ALGEBRA

Binomial formula. Induction

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1. Using the Newton Binomial formula, transform

$$(a)\,(2x-y)^4;\quad (b)\,\left(x+\frac{1}{x^3}\right)^3;\quad (c)\,(\sqrt{u}-\sqrt[4]{v})^8;\quad (d)\,\left(x+\frac{1}{x^3}\right)^6.$$

2. Find the coefficient at term t in the expansion

$$(a)\,(2p-3q)^7,\ t=p^2q^5;\quad (b)\,\left(\sqrt[4]{b^5}-\frac{3}{b^3}\right)^7,\ t=\sqrt[4]{b}\quad (c)\,\left(2x-\frac{1}{x}\right)^6\left(x+\frac{1}{2x}\right)^6,\ t=x^0.$$

3. Using the mathematical induction, prove the equalities:

(a) $1+2+3+\cdots+n=\frac{n(n+1)}{2}, n\in\mathbf{N}$;

(b) $1+3+5+\cdots+(2n-1)=n^2, n\in\mathbf{N}$;

(c) $1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}, n\in\mathbf{N}$.

4. Using the mathematical induction, prove the inequalities:

(a) $n^3<3^n, n\in\mathbf{N}$;

(b) $\frac{1}{\sqrt{1}}+\ldots\frac{1}{\sqrt{n}}\geqslant\sqrt{n}, n\in\mathbf{N}$;

(c) $(1+a)^n\geqslant 1+na$, for any $a\geqslant -1$ and $n\in\mathbf{N}$.

Complex numbers

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5. Perform the algebraic operations and write the result in the Carthesian form $x+iy$:

$$(a)\,(i+3)-(2-3i);\quad (b)\,(1-i)(2+5i);\quad (c)\,\frac{1-3i}{2+3i};\quad (d)\,(1-i)^4.$$

6. Find all complex numbers z which satisfy the following conditions:

$$(a)\,\mathrm{Re}\,z+\mathrm{Im}\,z=3;\quad (b)\,\mathrm{Re}\,(-iz)\leqslant 1;\quad (c)\,\mathrm{Im}\,((1+i)z)\leqslant 2.$$

Indicate the solution on the complex plane.

7. Comparing the real and imaginary parts of both sides of the equations, solve them for real x, y:

(a) $(1-i)x+(2-i)y = 1+i$; (b) $\dfrac{x}{1-i}+\dfrac{y}{1+i} = 1+i$; (c) $2x^2+iy^2 = 3$; (d) $3x^2-2iy^2 = (1+i)(i-2)$.

8. Writing z in the algebraic form $z = x + iy$, solve the equations

$$(a)\ z^2 = -i; \quad (b)\ (3-2i)z = (2+i); \quad (c)\ \frac{z+1}{2+i} = \frac{3-z}{3-2i}; \quad (d)\ z^2 - 4z + 5;$$

$$(e)\ z(1+i) + \overline{z}(2-i) = 1+i; \quad (f)\ i\mathrm{Re}\, z + \mathrm{Im}\, z = 1+2i; \quad (g)\ z\overline{z} = (\overline{z})^2.$$

9. Using the Carthesian form of complex numbers, compute the following roots:

$$(a)\ \sqrt{1-2i}; \quad (b)\ \sqrt{5-i}.$$

10. Write the following numbers in the trigonometric form:

$$(a)\ 2i; \quad (b)\ -1+\sqrt{3}i; \quad (c)\ -2\sqrt{3}-2i; \quad (d)\ \left(\frac{1-\sqrt{3}i}{2+2\sqrt{3}i}\right)^5.$$

11. Draw on the complex plane the sets of complex numbers satisfying the following conditions:

$$(a)\ |2z+i| = 6; \quad (b)\ |3z-1| < 3; \quad (c)\ 2 \leqslant |2z+i| \leqslant 4; \quad (d)\ |z-2i| = |z+i|;$$

$$(e)\ \mathrm{Im}\,(z^3) < 0; \quad (f)\ \mathrm{Re}\,(z^4) \geqslant 0; \quad (h)\ |z+1| \leqslant |\overline{z}+i|.$$

12. Using de Moivre's formula, compute the following powers:

$$(a)\ (1-i)^{13}; \quad (b)\ (-1+\sqrt{3}i)^{15}; \quad \left(\frac{1+i}{-1+i\sqrt{3}}\right)^{17}.$$

Give the answers in the Carthesian form.

13. Using the trigonometric form of complex numbers, compute the following roots:

$$(a)\ \sqrt[6]{-1}; \quad (b)\ \sqrt[3]{-\sqrt{3}+i}; \quad (c)\ \sqrt[6]{-64}.$$

Give the answers in the Carthesian form.

14. Solve the following equations:

$$(a)\ (z+1)^3 = (z-2)^3; \quad (b)\ (z+i)^4 = (1-z)^4; \quad (c)\ (2z-1)^3 = (z+i)^3.$$

Give the answers in the Carthesian form.

