property of having an invariant Sinai–Ruelle–Bowen measure $\mu_p \times \nu$ for $p \neq 1/2$ can be physically essential (see Theorem 3.1). As an example, we construct a family of skew products such that if S is from this family then $\mu_{0.5} \times \Lambda$ is S-invariant and simultaneously $\mu_{p^*} \times \nu_{p^*}$ is S-invariant for some $p^* > 1/2$ and ν_{p^*}. Here ν_{p^*} is a probability measure equivalent to Λ. The construction is presented in Sections 3 and 6. Moreover, it appears (see Theorem 3.1) that for every $p \in [1/2, p^*]$ there exists a continuous measure ν_p on I such the $\mu_p \times \nu_p$ is S-invariant and ergodic. Additionally

$$\frac{1}{n} \sum_{k=0}^{n-1} 1_{B \times J}(S^k(\omega, x)) \to \mu_p(B)\nu_p(J) \quad \text{as } n \to \infty$$

for μ_p-almost every $\omega \in \Omega$, every $x \in (0,1)$, any cylinder set $B \subset \Omega$ and any interval $J \subset I$. So the last property is physically essential for the above walk. Moreover the walk S is uniquely determined by ν_{p^*}.

Let S be a random walk as in the abstract and let $M_p(S)$ denote the set of S-invariant probability measures m on $\Omega \times I$ such that $m|\mathcal{B} \times \{\emptyset, I\} = \mu_p$. In Section 2 we present more general conditions describing $M_p(S)$ than those in [7, Theorem 1], and as a consequence we get asymptotic properties of random walks. For S as in Section 3 we show that $(S, \mu_{0,5} \times \Lambda)$, $(S, \mu_{p^*} \times \nu_{p^*})$ have natural extensions to K-automorphisms. In Section 4 we interpret our construction in terms of a quantum simple harmonic oscillator. The solution of the Schrödinger equation contains a distribution function which characterizes the motion of a particle in n-quantum state. If we assume that the random walk of a particle on $\mathbb{R}$ comes from a physically essential random walk on I then it is partially determined by two successive distributions for $n-1$- and n-quantum states, $n \geq 1$. We apply the above to describe the motion of the particle in 0-, 1- and 2-quantum state. Section 5 is devoted to a binomial model for asset prices. We conclude that circumstances when the asset prices change in chaotic way may be persistent. In Section 7 we present a numerical construction of S and apply it to obtain an explicit, given by parabolic maps, skew product for which the property of having an invariant Sinai–Ruelle–Bowen measure is physically essential.

2. Ergodic properties. Let us denote by $\mathcal{D}$ the set of distribution functions of probability measures on I. Let ν_G denote the measure determined by $G \in \mathcal{D}$. We also add new assumptions about S_i, $i = 0, 1$. Namely

$$S_i^{-1}(x) = (1-\epsilon_i)x + \epsilon_i g(x), \qquad i = 0, 1,$$

for some reals $\epsilon_0 < 0$ and $\epsilon_1 > 0$. Here g is a $C^1(0,1)$ homeomorphism of I and $g(x) < x$ for $x \in (0,1)$.

[7, Theorem 1] has the following extension.

Theorem 2.1. *If $\mu_p \times \mu \in M_p(S)$ with $\mu(\{0\}) = \mu(\{1\}) = 0$ then $\mu = \mu_G$ where G is a homeomorphism of I. Moreover $\mu_p \times \mu_G$ is ergodic and*

$$M_p(S) = \operatorname{conv}\{\mu_p \times \delta_{\{0\}}, \mu_p \times \delta_{\{1\}}, \mu_p \times \mu_G\}.$$

Proof. By ergodic decomposition [5, Theorem 1.1, p. 193] of $\mu_p \times \mu$ there exists $G \in \mathcal{D}$ such that $\mu_p \times \nu_G$ is ergodic and

$$\nu_G \notin \operatorname{conv}\{\delta_{\{0\}}, \delta_{\{1\}}\}.$$

Therefore by [8, Lemma 3], G is continuous and increasing. An application of [7, Theorem 1] completes the proof. ∎

We also have

$$\frac{1}{n}\sum_{k=0}^{n-1} 1_{B\times J}(S^k(\omega, x)) \to \mu_p(B)\mu(J) \qquad \text{as } n \to \infty$$

for μ_p-almost every $\omega \in \Omega$, every $x \in (0,1)$, any cylinder set $B \subset \Omega$ and any interval $J \subset I$, by repeating the reasoning in [10, proof of Theorem 2]. So $\mu_p \times \mu$ is a kind of Sinai–Ruelle–Bowen measure. In particular

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu_p\{\omega : S_{\omega(k)} \circ \cdots \circ S_{\omega(0)}(x) \in J\} \to \mu(J) \qquad \text{as } n \to \infty$$

for every $x \in (0,1)$ and any interval $J \subset I$.

Corollary 2.2. *If $\mu_p \times \Lambda \in M_p(S)$, i.e.*

$$p = \frac{\epsilon_1}{\epsilon_1 - \epsilon_0},$$

then $(S, \mu_p \times \Lambda)$ is ergodic.

To obtain the ergodicity of $(S, \mu_p \times \Lambda)$ in previous papers, we have assumed that $S_0 \in C^2(I)$ and there exists exactly one $x_0 \in I$ such that $S_0'(x_0) = 1$.

In the current mathematical literature, $\mu_p \times \Lambda$ is the only known product measure $\mu_p \times \mu \in M_p(S)$ such that μ is absolutely continuous with respect to Λ. Theorem 6.6 below changes this situation and that is why we have to present the results such as the following.

Let us denote by $(\bar{S}, \overline{\mu_p \times \mu})$ the natural extension of $(S, \mu_p \times \mu)$ to an automorphism.

Definition 2.3. $(S, \mu_p \times \mu)$ is said to have the *K-property* if $(\bar{S}, \overline{\mu_p \times \mu})$ is a K-automorphism, i.e. there exists a sub-σ-algebra $\bar{\mathcal{D}}$ of $\overline{\mathcal{B} \times \mathcal{A}}$ such that

$$\bigvee_{n=-\infty}^{\infty} \bar{S}^n(\bar{\mathcal{D}}) = \overline{\mathcal{B} \times \mathcal{A}} \quad \text{and} \quad \bigwedge_{n=-\infty}^{\infty} \bar{S}^n(\bar{\mathcal{D}}) = \bar{\mathcal{R}},$$

where $\bar{\mathcal{R}}$ is trivial in the sense that it contains only sets of measure 0 or 1.

Theorem 2.4. *If $\mu_p \times \mu \in M_p(S)$ and $\mu \equiv \Lambda$ then $(S, \mu_p \times \mu)$ has the K-property.*

Proof. If

$$p = p_0 = \frac{\epsilon_1}{\epsilon_1 - \epsilon_0}$$

then by ergodicity of $(S, \mu_{p_0} \times \Lambda)$ and by using the reasoning from the proof of [6, Theorem 2] we get the K-property of $(S, \mu_{p_0} \times \Lambda)$. Now, let $\mu_p \times \mu \in M_p(S)$ and $\mu \equiv \Lambda$. To get prove the K-property of $\mu_p \times \mu$ it is enough to prove the total ergodicity of $(S, \mu_p \times \mu)$ by [6, Theorem 1]. Let $f \circ S = af$ $\mu_p \times \mu$-a.e. for $f \in L_1(\mu_p \times \mu)$ and $|a| = 1$. Then $f(\omega, x) = f(x)$ $\mu_p \times \mu$ a.e. by [12, Theorem 3.2]. Therefore $f \circ S_i = af$ for $i = 0, 1$ μ-a.e. or Λ-a.e. Hence $f \circ S = af$ $\mu_{p_0} \times \Lambda$ a.e. The K-property of $\mu_{p_0} \times \Lambda$ implies $f = \text{const}$ Λ-a.e., which yields $f = \text{const}$ $\mu_p \times \mu$-a.e. ∎

Let us consider the random walk $X_0 = x_0$, $X_1 = S_{\omega(0)}(x_0)$, $X_2 = S_{\omega(1)} \circ S_{\omega(0)}(x_0), \ldots$, inductively

$$X_{n+1} = S_{\omega(n)}(X_n).$$

We apply the dual skew product method for $(S, \mu_p \times \mu)$, where $\mu \equiv \Lambda$, much as for $(S, \mu_{p_0} \times \Lambda)$ in [9]. As a consequence we get

$$\int |\mu_p \{X_n(\omega, x) \in A\} - \mu(A)| \, d\mu(x) \to 0$$

as $n \to \infty$ for $A \in \mathcal{A}$.

