MA1_PC1

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Problem Class #1, 13.10/14.10

1.1 (a) $a_n = \frac{n}{2n+1}$ - monotone and bounded?

$\underline{0} \le a_n \le \frac{n}{2n} = \underline{\frac{1}{2}}$, bounded

$a_n = \frac{1}{2+\frac{1}{n}}$ $\qquad \frac{1}{n}\downarrow,\ 2+\frac{1}{n}\downarrow,\ \frac{1}{2+\frac{1}{n}}\nearrow$

1.1(c) $c_n = \frac{(n!)^2}{(2n)!}$ $\qquad n! = 1\cdot 2\cdot \ldots \cdot n$

$(n+1)! = (n+1)\cdot \cancel{n!}$

$(2n+2)! = (2n+2)(2n+1)\cancel{(2n)!}$

$\frac{c_{n+1}}{c_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n+1}{2n+1}\cdot\frac{1}{2} = \frac{n+1}{\underset{\underset{2n+2}{\downarrow}}{4n+2}} < \frac{1}{2} < 1$

$\Rightarrow c_{n+1} < c_n$, $\downarrow$.

$\frac{(1!)^2}{2!} = \frac{1}{2}$

Bounded: $\underset{\text{lower}}{\underline{0}} \le c_n \le \underset{\text{upper}}{\underline{c_1}}$

1.1(f) $f_n = \sqrt{n+8} - \sqrt{n+3} = \frac{(n+8)-(n+3)}{\sqrt{n+8}+\sqrt{n+3}} = \frac{5}{\sqrt{n+8}+\sqrt{n+3}}$

Multiplication by conjugate trick

$\sqrt{n+8}\nearrow,\ \sqrt{n+3}\nearrow$ $\qquad f_n\downarrow$.

$0 \le f_n \le f_1$, bounded.

1.2 (b.) $\frac{n^2+1}{2n} \to +\infty$, $n \to \infty$.

We have to find, for any given $A > 0$, $N = N_A$ such that $\frac{n^2+1}{2n} > A$, $n \geq N$

$\boxed{\frac{n^2+1}{2n} > A} \Leftrightarrow n^2+1 > 2An$, follows from

$\Uparrow$ $n^2 > 2An \Leftrightarrow n > 2A$. , $n \geq N$

Take $N_A = [2A] + 1 \Rightarrow$ the required ineq. holds

$\frac{n^2+1}{2n} \geq \frac{n}{2} \to +\infty$

Prove by definition!

1.3

$\frac{1/n}{1/n^2} = \frac{n^2}{n} = n$

- $a_n = \frac{1}{n} \to 0$
  $b_n = \frac{1}{n^2} \to 0$

  $\frac{a_n}{b_n} = n \to +\infty$; $\frac{b_n}{a_n} = \frac{1}{n} \to 0$.

  $\frac{0}{0} = \begin{cases} +\infty, & \text{1st case} \\ 0, & \text{2nd case} \end{cases}$

- $a_n = \frac{x}{n}$, $b_n = \frac{1}{n} \Rightarrow \frac{0}{0} = x \in \mathbb{R}$

- $a_n = \frac{(-1)^n}{n}$, $b_n = \frac{1}{n} \Rightarrow \frac{a_n}{b_n} = (-1)^n$ does not converge

For $\frac{0}{0}$, depending on the choice of seq. in fraction, we can have

$\begin{cases} -\infty \\ x \in \mathbb{R} \\ +\infty \\ \text{does not converge} \end{cases}$

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$+\infty \cdot \frac{a}{0} = +\infty$

1.4(b) $b_n = \frac{n^2+3n-8}{2n+5} = n \boxed{\frac{1+\frac{3}{n}-\frac{8}{n^2}}{2+\frac{5}{n}}} \xrightarrow[n\to\infty]{} +\infty$

