MA1_PC1

Wednesday, October 14, 2020 9:13 AM

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Problem Class #1, 13.10/14.10

$2n+1 > 2n$

$n \geq 1$

$\frac{n}{2n+1} < \frac{n}{2n}$

1.1 (a) $a_n = \frac{n}{2n+1}$ - monotone and bounded?

$0 \leq a_n \leq \frac{n}{2n} = \frac{1}{2}$, bounded

$\frac{(n+1)!}{n!} = (n+1)$

$a_n = \frac{1}{2+\frac{1}{n}}$ $\quad \frac{1}{n}\downarrow$, $2+\frac{1}{n}\downarrow$, $\frac{1}{2+\frac{1}{n}}\nearrow$

$(n+1)! = (n+1)\cdot n!$

$(2n+2)! = (2n+2)(2n+1)(2n)!$

1.1(c) $c_n = \frac{(n!)^2}{(2n)!}$ $\quad n! = 1\cdot 2\cdot\ldots\cdot n$

$2n+2 = 2(n+1)$

$\frac{c_{n+1}}{c_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n+1}{2n+1}\cdot\frac{1}{2} = \frac{n+1}{4n+2} < \frac{1}{2} < 1$

$\frac{n+1}{4n+2} < \frac{n+1}{2n+2} = \frac{1}{2}$

$\Rightarrow c_{n+1} < c_n$, $\downarrow$.

Bounded: $\quad 0 \leq c_n \leq c_1 = \frac{(1!)^2}{2!} = \frac{1}{2}$

1.1(f) $f_n = \sqrt{n+8} - \sqrt{n+3} = \frac{(n+8)-(n+3)}{\sqrt{n+8}+\sqrt{n+3}} = \frac{5}{\sqrt{n+8}+\sqrt{n+3}}$ $\nearrow$ $\downarrow$

$\sqrt{n+8}\nearrow$, $\sqrt{n+3}\nearrow$ $\quad f_n\downarrow$.

$0 \leq f_n \leq f_1$, bounded.

$\sqrt{9}-\sqrt{4} = 3-2 = 1$

$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = a-b$

$\sqrt{a}-\sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$

1.2 (b.) $\frac{n^2+1}{2n} \to +\infty$, $n \to \infty$.

We have to find, for any given $\boxed{A>0,}$

$N = N_A$ such that $\frac{n^2+1}{2n} > A$ for $n \geq N$

True $\frac{n^2+1}{2n} > A \iff$ True $n^2+1 > 2An$, follows from

$\Uparrow$ $n^2 > 2An \iff n > 2A$. True

Take $N_A = [2A]+1 \Rightarrow$ the required ineq. holds

$N_A > 2A$; for $n \geq N$ $n > 2A$

1.3

$$\frac{a_n}{b_n} \to ?$$ $$a_n \to 0$$ $$b_n \to 0$$

- $a_n = \frac{1}{n} \to 0$
  $b_n = \frac{1}{n^2} \to 0$

$$\frac{1/n}{1/n^2} = \frac{n^2}{n} = n \to +\infty$$

$$\frac{1/n^2}{1/n} = \frac{1}{n} \to 0$$

$$\frac{a_n}{b_n} = n \to +\infty \,; \quad \frac{b_n}{a_n} = \frac{1}{n} \to 0 .$$

$$\frac{0}{0} = \begin{cases} +\infty, & \text{1st case} \\ 0, & \text{2nd case} \end{cases}$$

$$\frac{a_n}{b_n} = \frac{x/n}{1/n} = x$$

- $a_n = \frac{x}{n}$, $b_n = \frac{1}{n} \Rightarrow \frac{0}{0} = x \in \mathbb{R}$

- $a_n = \frac{(-1)^n}{n}$, $b_n = \frac{1}{n} \Rightarrow \frac{a_n}{b_n} = (-1)^n$ does not converge