3. Construction. Assume for simplicity that $\epsilon_0 = -1$ and $\epsilon_1 = 1$. Let g be a homeomorphism of I such that $g \in C^1(0, 1)$, $g(x) < x$ for $x \in (0, 1)$, and $2x - g(x)$ is a homeomorphism too. Then g determines S where

$$S_0^{-1}(x) = 2x - g(x) \quad \text{and} \quad S_1^{-1}(x) = g(x).$$

We also introduce the operator $\mathcal{A}_p : \mathcal{D} \to \mathcal{D}$ for $p \in (0,1)$ such that

$$\mathcal{A}_p F(x) = pF(S_0^{-1}(x)) + (1-p)F(S_1^{-1}(x)).$$

It is easy to see that the measure $\mu_p \times \mu_F$ is S-invariant if and only if $\mathcal{A}_p F = F$. Obviously $\mu_{0.5} \times \Lambda$ is S-invariant. Assume that F is a homeomorphism of I, $x < F(x)$ for $x \in (0,1)$ and $\mathcal{A}_{p^*} F = F$ for some $p^* \in (1/2, 1)$.

THEOREM 3.1. *For every $p \in [1/2, p^*]$ there exists an S-invariant measure $\mu_p \times \mu_G$ such that $(S, \mu_p \times \mu_G)$ is ergodic and G is a homeomorphism of I.*

Proof. Let

$$F_p(x,y) = pF(2x-y) + (1-p)F(y) - F(x)$$

for $(x,y) \in I \times I$ such that $2x - 1 \le y \le x$. Then

$$F_p(x,y) < F_{p^*}(x,y) \quad \text{for } p < p^* \text{ and } (x,y) \in I \times I,\ 2x-1 < y < x.$$

Therefore

$$\mathcal{A}_p F(x) - F(x) = F_p(x, g(x)) < F_{p^*}(x, g(x)) = 0 \text{ for } p < p^* \text{ and } x \in (0,1).$$

Simultaneously $\mathcal{I} \le F$, where $\mathcal{I}$ denotes the identity function. Hence

$$\mathcal{I} \le \mathcal{A}_p \mathcal{I} \le \mathcal{A}_p F \le F \quad \text{for } p \in (1/2, p^*).$$

Therefore

$$\mathcal{I} \le \mathcal{A}_p^n \mathcal{I} \le F,$$

and the sequence $\mathcal{A}_p^n \mathcal{I}(x)$ is non-decreasing for $x \in I,\ n = 1, 2, \ldots$. Let

$$G(x) = \lim_{n\to\infty} \mathcal{A}_p^n \mathcal{I}(x)$$

and $\bar{G}(x) = G(x^-)$ for every $x \in (0,1)$. Then $\mu_p \times \mu_{\bar{G}}$ is S-invariant since $\mathcal{A}_p \bar{G} = \bar{G}$. Moreover $\mu_{\bar{G}}(0) = \mu_{\bar{G}}(1) = 0$ since $\mathcal{I} \le \bar{G} \le F$. Now we are in a position to use Theorem 2.1. ■

Taking into consideration the results of Section 2 we see that the Sinai–Ruelle–Bowen measure $\mu_{p^*} \times \mu_F$ for (S, p^*) is physically essential.

As in Section 2 we consider

$$T_i^{-1}(x) = (1-\epsilon_i)x + \epsilon_i h(x), \qquad i = 0, 1,$$

for some reals $\epsilon_0 < 0$ and $\epsilon_1 > 0$. It is known that $\mathcal{I} \le \mathcal{A}_p \mathcal{I}$ for $p \ge \epsilon_1/(\epsilon_1 - \epsilon_0)$. Here $\mathcal{A}_p$ is determined by T. Therefore we can apply the reasoning from the proof of Theorem 3.1 to get:

COROLLARY 3.2. *If $T_0^{-1}(x) \le 2x - g(x)$ and $T_1^{-1}(x) \le g(x)$ for $x \in [0,1]$ then for every*

$$p \in [\epsilon_1/(\epsilon_1 - \epsilon_0), p^*]$$

there exists a T-invariant measure $\mu_p \times \mu_G$ such that $(T, \mu_p \times \mu_G)$ is ergodic and G is a homeomorphism of I.

Set
$$\|f\| = \sup\{|f(x)| : x \in I\}$$
and let G_p be the distribution function G given by Theorem 3.1 for $p \in [\frac{1}{2}, p^*]$.

PROPOSITION 3.3. *Let $p_0 \in [1/2, p^*]$. Then*
$$\lim_{p \to p_0} \|G_p - G_{p_0}\| = 0.$$

Proof. Let us consider $G_n = G_{p_n}$ such that $\lim_{n\to\infty} p_n = p_0$. Then by Helly's Theorem there exists a subsequence G_{n_k} and a non-decreasing function G such that
$$\lim_{k\to\infty} |G_{n_k}(x) - G(x)| = 0 \quad \text{for every } x \in I.$$
It is easy to see that $\mathcal{A}_{p_0} G = G$. Moreover $\mathcal{I} \le G \le F$ by the proof of Theorem 3.1. Let $\bar{G}(x) = G(x^-)$ for every $x \in (0,1)$. Then $\bar{G} \in \mathcal{D}$ and $\bar{G} = G_{p_0}$ by Theorem 2.1. Continuity of G_{p_0} implies uniform convergence of G_{n_k} to G_{p_0}. ∎

Let $F \in C^2(0,1)$ with $F(0) = 0$, $F(1) = 1$, $F'(0^+) = \infty$, $F'(1) = 0$ and $F''(x) < 0$ for $x \in (0,1)$. We will find g as above such that $\mathcal{A}_p F = F$ for some $p \in (0,1)$. In other words, we will find the implicit function given by $F_p(x,y) = 0$ where
$$F_p(x,y) = pF(2x - y) + (1-p)F(y) - F(x),$$
or we solve the equivalent differential equation. The detailed description and examples are contained in Section 6. Let S be the skew product determined by g which is given by Theorem 6.6 for some $p^* \in (1/2, 1)$. Then the conclusions of Theorem 3.1 and Proposition 3.3 hold for (S, p^*).

