LIST 1
Sequences

1.1. Show that the given sequences are monotone and bounded.

(a) $a_n = \dfrac{n}{2n+1}$, (b) $b_n = \dfrac{2^n}{3^n+2}$, (c) $c_n = \dfrac{(n!)^2}{(2n)!}$, (d) $d_n = \sin\dfrac{\pi}{2n+1}$,
(e) $e_n = \dfrac{(n+2)^2}{2^{n+2}}$, (f) $f_n = \sqrt{n+8} - \sqrt{n+3}$, (g) $g_n = \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n}$.

1.2. Using the definition of a limit of a sequence, show that

(a) $\lim\limits_{n\to\infty} \dfrac{n}{n+2} = 1$, (b) $\lim\limits_{n\to\infty} \dfrac{n^2+1}{2n} = +\infty$, (c) $\lim\limits_{n\to\infty} \dfrac{n+4}{n+2} \neq 2$.

1.3. By giving the appropriate examples, show that the following expressions are not well defined

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0\cdot\infty, \quad \infty-\infty, \quad 1^{\infty}, \quad \infty^0, \quad 0^0.$$

1.4. Find the limits of the sequences

(a) $a_n = \dfrac{2n-3}{3n+4}$, (b) $b_n = \dfrac{n^2+3n-8}{2n+5}$, (c) $c_n = \dfrac{n^2+n-3}{n^3+2n+1}$,

(d) $d_n = \dfrac{(2n^3+3)^8}{(2n^4+7)^6}$, (e) $e_n = \dfrac{n+\sqrt{n^3+7}}{\sqrt[3]{n^2+5}+4n}$, (f) $f_n = \dfrac{8^{n+2}+2^n}{2^{3n+1}+3^n+4}$,

(g) $g_n = \dfrac{1+2+3+\cdots+n}{n^2}$, (h) $h_n = \sqrt{n+8} - \sqrt{n+3}$,

(i) $i_n = \sqrt{n^2+4n+1} - \sqrt{n^2+3}$, (j) $j_n = \sqrt{2n+1} - \sqrt{n+23}$,

(k) $k_n = \sqrt{9^n+4\cdot3^n+1} - \sqrt{9^n+3}$, (l) $l_n = n^{30} - 2\cdot n^{21} - 3\cdot n^9 + 3$,

(m) $m_n = 7^n - 2\cdot5^{2n} + 3\cdot9^{n+5} + 4$, (n) $m_n = \left(\dfrac{n+4}{n+1}\right)^{n+3}$, (o) $o_n = \left(\dfrac{n^2+3}{n^2+1}\right)^{n^2}$,

(p) $p_n = \left(\dfrac{2n+1}{2n+5}\right)^{1-3n}$, (r) $r_n = \left(\dfrac{4n+1}{2n-1}\right)^{n+6}$, (s) $s_n = \left(\dfrac{3^n+2^n}{5^n+3^n}\right)^{n}$.

1.5. For a given sequence (a_n) find a sequence (b_n) of the form $b_n = n^p$ or $b_n = \alpha^n$ such that (a_n) i (b_n) are of the same order. (*We say that the sequences (a_n), (b_n) are of the same order if $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c$, for a certain positive number c.*)

(a) $a_n = \dfrac{1}{n^2+4n+3}$, (b) $a_n = \dfrac{n^2}{n^3+7}$, (c) $a_n = \sqrt{n+9} - \sqrt{n+1}$,

(d) $a_n = \dfrac{1}{3\cdot2^n+2\cdot3^n}$, (e) $a_n = \dfrac{3^n}{4^n+5^n}$, (f) $a_n = \dfrac{4^{n+2}}{5\cdot2^{n+1}+2\cdot3^n}$.