1.4(b) $$b_n = \frac{n^2+3n-8}{2n+5} = n \, \boxed{\frac{1+\frac{3}{n}-\frac{8}{n^2}}{2+5/n}} \to +\infty \quad n\to\infty$$

n^2, n, $\frac{n^2}{n}$, 1, 0, 0, $\to +\infty$, 2, 0, $\to 1/2$

1.4.(c) $$c_n = \frac{n^2+n-3}{n^3+2n+1} = \frac{1}{n} \, \boxed{\frac{1+\frac{1}{n}-\frac{3}{n^2}}{1+\frac{2}{n^2}+\frac{1}{n^3}}} \to 0\cdot 1 = 0 \quad n\to\infty$$

n^3, $\frac{n^2}{n^3}$, $\to 0$, $\to 0$, 0, 0, 1

1.4.(e) $$e_n = \frac{n+\sqrt{n^3+7}}{\sqrt[3]{n^2+5}+4n} = \frac{n^{3/2}}{n} \, \boxed{\frac{n^{-1/2}+\sqrt{1+\frac{7}{n^3}}}{\sqrt[3]{n^{-1}+5\cdot n^{-3}}+4}} \to +\infty$$

$= \sqrt{n}$, $\to 0$, $\to 0$, $\sqrt{1}$, 1

1.4.(e) $e_n = \sqrt[3]{n^2+5} + 4n = n\left(\sqrt[3]{n^{-1}+5\cdot n^{-3}} + 4\right)$, $n \to \infty$

$\to \frac{\sqrt{1}}{4} = \frac{1}{4}$

1.4(h) $h_n = \sqrt{n+8} - \sqrt{n+3} = \frac{5}{\sqrt{n+8}+\sqrt{n+3}} \to \boxed{0}$, $n \to \infty$

$\frac{1}{\sqrt{n}} \cdot \frac{5}{\sqrt{1+\frac{8}{n}}+\sqrt{1+\frac{3}{n}}} \to \frac{5}{2}$

$n+4 = n+1+3$

1.4.(n) $m_n = \left(\frac{n+4}{n+1}\right)^{n+3} = \left(1+\frac{3}{n+1}\right)^{\boxed{n+3}} \neq \to 0$

$= \left(1+\frac{3}{n+1}\right)^{n+1} \cdot \left(1+\frac{3}{n+1}\right)^2$

$\left(1+\frac{3}{n+1}\right)^{n+1} \to e^3$ because $n+1 \to \infty$

$\left(1+\frac{3}{n+1}\right) \to 0 \to 1$

$m_n \to e^3 \cdot 1 = e^3, \quad n \to \infty.$

$\left(1+\frac{a}{n}\right)^n \to e^a, \; n \to \infty.$

$\left(1+\frac{a}{n_k}\right)^{n_k} \to e^a, \; n_k \to \infty$

1.5 (c) $a_n = \sqrt{n+9} - \sqrt{n+1} = \dfrac{8}{\sqrt{n+9}+\sqrt{n+1}} = \dfrac{1}{\sqrt{n}} \boxed{\dfrac{8}{\sqrt{1+\frac{9}{n}}+\sqrt{1+\frac{1}{n}}}} \to \dfrac{8}{2} = 4$

Has same order with $b_n = n^{-\frac{1}{2}} = \dfrac{1}{\sqrt{n}}$

1.5 (e) $a_n = \dfrac{3^n}{4^n+5^n} = \dfrac{1}{5^n} \cdot \dfrac{3^n}{1+\left(\frac{4}{5}\right)^n} = \left(\dfrac{3}{5}\right)^n \cdot \boxed{\dfrac{1}{1+\left(\frac{4}{5}\right)^n}} \to 1$

A, 1, $\left(\frac{4}{5}\right)^n \to 0$

Has same order with $\boxed{b_n = \left(\frac{3}{5}\right)^n}$

$A < B \qquad \frac{A}{B} < 1$

$A^n \ll B^n$

$\left(\frac{A}{B}\right)^n \xrightarrow[n\to\infty]{} 0$