There is another way to obtain S which possesses two invariant probability measures, namely, such that $\mu_p \times \Lambda \in M_p(S)$ for some $p \ne 1/2$ and $\nu^* \in M_{0.5}(S)$, where ν^* is non-trivial, i.e. $\nu^* \notin \mathrm{conv}\{\mu_{0.5} \times \delta\nu_{\{0\}}, \mu_{0.5} \times \delta_{\{1\}}\}$, by using results of [3] and [2] (see also [4]). To get this we consider
$$S_i^{-1}(x) = (1 - \epsilon_i)x + \epsilon_i x^2, \quad i = 0, 1,$$
where $\epsilon_0 + \epsilon_1 \ne 0$, $(1 - \epsilon_0)(1 - \epsilon_1) < 1$ and $(1 + \epsilon_0)(1 + \epsilon_1) < 1$. Then S has a non-trivial measure $\nu^* \in M_{0.5}(S)$ by [3, Theorem 5.1]. Moreover ν^* is a product measure by [2, Theorem 4.2], i.e. $\nu^* = \mu_{0.5} \times \nu$, where ν is a probability measure on $(0,1)$. Therefore S has invariant measures $\mu_p \times \Lambda$ for $p = \epsilon_1/(\epsilon_1 - \epsilon_0)$ and $\mu_{0.5} \times \nu$ for $p = 1/2$. But we do not know when ν is an absolutely continuous measure.

4. Quantum harmonic oscillator. Results of the Appendix (Section 6) show that the shape of the distribution function uniquely determines a random walk which is physically essential. This is the justification for applying the above to the quantum harmonic oscillator. A one-dimensional quantum harmonic oscillator Ψ satisfies the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$$

where $H = \frac{1}{2}(P^2 + Q^2)$ is the Hamiltonian. Here $P = -i\frac{d}{dx}$ and Q is multiplication by x. Let us consider the ground state solution

$$\Psi_0(x,t) = \frac{1}{\sqrt[4]{\pi}}\exp\left(-\frac{i}{2\hbar}t\right)\exp\left(-\frac{x^2}{2}\right).$$

The quantum interpretation of $|\Psi_0(x,t)|^2 = \frac{1}{\sqrt{\pi}}\exp(-x^2)$ is the following: The probability that a particle is in the set $A \subset \mathbb{R}$ at time t is

$$\frac{1}{\sqrt{\pi}}\int_A \exp(-x^2)\,dx.$$

We construct a discrete time random walk $\tilde{X}_n(\omega,x)$ on $\mathbb{R}$ which asymptotically imitates the motion of a particle in the ground state.

Let S be the skew product determined by $g(x)$ given by Theorem 6.6 for some $p^* \in (1/2,1)$. By using the map $\Xi : \Omega\times\mathbb{R} \to \Omega\times I$ where $\Xi(\omega,x) = (\omega,\phi_0(x))$ and

$$\phi_0(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x}\exp(-t^2)\,dt,$$

we define

$$\tilde{S}(\omega,x) = (\sigma(\omega),\tilde{S}_{\omega(0)}(x)),$$

a skew product on $\Omega\times\mathbb{R}$. Here $\tilde{S}_i(x) = \phi_0^{-1}(S_i(\phi_0(x)))$ for $i = 0,1$. The map $\tilde{S}$ preserves the measures $\mu_{0.5}\times\tilde{\mu}_\Lambda$ and $\mu_{p^*}\times\tilde{\mu}_F$ where $\tilde{\mu}_\Lambda$ has the normal distribution with density $\frac{1}{\sqrt{\pi}}\exp(-x^2)$ and $\tilde{\mu}_F$ has distribution $F(\phi_0(x))$. Moreover $\tilde{S}$ preserves the ergodic measures $\mu_p\times\tilde{\mu}_{G_p}$ for $p \in (1/2,p^*)$ where $\tilde{\mu}_{G_p}$ has distribution $G_p(\phi_0(x))$. The distributions G_p are provided by Theorem 3.1. Denote by $\tilde{X}_n(\omega,x)$ the random walk determined by $\tilde{S}$, i.e. $\tilde{X}_0(\omega,x_0) = x_0, \tilde{X}_1(\omega,x_0) = \tilde{S}_{\omega(0)}(x_0),\ldots,$ inductively

$$\tilde{X}_{n+1}(\omega,x_0) = \tilde{S}_{\omega(n)}(\tilde{X}_n(\omega,x_0)).$$

Here $\tilde{X}_n(\omega,x) = \phi_0^{-1}(X_n(\omega,\phi_0(x)))$. The real p^* and the distribution F uniquely determine the random walk $\tilde{X}_n$ by Theorem 6.6. The convergences

$$\mu_{0.5}\{X_n(\omega,x)\in A\} \to \Lambda(A) \quad \text{in } L_1(\Lambda)$$

and

$$\mu_{p^*}\{X_n(\omega,x)\in A\} \to \mu_F(A) \quad \text{in } L_1(\mu_F)$$

for $A \in \mathcal{A}$, as has been observed in Section 2, imply

$$\mu_{0.5}\{\tilde{X}_n(\omega,x)\in B\} \to \tilde{\mu}_\Lambda(B) \quad \text{in } L_1(\tilde{\mu}_\Lambda)$$

and

$$\mu_{p^*}\{\tilde{X}_n(\omega,x)\in B\}\to\tilde{\mu}_F(B)\quad\text{ in } L_1(\tilde{\mu}_F)$$

for $B\in\mathcal{B}(\mathbb{R})$.

The random walk for the nth quantum state solution where $n>0$ is more complicated. Let us consider the case $n=1$. The 1-state solution

$$\Psi_1(x,t)=\frac{\sqrt{2}}{\sqrt[4]{\pi}}\exp\left(-\frac{3i}{2\hbar}t\right)x\exp\left(-\frac{x^2}{2}\right)$$

gives the distribution function

$$\phi_1(x)=\frac{2}{\sqrt{\pi}}\int\limits_{-\infty}^{x}t^2\exp(-t^2)\,dt.$$

We obtain a physically essential random walk S on I determined by ϕ_1 as follows. By the equality

$$\phi_1(x)=\phi_0(x)-\frac{1}{\sqrt{\pi}}x\exp(-x^2)\quad\text{ for } x\in\mathbb{R}$$

we get

$$F^{(1)}(x)=x-\frac{1}{\sqrt{\pi}}\phi_0^{-1}(x)\exp(-[\phi_0^{-1}(x)]^2)\quad\text{ for } x\in I.$$

Here $F^{(1)}(\phi_0(x))=\phi_1(x)$ for $x\in\mathbb{R}$. The function $F^{(1)}$ is an increasing homeomorphism of I, $F^{(1)}(1/2)=1/2$, $F^{(1)}$ is concave on $[0,1/2]$ and convex on $[1/2,1]$. Moreover $(F^{(1)'})(0)=(F^{(1)'})(1)=\infty$ and $(F^{(1)'})(1/2)=0$ as $(F^{(1)'})(x)=2[\phi_0^{-1}(x)]^2$. In fact $F^{(1)}(x)=1-F^{(1)}(1-x)$ for $x\in[0,1]$. Moreover $F^{(1)}$ satisfies the assumptions of Hypothesis 6.11 for the interval $[0,1/2]$ instead of $[0,1]$. By hypothesis we can modify $F^{(1)}$ to $\check{F}^{(1)}$ for every $\epsilon>0$ such that $\|F^{(1)}-\check{F}^{(1)}\|<\epsilon$. The equality $\check{F}^{(1)}_{p^*}(x,y)=0$, where

$$\check{F}^{(1)}_{p^*}(x,y)=p^*\check{F}^{(1)}(2x-y)+(1-p^*)\check{F}^{(1)}(y)-\check{F}^{(1)}(x)$$

and $p^*\in(1/2,1/\sqrt[3]{4})$, determines an increasing homeomorphism $h(x)$ of I such that

$$h(x)=\begin{cases} g(x) & \text{for } x\in[0,1/2],\\ 1-g(1-x) & \text{for } x\in(1/2,1].\end{cases}$$