$\frac{n^2}{n} = n$, $n \to +\infty$, $\to \frac{1}{2} > 0$

1.4.(c) $c_n = \frac{n^2+n-3}{n^3+2n+1} = \frac{1}{n} \boxed{\frac{1+\frac{1}{n}-\frac{3}{n^2}}{1+\frac{2}{n^2}+\frac{1}{n^3}}} \xrightarrow[n\to\infty]{} 0 = 0 \cdot 1$

1.4.(c) $c_n = \dfrac{\ldots}{n^3+2n+1} = \dfrac{n^3 \ldots}{n^3\left(1 + \frac{2}{n^2} + \frac{1}{n^3}\right)} \to 0$

$n \to \infty$

$\dfrac{1+0-0}{1+0+0} = 1$

1.4.(e) $e_n = \dfrac{n + \sqrt{n^3+7}}{\sqrt[3]{n^2+5} + 4n} = \dfrac{n^{3/2}\left(n^{-1/2} + \sqrt{1+7/n^3}\right)}{n\left(\sqrt[3]{\frac{1}{n} + \frac{5}{n^3}} + 4\right)} = n^{1/2} \cdot \dfrac{\sqrt{1+7/n^3} + \frac{1}{\sqrt{n}}}{4 + \sqrt[3]{1/n + 5/n^3}}$

$n^{1/2} \to +\infty$, $\quad \dfrac{1}{4}$, $\quad \to +\infty$

$n = n^1$, $\quad \sqrt{n^3} = n^{3/2}$, $\quad \sqrt{n^3 + \ldots}$

1.4(h) $h_n = \sqrt{n+8} - \sqrt{n+3} = \dfrac{5}{\sqrt{n+8} + \sqrt{n+3}} \to 0, \; n \to \infty$

$= \dfrac{1}{\sqrt{n}} \cdot \dfrac{5}{\sqrt{1+\frac{8}{n}} + \sqrt{1+\frac{5}{n}}} \to 0 \cdot \dfrac{5}{2} = 0.$

$\dfrac{5}{\sqrt{1}+\sqrt{1}} = \dfrac{5}{2}$

1.4.(n) $m_n = \left(\dfrac{n+4}{n+1}\right)^{n+3} = \left(1 + \dfrac{3}{n+1}\right)^{n+3} =$

$n+3 = (n+1)+2$

$= \left(1+\dfrac{3}{n+1}\right)^{n+1} \cdot \left(1 + \dfrac{3}{n+1}\right)^2$

$\left(1 + \dfrac{3}{n+1}\right)^{n+1} \to e^3$ because $n+1 \to \infty$

$m_n \to e^3 \cdot 1 = e^3, \; n \to \infty.$

$\left(1+\dfrac{a}{n}\right)^n \to e^a$

For any sequence $n_k \to +\infty$, $k \to \infty$

$\left(1 + \dfrac{a}{n_k}\right)^{n_k} \to e^a$

1.5 (c) $a_n = \sqrt{n+9} - \sqrt{n+1} = \dfrac{8}{\sqrt{n+9}+\sqrt{n+1}} = \dfrac{1}{\sqrt{n}} \dfrac{8}{\sqrt{1+\frac{9}{n}}+\sqrt{1+\frac{1}{n}}} \to \dfrac{8}{2} = 4$

$a_n \asymp \dfrac{1}{\sqrt{n}} = n^{-1/2}$

Has same order with $b_n = n^{-\frac{1}{2}} = \dfrac{1}{\sqrt{n}}$

$a > b$

1.5 (e) $a_n = \dfrac{3^n}{4^n+5^n} = \dfrac{1}{5^n} \cdot \dfrac{3^n}{1+\left(\frac{4}{5}\right)^n} = \left(\dfrac{3}{5}\right)^n \cdot \boxed{\dfrac{1}{1+\left(\frac{4}{5}\right)^n}} \to 1$

$\left(\frac{4}{5}\right)^n \to 0$

$a^n >> b^n$

$a^n + b^n = a^n\left(1+\left(\frac{b}{a}\right)^n\right)$

$\frac{b}{a} < 1$

Has same order with $b_n = \left(\frac{3}{5}\right)^n$

$\left(\frac{3}{5}\right)^n$