Here $g(x)$ is given by Theorem 6.8 for $\check{F}^{(1)}$ restricted to $[0,1/2]$. The random walk S determined by $h(x)$ has the following properties: $\mu_{0.5}\times\Lambda\in M_{0.5}(S)$ and $\mu_{p^*}\times\mu_{\check{F}^{(1)}}\in M_{p^*}(S)$. It is easy to see that S is not ergodic. There are two ergodic components, $\Omega\times[0,1/2]$ and $\Omega\times[1/2,1]$. Set $S^{(1)}=S|\Omega\times[0,1/2]$ and $S^{(2)}=S|\Omega\times[1/2,1]$. The walks $S^{(1)}$ and $S^{(2)}$ move in the opposite directions, i.e. $S^{(1)}_{\omega(0)}(x)-x$ has opposite sign to $S^{(2)}_{\omega(0)}(y)-y$ for $x\in(0,1/2)$ and $y\in(1/2,1)$. Moreover, $S^{(i)}$, $i=1,2$, have the properties given by

Theorems 2.4 and 3.1. Hence the property that

$$\frac{1}{n}\sum_{k=0}^{n-1}\mu_{p^*}\{X_k(\omega,x)\in J\}\to 2\mu_{\check F^{(1)}}(J\cap[0,1/2])1_{[0,1/2]}(x) \\ + 2\mu_{\check F^{(1)}}(J\cap[1/2,1])1_{[1/2,1]}(x)$$

as $n\to\infty$ for every $x\in(0,1)$ and any interval $J\subset[0,1]$ is physically essential. By using the map $\Xi_1:\Omega\times\mathbb{R}\to\Omega\times I$ where $\Xi_1(\omega,x)=(\omega,\phi_0(x))$ we define

$$\tilde S(\omega,x)=(\sigma(\omega),\tilde S_{\omega(0)}(x)),$$

a skew product on $\Omega\times\mathbb{R}$. Then $\tilde S$ preserves the measures $\mu_{0.5}\times\tilde\mu_\Lambda$ and $\mu_{p^*}\times\tilde\mu_{\check F^{(1)}}$ where $\tilde\mu_\Lambda$ has distribution $\phi_0(x)$ and $\tilde\mu_{\check F^{(1)}}$ has distribution $\check F^{(1)}(\phi_0(x))$. Since $\check F^{(1)}(\phi_0(x))\approx\phi_1(x)$, we have

$$\phi_1(x)\approx p^*\phi_1(\tilde S_0(x))+(1-p^*)\phi_1(\tilde S_1(x))\quad\text{ for } x\in\mathbb{R}.$$

The process $\tilde X_n$ determined by $\tilde S$ has the property

$$\mu_{p^*}\{\tilde X_n(\omega,x)\in B\}\to 2\tilde\mu_{\check F^{(1)}}(B\cap(-\infty,0])1_{(-\infty,0]}(x) \\ + 2\tilde\mu_{\check F^{(1)}}(B\cap[0,\infty))1_{[0,\infty)}(x)$$

as $n\to\infty$ in $L_1(\tilde\mu_{\check F^{(1)}})$ for $B\in\mathcal{B}(\mathbb{R})$.

The above suggests the physical interpretation of the 1-quantum state as existence of two particles which move in the opposite directions towards each other or one particle which consists of two components as above.

For $n=2$ the distribution function

$$\phi_2(x)=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{x}(2t^2-1)^2\exp(-t^2)\,dt$$

satisfies the equation

$$\phi_2(x)=\phi_1(x)-\frac{1}{\sqrt{\pi}}\left(x^3-\frac{1}{2}x\right)\exp(-x^2)\quad\text{ for } x\in\mathbb{R}.$$

Hence

$$F^{(2)}(x)=x-\frac{1}{\sqrt{\pi}}\phi_1^{-1}(x)([\phi_1^{-1}(x)]^2-1/2)\exp(-[\phi_1^{-1}(x)]^2)\quad\text{ for } x\in I$$

and $F^{(2)}(\phi_1(x))=\phi_2(x)$. Here $F^{(2)}(x_1)=x_1$, $F^{(2)}(1/2)=1/2$ and $F^{(2)}(1-x_1)=1-x_1$ where $\phi_1^{-1}(x_1)=-\sqrt{2}/2$. Equivalently $\phi_1(-\sqrt{2}/2)=\phi_2(-\sqrt{2}/2)$ $=x_1$, $\phi_1(0)=\phi_2(0)=1/2$ and $\phi_1(\sqrt{2}/2)=\phi_2(\sqrt{2}/2)=1-x_1$.

We repeat the reasoning similar to the case $n=1$ and finish with the interpretation via the existence of four particles or two particles which consist of two opposite components. Here we use $\Xi_2(\omega,x)=(\omega,\phi_1(x))$.

Let us consider the general case. Here

$$\phi_n(x) = \frac{1}{n!2^n\sqrt{\pi}} \int_{-\infty}^{x} H_n^2(t)e^{-t^2}\,dt$$

where H_n is the nth Hermite polynomial, i.e.

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},$$

and $F^{(n+1)}(x) = \phi_{n+1}(\phi_n^{-1}(x))$ for $n = 0, 1, \ldots$. Let

$$C_n = \{x : \phi_n(x) = \phi_{n+1}(x)\}.$$

THEOREM 4.1. *The cardinality of* C_n *is* $2n+1$ *and* $\varphi_n(C_n)$ *is the set of inflexion points of* $F^{(n+1)}$ *for* $n = 0, 1, \ldots$.

Proof. We start with the identity (see [13])

$$H_m H_n = \sum_{l=0}^{\min(m,n)} \binom{m}{l}\binom{n}{l} 2^l l! H_{m+n-2l}.$$

Hence

$$H_n^2 = \sum_{l=0}^{n} \binom{n}{l}^2 2^l l! H_{2(n-l)}.$$

Therefore

$$\begin{aligned}
&\phi_n(x) = \phi_{n+1}(x)\\
&\quad\Leftrightarrow \sum_{l=0}^{n} \binom{n+1}{l}^2 2^l l! H_{2(n-l)+1}(x) = 2(n+1)\sum_{l=0}^{n} \binom{n}{l}^2 2^l l! H_{2(n-l)-1}\\
&\quad\Leftrightarrow H_{n+1}(x)H_n(x) = 0.
\end{aligned}$$

Hence the cardinality of C_n is $2n+1$. The second part of the conclusion follows by calculating $\frac{d^2}{dx^2}F^{(n+1)}$ and using Turán's inequality

$$H_n^2 - H_{n-1}H_{n+1} > 0. \blacksquare$$

To end this section we present some ideas which lead to this example. The set C_n determines a partition of $\mathbb{R}$ into $2(n+1)$ intervals with measures given by ϕ_{n+1} the same as those given by ϕ_n. We postulate that every interval is occupied by a particle or every pair of neighboring intervals is occupied by a compound particle. The dynamics of these particles is determined by intervals of convexity and concavity of $F^{(n+1)}$ and by the shape of ϕ_{n+1}. Moreover the direction of motion of a particle depends on probability. If p^* is less than $1/2$ then the direction is reversed (see Remark 6.7).

5. Binomial model for asset prices. The existence of a random walk described in Section 3 indicates a new property of a one-asset binomial model considered in [9]. For the convenience of the reader we recall the description of the model.

At each time there are two possibilities: the security price may go up by a factor of $u(x)$ or it may go down by a factor of $d(x)$. The factors u and d are functions of the prices, $u : (0,1) \to (1,\infty)$ and $d : (0,1) \to (0,1)$. Given the functions $u(x), d(x)$ and the probabilities $p = p_d, q = p_u = 1 - p_d$, we can construct a random map S which consists of the transformations S_d, S_u given by

$$S_d(x) = d(x)x \quad \text{and} \quad S_u(x) = u(x)x.$$

Here $S_d, S_u : [0,1] \to [0,1]$ are continuous maps with

$$\forall_{x\in[0,1]}\ S_d(x) \le x \text{ and } S_u(x) \ge x.$$

We will assume that S_d and S_u are homeomorphisms, so S is given by (1). The subscript u of S_u indicates that S_u is the law which moves the price up, and similarly S_d moves the price down. The process determined by the random walk S and starting from $x_0 \in (0,1)$ can be written as $X_n(\omega, x_0)$. If S_u and S_d commute, then there exists exactly one $p \in (0,1)$ such that X_n has chaotic behavior (see [9, Example 1]). By uniqueness of p this behavior is not physically essential, and so is not observed in practice. If the random walk S is determined by g given by Theorems 6.6 and 6.8 then X_n has chaotic behavior for $p \in [1/2, 1/\sqrt[3]{4}]$. See also Corollary 3.2.

COROLLARY 5.1. *The state when asset prices behave chaotically can be persistent.*

6. Appendix. Let $F \in C^2(0,1) \cap C[0,1]$ with $F(0) = 0$, $F(1) = 1$, $F'(0^+) = \infty$, $F'(1^-) = 0$ and $F''(x) < 0$ for $x \in (0,1)$. Here F' denotes $\frac{d}{dx}F$. We will consider the implicit equation $F_p(x,y) = 0$ where

$$F_p(x,y) = pF(2x-y) + (1-p)F(y) - F(x).$$

We are looking for solutions $y = g(x)$ in the set

$$D = \{(x,y) : 0 < y < x \text{ for } x \in (0,1/2] \text{ and } 2x-1 < y < x \text{ if } x \in (1/2,1)\}$$

and for $p \in (1/2,1)$ by concavity of F. Simultaneously $h(x) = 2x - g(x)$ is the implicit function for $1-p$ and its graph lies in

$$\{(x,y) : x < y < 2x \text{ for } x \in (0,1/2] \text{ and } x < y < 1 \text{ if } x \in (1/2,1)\}.$$

From now on we assume that p is fixed and denote $F_p(x,y)$ by $F(x,y)$. We will write

$$F_x = \frac{\partial F}{\partial x} \quad \text{and} \quad F_y = \frac{\partial F}{\partial y}.$$

Then

$$F_x = 2pF'(2x-y) - F'(x), \qquad F_y = -pF'(2x-y) + (1-p)F'(y).$$

Hence

$$F_y = 0 \;\Leftrightarrow\; x = \psi_p(y) = \frac{1}{2}y + \frac{1}{2}F'^{-1}\left(\frac{1-p}{p}F'(y)\right).$$

Here F'^{-1} is the inverse function to F'. We will denote $\psi_p(y)$ by $\psi(y)$ throughout this section. By the definition of ψ we see that ψ is increasing, $\psi(0) = 0$, $\psi(1) = 1$, and $(\psi(y), y) \in D$ for $y \in (0, 1)$. Next,

$$F_x = 0 \;\Leftrightarrow\; y = \varphi_p(x) = 2x - F'^{-1}\left(\frac{1}{2p}F'(x)\right).$$

In this section we will also denote $\varphi_p(x)$ by $\varphi(x)$. Here $\varphi(0) = 0$, $\varphi(1) = 1$, and $(x, \varphi(x)) \in D$ for $x \in (1/2, 1)$. It remains to check that $\varphi(x) > 0$ for $x \in (0, 1/2)$, which is true when φ is an increasing function. If we change p to $1 - p$ then we need to check $\varphi_{1-p}(x) < 1$ for $x \in (1/2, 1)$. Denote

$$\chi_1(x) = \min\{\varphi(x), \psi^{-1}(x)\}, \qquad \chi_0(x) = \max\{0, 2x - 1\} \qquad \text{for } x \in (0, 1).$$

Lemma 6.1. *Assume that φ and φ_{1-p} are increasing functions. Then $F(x, \psi^{-1}(x)) > 0$ and $F(x, \chi_0(x)) < 0$ for $x \in (0, 1)$.*

Proof. Observe that $F(x, \psi^{-1}(x)) > 0$ for $x \in (0, 1)$ since $F(x, x) = 0$ and $F_y(x, y) < 0$ for $\psi^{-1}(x) < y < x$. Simultaneously $F(x, \chi_0(x)) < 0$ became $2pF'(2x) < F'(x)$ for $x \in (0, 1/2)$ and by $2(1-p)F'(2x-1) > F'(x)$ for $x \in (1/2, 1)$, which follows from the assumptions about φ and φ_{1-p}. ∎

Remark 6.2. The conclusion of Lemma 6.1 is true when we replace p by $1 - p$. Here

$$\chi_1(x) = \max\{\varphi_{1-p}(x), \psi^{-1}_{1-p}(x)\}, \qquad \chi_0(x) = \min\{1, 2x\} \qquad \text{for } x \in (0, 1).$$

Lemma 6.3.

$$F_y(x, y) < 0 \text{ for } \psi^{-1}(x) < y < x, \qquad F_y(x, y) > 0 \quad \text{for } 0 < y < \psi^{-1}(x).$$

Similarly

$$F_x(x, y) > 0 \text{ for } \varphi(x) < y < x, \qquad F_x(x, y) < 0 \quad \text{for } 0 < y < \varphi(x).$$

Proof. This is easy to see from $F_x(x, x) > 0$, $F_y(x, x) < 0$, and

$$F_{yy}(x, y) < 0 \text{ and } F_{xy}(x, y) > 0 \quad \text{for } 0 < y < x. \;\blacksquare$$

Lemma 6.4. *For every $x \in (0, 1)$ there exists exactly one y such that $F(x, y) = 0$. Here $\chi_0(x) < y < \psi^{-1}(x)$. Moreover there exists (x_0, y_0) such that $F(x_0, y_0) = 0$ and $\chi_0(x_0) < y_0 < \chi_1(x_0)$.*

Proof. Fix $x \in (0, 1)$. Since $F(x, x) = 0$ and $F_y(x, y) < 0$ for $\psi^{-1}(x) < y < x$ with $x \in (0, 1)$, we see that $F(x, y) > 0$ for $\psi^{-1}(x) \le y < x$. Moreover $F(x, \chi_0(x)) < 0$. So there exists $y \in (\chi_0(x), \psi^{-1}(x))$ such that $F(x, y) = 0$. The uniqueness of y follows from Lemma 6.3.

To prove the "moreover" part, it is enough to show that there exists $x_0 \in (0, 1)$ such that $\varphi(x_0) = \psi^{-1}(x_0)$. Observe that $F(0, \varphi(0)) = F(1, \varphi(1)) = 0$.

Therefore there exists $x_0 \in (0,1)$ such that $\frac{d}{dx}F(x,\varphi(x))\big|_{x=x_0} = 0$. Let us compute

$$\frac{d}{dx}F(x,\varphi(x)) = F_x(x,\varphi(x)) + F_y(x,\varphi(x))\frac{d\varphi}{dx}(x) = F_y(x,\varphi(x))\frac{d\varphi}{dx}(x).$$

Hence $F_y(x,\varphi(x)) = 0$ at some point x_0 since $\frac{d\varphi}{dx}(x) > 0$ for $x \in (0,1)$. ∎

REMARK 6.5. The conclusion of Lemma 6.4 is true when we replace p by $1-p$.

THEOREM 6.6. *Let $F \in C^2(0,1) \cap C[0,1]$ with $F(0) = 0$, $F(1) = 1$, $F'(0^+) = \infty$, $F'(1^-) = 0$ and $F''(x) < 0$ for $x \in (0,1)$. Moreover, let $p^* \in (1/2,1)$ be such that φ and φ_{1-p^*} are increasing functions. Then there exists exactly one implicit function $y = g(x)$ solving $F(x,y) = 0$ such that $g(x) < x$ for $x \in (0,1)$. The function g is a homeomorphism of I and $g \in C^1(0,1)$. Moreover $2x - g(x)$ is a homeomorphism too.*

Proof. Let

$$f(x,y) = -\frac{F_x(x,y)}{F_y(x,y)} \quad \text{for } x \neq \psi(y).$$

To determine the implicit function given by $F(x,y) = 0$ let us take (x_0,y_0) given by Lemma 6.4 and solve the differential equation

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0. \tag{2}$$

Let (a,b) denote the maximal interval in which the function $y(x)$ is defined. We show that $(a,b) = (0,1)$. Firstly we prove that $a = 0$. We consider three cases.

(I) *There exist $\delta > 0$ and x such that $y(s) \in (\varphi(s), \psi^{-1}(s))$ for $s \in (x, x+\delta) \subset (a,b)$ and $y(x) = \varphi(x)$.* Then $y(s)$ is decreasing on $(x,x+\delta)$ by Lemma 6.3 (here $f(s,y(s)) < 0$). This contradicts $\varphi(s)$ being strictly increasing.

(II) *$y(s) \in (\varphi(s), \psi^{-1}(s))$ for $s \in (x-\delta, x) \subset (a,b)$ and $y(x) = \varphi(x)$.* Then by (I), $y(s) \in (\varphi(s), \psi^{-1}(s))$ for $s \in (a,x)$. Therefore $y(s)$ is decreasing on (a,x).

(III) *$\chi_0(x) < y(x) \le \chi_1(x)$ for every $x \in (a,b)$.* Then $y(x)$ is strictly increasing on (a,b) as $f(x,y) > 0$ for $\chi_0(x) < y < \chi_1(x)$ (by Lemma 6.3) and since $\varphi(x)$ is strictly increasing.

Hence $\lim_{x\to a^+} y(x) = y(a^+)$ exists. We have $F(a, y(a^+)) = 0$ and $\chi_0(a) < y(a^+) < \psi^{-1}(a)$ as $F(a,\chi_0(a)) < 0$ and $F(a,\psi^{-1}(a)) > 0$. Therefore there exists a solution of $\frac{dy}{dx} = f(x,y)$ with $y(a) = y(a^+)$ on the interval $(a-\delta, a+\delta)$ for some $\delta > 0$. This contradicts the maximality of (a,b). Therefore $a = 0$. The case $b = 1$ is similar. We extend the definition of $y(x)$ on $[0,1]$ by setting

$$y(1) = \lim_{x\to 1^-} y(x) \quad \text{and} \quad y(0) = \lim_{x\to 0^+} y(x).$$

Then $y(0) = 0$ as $F(0,y(0)) = 0$, and $y(1) = 1$ because $F(1,y(1)) = 0$.

We see that case (II) does not hold, as $y(0) = \varphi(0) = 0$. Hence by (III) $g(x) = y(x)$ is homeomorphism, $g \in C^1(0,1)$ and $g(x) < x$. Moreover g is a unique implicit function by Lemma 6.4. We next show that $h(x) = 2x - g(x)$ is increasing. As h is an implicit function given by $F_{1-p^*}(x, y) = 0$, by using Remarks 6.2 and 6.5 we can apply the reasoning as for p^*. ∎

If we consider the symmetrical case with respect to the diagonal then $y = 1 - g(1-x)$ satisfies $H_p(x, 1-g(1-x)) = 0$ for $x \in [0,1]$ where $H(x) = 1 - F(1-x)$.

The assumptions of Theorem 6.6 are not satisfied for $F = F^{(1)}$ (see Section 4 for the definition of $F^{(1)}$). Namely φ_{1-p} is increasing for every $p \in (1/2, 7/8)$ but φ is not $1-1$ for any $p \in (1/2, 1)$. Here we consider the interval $[0, 1/2]$ instead of $[0,1]$.

Remark 6.7. Assume that G is an increasing homeomorphism of I. Moreover, suppose $G(x_0) = x_0$ for some $x_0 \in (0,1)$ and G is concave on $(0, x_0)$ and convex on $(x_0, 1)$. If $y = g(x)$ is an increasing homeomorphism such that $g(x) \neq x$ for $x \notin \{0, x_0, 1\}$ and

$$pG(2x - g(x)) + (1-p)G(g(x)) - G(x) = 0$$

for some $p \neq 1/2$ and every $x \in I$ then $g(x_0) = x_0$. Moreover $g(x) < x$ for $x \in (0, x_0)$ and $g(x) > x$ for $x \in (x_0, 1)$ if $p \in (1/2, 1)$.

Proof. Let $p \in (1/2, 1)$ and $x \in (0, x_0)$. We have

$$1/2 < p = \frac{G(x) - G(g(x))}{G(2x - g(x)) - G(g(x))} < 1.$$

If $x < g(x) < x_0$ then

$$G(x) < 1/2G(2x - g(x)) + 1/2G(g(x)) < G(x)$$

as G is concave on $(0, x_0)$. But this is impossible. Let $x_1 = \sup\{x < x_0 : g(x) < x\}$. Then $x_0 = x_1$ under our assumptions. Hence $g(x) < x$ for $x \in (0, x_0)$. Similarly we show that $g(x) > x$ for $x \in (x_0, 1)$. This implies that $g(x_0) = x_0$. For $p \in (0, 1/2)$ the reasoning is analogous. ∎

Example. We apply Theorem 6.6 to $F(x) = (1 - (1-x)^\alpha)^{1/\alpha}$, where $\alpha > 1$. We have $F'(x) = ((1-x)^{-\alpha} - 1)^{(1-\alpha)/\alpha}$ and $F'^{-1}(x) = 1 - (1 + x^{\alpha/(1-\alpha)})^{-1/\alpha}$ for $x \in (0,1)$. Obviously $F \in C^2(0,1)$, $F(0) = 0$, $F(1) = 1$, $F'(0^+) = \infty$, $F'(1) = 0$ and $F''(x) < 0$ for $x \in (0,1)$. To show that

$$\begin{aligned}\varphi(x) &= 2x - F'^{-1}(b^{-1}F'(x)) \\ &= 2x - 1 + \left(1 + b^{\alpha/(\alpha-1)}((1-x)^{-\alpha} - 1)\right)^{-1/\alpha}\end{aligned}$$

is increasing for $b = 2p$ and $b = 2(1-p)$, we check the inequality $\varphi'(x) > 0$

for $x \in (0,1)$. By simple computation we get

$$2^{-1} < p < \min\{1-2^{-\alpha}, 2^{-1/\alpha}\} \ \Rightarrow\ \varphi'(x) > 0$$

for $x \in (0,1)$. For example if $\alpha = 2$ then $1/2 < p < \sqrt{2}/2$.

It appears that regularity of F'' at 0 and 1 ensures that the assumptions of Theorem 6.6 hold.

THEOREM 6.8. *Let* $F \in C^2(0,1) \cap C[0,1]$ *with* $F(0) = 0$, $F(1) = 1$, $F'(0^+) = \infty$, $F'(1^-) = 0$ *and* $F''(x) < 0$ *for* $x \in (0,1)$. *If* F'' *is strictly increasing on* $(0,1)$, $F''(1^-) < 0$, $0 < (F^{-1})''(0^+) < \infty$ *then there exists* $p^* \in (1/2, 1/\sqrt[3]{4})$ *such that* φ *and* φ_{1-p} *are strictly increasing on* $[0,1]$ *for* $p \in (1/2, p^*)$.

The proof is preceded by two lemmas.

LEMMA 6.9. *If* F'' *is strictly increasing on* $(0,1)$ *then* φ_{1-p} *is strictly increasing on* $[0,1]$ *for* $p \in (1/2, 3/4)$.

Proof. The strict increasing of φ_{1-p} is equivalent to

$$4(1-p)F''\bigg(F'^{-1}\bigg(\frac{1}{2(1-p)}F'(x)\bigg)\bigg) < F''(x) \quad \text{for } x \in (0,1).$$

An analogous condition holds for $\varphi = \varphi_p$. Note that $F'(x) < \frac{1}{2(1-p)}F'(x)$ implies $F'^{-1}\big(\frac{1}{2(1-p)}F'(x)\big) < x$. Hence

$$F''\bigg(F'^{-1}\bigg(\frac{1}{2(1-p)}F'(x)\bigg)\bigg) < F''(x),$$

and consequently

$$4(1-p)F''\bigg(F'^{-1}\bigg(\frac{1}{2(1-p)}F'(x)\bigg)\bigg) < F''(x) \quad \text{for } p < 3/4. \ ■$$

LEMMA 6.10. *If* F'' *is strictly increasing on* $(0,1)$ *and* $F''(1^-) < 0$ *then*

$$\forall_{\epsilon\in(0,1)}\ \exists_{p_0\in(1/2,1)}\ \forall_{p\in(1/2,p_0)}\ \forall_{x\in[\epsilon,1)}\ F''\bigg(F'^{-1}\bigg(\frac{1}{2p}F'(x)\bigg)\bigg) < \frac{1}{2}F''(x).$$

Proof. Fix $\epsilon \in (0,1)$. If we define $F''(1) = F''(1^-)$ then $F'' \in C[\epsilon,1]$. Observe that

$$\lim_{p\to(1/2)^+} F'^{-1}\bigg(\frac{1}{2p}F'(x)\bigg) = x$$

uniformly as $F'^{-1}\big(\frac{1}{2p}F'(x)\big)$ is strictly increasing. Therefore

$$\lim_{p\to(1/2)^+} F''\bigg(F'^{-1}\bigg(\frac{1}{2p}F'(x)\bigg)\bigg) = F''(x)$$

uniformly too. Hence there exists $p_0 \in (1/2, 1)$ such that

$$\forall_{x\in[\epsilon,1)}\ F''\Big(F'^{-1}\Big(\frac{1}{2p}F'(x)\Big)\Big) < F''(x) - \frac{1}{2}F''(1) < \frac{1}{2}F''(x).$$

Hence the inequality above holds for every $p \in (1/2, p_0)$ and $x \in [\epsilon, 1)$. ∎

Proof of Theorem 6.8. Let $(F^{-1})''(0^+) = a$. Then

$$\lim_{x\to 0^+}(F^{-1})''(x) = \lim_{x\to 0^+} -\frac{F''(F^{-1}(x))}{[F'(F^{-1}(x))]^3} = \lim_{u\to 0^+} -\frac{F''(u)}{[F'(u)]^3} = a.$$

Hence

$$\lim_{x\to 0^+}\frac{F''\big(F'^{-1}\big(\frac{1}{2p}F'(x)\big)\big)}{\big[F'\big(F'^{-1}\big(\frac{1}{2p}F'(x)\big)\big)\big]^3} = \lim_{x\to 0^+}\frac{F''\big(F'^{-1}\big(\frac{1}{2p}F'(x)\big)\big)}{F''(x)}\,\frac{F''(x)}{\big(\frac{1}{2p}\big)^3[F'(x)]^3} = -a.$$

Therefore

$$\lim_{x\to 0^+}\frac{F''\big(F'^{-1}\big(\frac{1}{2p}F'(x)\big)\big)}{F''(x)} = \frac{1}{(2p)^3}.$$

Here $1/(2p)^3 > 1/2 \iff p < 1/\sqrt[3]{4}$. Let $p_1 \in (1/2, 1/\sqrt[3]{4})$. We take $\epsilon \in (0,1)$ such that

$$\forall_{x\in(0,\epsilon]}\ F''\Big(F'^{-1}\Big(\frac{1}{2p}F'(x)\Big)\Big) < \frac{1}{2}F''(x).$$

Hence the above inequality holds for every $p \in (1/2, p_1)$. Let $p^* = \min\{p_1, p_0\}$ where p_0 is given by Lemma 6.10 for ϵ. Then

$$4pF''\Big(F'^{-1}\Big(\frac{1}{2p}F'(x)\Big)\Big) < 2pF''(x) < F''(x)$$

for $p \in (1/2, p^*)$ and every $x \in (0,1)$. By Lemma 6.9 this finishes the proof. ∎

It seems that the following hypothesis is true.

Hypothesis 6.11. Let $F \in C^2(0,1) \cap C[0,1]$ with $F(0) = 0$, $F(1) = 1$, $F'(0^+) = \infty$, $F'(1^-) = 0$, $F''(x) < 0$ for $x \in (0,1)$ and F'' strictly increasing on $(0,1)$. Then for every $\epsilon > 0$ there exists a function H satisfying the assumptions of Theorem 6.8 such that $\|F - H\| < \epsilon$.

The last hypothesis is valid for $F = F^{(1)}$ in the case of the interval $[0,1/2]$ instead of $[0,1]$ (see Section 4 for the definition of $F^{(1)}$).

By using the Example it is easy to see that the assumptions: F'' is strictly increasing on $(0,1)$, $F''(1^-) < 0$ and $0 < (F^{-1})''(0^+) < \infty$, are not necessary. To see this, set $F_\alpha(x) = (1-(1-x)^\alpha)^{1/\alpha}$ for $\alpha > 1$. Then F_α'' is not monotonic for $1 < \alpha < 2$, $F_\alpha''(1^-) = 0$ for $\alpha > 2$ and $(F_\alpha^{-1})''(0^+) = \infty$ for $1 < \alpha < 2$.

To end this section we present some necessary conditions for F to determine g such as in Theorem 6.6 for some $p \in (1/2, 1)$.

Lemma 6.12. *Let $F \in C^1(0,1) \cap C[0,1]$ with $F(0) = 0$ and $F(1) = 1$, and suppose $F'(0^+)$ and $F'(1^-)$ exist and $F'(x) > 0$ for $x \in (0,1)$. If $A_pF = F$*

for some $p \in (1/2, 1)$ *and some homeomorphism* g *such that* $g(x) < x$ *for* $x \in (0,1)$ *and* $g \in C^1[0,1]$ *then* $F'(0^+) = \infty$, $F'(1^-) = 0$ *and* $F(x) > x$ *for* $x \in (0,1)$.

Proof. Let P be the Frobenius–Perron operator for $(S, \mu_p \times \Lambda)$. Here S is determined by g. The measure $\mu_p \times \nu_F \equiv \mu_p \times \Lambda$ is S-invariant and ergodic (see Theorem 2.1). Therefore

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k f = F'$$

with L^1 convergence for every $0 \le f \in L^1(\Lambda)$ and $\int_0^1 f\,dx = 1$ (see [11, Theorem 5.2.2]). In particular

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P^k 1 = F' \quad \text{ in } L^1.$$

From the equality $\int_0^x P^n 1\,dx = \mathcal{A}_p^n \mathcal{I}(x)$ and the inequality $\mathcal{I}(x) < \mathcal{A}_p \mathcal{I}(x)$ for $x \in (0,1)$ we get

$$\lim_{n\to\infty} \mathcal{A}_p^n \mathcal{I}(x) = F(x) \quad \text{ for } x \in [0,1].$$

Therefore $x < F(x)$ for $x \in (0,1)$. Hence $F'(0^+) > 0$. We also have $0 < g'(0) < 1$. Since $PF'(x) = F'(x)$ for $x \in [0,1]$ we get $F'(0^+) = (2p + (1-2p)g'(0))F'(0^+)$, which implies $F'(0^+) = \infty$. Similarly, we prove that $F'(1^-) = 0$. ■

In view of Corollary 3.2 we consider more general random walks:

$$S_0^{-1}(x) = (1+\epsilon)x - \epsilon g(x), \qquad S_1^{-1}(x) = g(x) \quad \text{ for } \epsilon > 0. \tag{3}$$

It is easy to see that the pair of homeomorphisms

$$T_i^{-1}(x) = (1-\epsilon_i)x + \epsilon_i h(x), \qquad i = 0,1,$$

where $\epsilon_0 < 0, \epsilon_1 > 0$ can be written as in (3) for $\epsilon = -\epsilon_0/\epsilon_1$ and $g = T_1^{-1}$.

For $p > \frac{1}{1+\epsilon}$ we consider

$$F_p^\epsilon(x,y) = pF((1+\epsilon)x - \epsilon y) + (1-p)F(y) - F(x)$$

for

$$(x,y) \in \left\{(x,y) : 0 < y < x \text{ for } x \in \left(0, \frac{1}{1+\epsilon}\right]\right\} \cap \left\{\frac{1+\epsilon}{\epsilon}x - \frac{1}{\epsilon} < y < x \text{ for } x \in \left(\frac{1}{1+\epsilon}, 1\right)\right\},$$

and

$$F_{1-p}^\epsilon(x,y) = (1-p)F\left(\frac{1+\epsilon}{\epsilon}x - \frac{1}{\epsilon}y\right) + pF(y) - F(x)$$

for

$$(x,y) \in \left\{(x,y) : x < y < (1+\epsilon)x \text{ for } x \in \left(0, \frac{1}{1+\epsilon}\right]\right\}$$
$$\cap \left\{x < y < 1 \text{ for } x \in \left(\frac{1}{1+\epsilon}, 1\right)\right\}.$$

The conditions $\frac{\partial F^\epsilon_p}{\partial x} = 0$, $\frac{\partial F^\epsilon_{1-p}}{\partial x} = 0$ determine the functions

$$\varphi^\epsilon_p(x) = \frac{1+\epsilon}{\epsilon}x - \frac{1}{\epsilon}F'^{-1}\left(\frac{1}{(1+\epsilon)p}F'(x)\right)$$

and

$$\varphi^\epsilon_{1-p}(x) = (1+\epsilon)x - \epsilon F'^{-1}\left(\frac{\epsilon}{(1+\epsilon)(1-p)}F'(x)\right).$$

By using much the same reasoning as in the proofs of Theorems 6.6 and 6.8 we get

THEOREM 6.13. *Let $F \in C^2(0,1) \cap C[0,1]$ with $F(0) = 0$, $F(1) = 1$, $F'(0^+) = \infty$, $F'(1^-) = 0$ and $F''(x) < 0$ for $x \in (0,1)$. Moreover, let $p^* \in (1/(1+\epsilon), 1)$ be such that $\varphi^\epsilon_{p^*}$ and $\varphi^\epsilon_{1-p^*}$ are increasing functions. Then there exists exactly one implicit function $y = g(x)$ such that $g(x) < x$ for $x \in (0,1)$. The function g is a homeomorphism of I and $g \in C^1(0,1)$. Moreover $(1+\epsilon)x - \epsilon g(x)$ is a homeomorphism too.*

THEOREM 6.14. *Let $F \in C^2(0,1) \cap C[0,1]$ with $F(0) = 0$, $F(1) = 1$, $F'(0^+) = \infty$, $F'(1^-) = 0$ and $F''(x) < 0$ for $x \in (0,1)$. If F'' is strictly increasing on $(0,1)$, $F''(1^-) < 0$, and $0 < (F^{-1})''(0^+) < \infty$ then there exists p^* where*

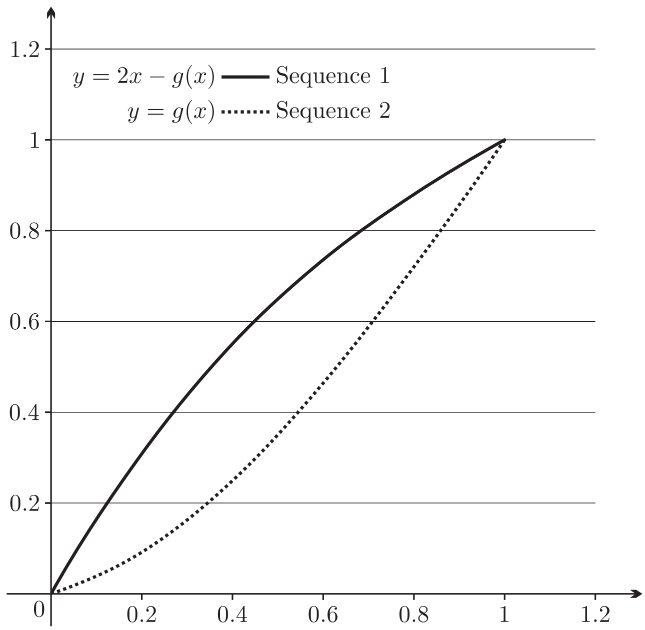
$$\frac{1}{1+\epsilon} < p^* < \min\left\{\frac{1}{(1+\epsilon)^{2/3}}, \frac{1+2\epsilon}{(1+\epsilon)^2}\right\}$$

such that φ^ϵ_p and φ^ϵ_{1-p} are strictly increasing on $[0,1]$ for $p \in \left(\frac{1}{1+\epsilon}, p^*\right)$.

7. Numerical solution. We apply the Runge–Kutta method of rank 4 to solve (2) for $F(x) = (1-(1-x)^2)^{1/2}$. Let $p = 0.6$, $x_0 = 0.01$ and $y_0 \in (y_1, y_2)$ where $y_1 = 0.00297$, $y_2 = 0.002975$. Here $F(x_0, y_1) < 0$ and $F(x_0, y_2) > 0$. We start at (x_0, y_1) and next at (x_0, y_2) respectively. After 990 steps with step $h = 0.001$ we get two sequences of points (x_n, y^1_n) and (x_n, y^2_n). It appears that $F(x_n, y^1_n) < 0$ and $F(x_n, y^2_n) > 0$ for $n = 0, \ldots, 989$ and

$$\max\{y^2_n - y^1_n : n = 0, \ldots, 990\} = 0.007282.$$

The solution $y = g(x)$ of (2) satisfies $y^1_n < g(x_n) < y^2_n$ for $n = 0, \ldots, 989$ by Theorem 6.6. Below we present the point graphs of (x_n, y^1_n) and $(x_n, 2x_n - y^1_n)$ which approximate $g(x)$ and $2x - g(x)$ respectively.



Now we are in a position to apply Corollary 3.2. Let S be given by

$$S_0^{-1}(x) = 1.307x - 0.307x^2 \quad \text{and} \quad S_1^{-1}(x) = 0.26x + 0.74x^2.$$

Then $S_1^{-1}(0.01) < 0.00297 < g(0.01)$ and $F_{0.6}(x, S_1^{-1}(x)) < 0$ for $x \in (0,1)$. Therefore $S_1^{-1}(x) < g(x)$ for every $x \in (0,1)$. Similarly $S_0^{-1}(0.01) < 0.02 - 0.002975 < 0.02 - g(0.01)$ and $F_{0.4}(x, S_0^{-1}(x)) > 0$ for $x \in (0,1)$. Hence $S_0^{-1}(x) < 2x - g(x)$ for every $x \in (0,1)$. Then for every $p \in [0.706781, \sqrt{2}/2)$, S possesses ergodic invariant measure $\mu_p \times \mu_G$ by Corollary 3.2.

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