15. Solve the equations for complex z:

(a) $z^2-z+1 = 0$; (b) $z^2+16 = 0$; (c) $z^4-3z^2+2 = 0$; (d) $z^2+(1-i)z+2i = 0$; (e) $z^4 = -1$;

$$(f)\ z^2+4iz+1 = 0; \quad (g)\ z^3 = (1+i)^3; \quad (h)\ (z-i)^4 = (2z+1)^4.$$

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16. Find all integer roots of the following real polynomials:

$$(a)\, x^3 + x^2 - x + 2; \quad (b)\, x^4 - 3x^3 + 5x^2 - 9x + 6; \quad (c)\, x^4 + x^2 - 2.$$

17. Find all rational roots of the following real polynomials:

$$(a)\, 6x^4 - x^3 + 11x^2 - 2x - 2; \quad (b)\, x^4 - 5x^2 + 4; \quad (c)\, 4x^4 + 7x^2 - 2.$$

18. Perform the long division and find $Q(x), R(x)$ such that $P(x) = D(x)Q(x) + R(x)$, $\deg(R) < \deg(D)$ for

$$(a)\, P(x) = x^{12} - 3x^{10} + 2x^7,\ D(x) = x^3 + 1; \quad (b)\, P(x) = 2x^8 - 4x^3 + 5x,\ D(x) = x^2 + x + 1.$$

19. Find all roots of the following real polynomials:

$$(a)\, x^4 - 6x^2 - 3x + 2; \quad (b)\, x^4 - 3x^3 - 2x^2 + 2x + 12.$$

20. Find all roots of the following complex polynomials, knowing one of their roots:

$$(a)\, z^4 + 2z^3 + 4z^2 + 3z + 2, z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i; \quad (b)\, z^4 + 3z^3 + 9z^2 + 12z + 10, z_1 = -1 - i.$$

21. Factor the following real polynomials into irreducible real factors:

$$(a)\, x^3 - x^2 + x - 1; \quad (b)\, x^6 + 8; \quad (c)\, x^4 + 3x^2 + 2.$$

22. Factor the following complex polynomials into irreducible complex factors:

$$(a)\, z^3 - z^2 + z - 1; \quad (b)\, z^4 + 3z^2 + 2; \quad (c)\, z^4 + 1.$$

23. Decompose the following real rational functions into real partial fractions:

$$(a)\, \frac{x}{(x^2-1)(x+2)}; \quad (b)\, \frac{x^4 - 2}{x^3 + 1}; \quad (c)\, \frac{1}{(x^2-1)(x+1)(x-2)}.$$

24. Decompose the following complex rational functions into complex partial fractions:

$$(a)\ \frac{1}{z^3 - z^2 + 4z - 4};\quad (b)\ \frac{z-1}{z^3+1};\quad (c)\ \frac{1}{(z^2+2)(z+1)}.$$

Matrices. Determinants. Inverse matrices

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25. For the matrices A, B, C given below, which of the matrices: $A+B, A+C, 2A, AB, BA, AC, CA, A^2, C^2$ are well defined? Compute the matrices which are well defined.

(a)

$$A = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 4 & 3 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \end{pmatrix};$$

(b)

$$A = \begin{pmatrix} 2 & 1 \\ -1 & -2 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

(c)

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ -1 & 2 \\ 1 & -2 \end{pmatrix}.$$

26. Let

$$A = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & 1 \\ 3 & 5 \end{pmatrix}$$

Compute AB, and then solve the matrix equations

(a) $AX = C$;

(b) $XA = C$;

(c) $AXB = C$.

27. It is known that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6.$$

Find the following determinants:

$$\text{(a)} \begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}, \quad \text{(b)} \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}, \quad \text{(c)} \begin{vmatrix} g & d & a \\ h & e & b \\ i & f & c \end{vmatrix}, \quad \text{(d)} \begin{vmatrix} 3a & -b & c+4a \\ 3d & -e & f+4d \\ 3g & -h & i+4g \end{vmatrix}.$$

28. Let A be a real 7×7-matrix with $\det A = 2$. Find the determinants of the following matrices: (a) $2A$; (b) $-5A$; (c) $-A^3$; (d) AA^T.

29. Write the Laplace expansions of the given determinants along indicated rows or columns

$$\text{(a)}\ \begin{pmatrix} 2 & 1 & \mathbf{2} \\ 3 & 2 & \mathbf{1} \\ 4 & 3 & \mathbf{-1} \end{pmatrix}, \quad \text{(b)}\ \begin{pmatrix} 2 & 1 & 3 & 1 \\ \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} \\ 1 & -1 & 1 & 3 \\ 2 & -2 & 1 & -3 \end{pmatrix}, \quad \text{(c)}\ \begin{pmatrix} 1 & -3 & 1 & \mathbf{2} \\ 2 & 3 & -2 & \mathbf{1} \\ -2 & 1 & 1 & \mathbf{0} \\ 1 & 4 & 3 & \mathbf{0} \end{pmatrix}.$$

30. Calculate the determinants from the previous problem.

31. Calculate the determinants

$$\text{(a)}\ \begin{pmatrix} 1 & 2 & -3 \\ -2 & 3 & -1 \\ 3 & 2 & 1 \end{pmatrix}, \quad \text{(b)}\ \begin{pmatrix} 2 & 0 & 2 & 3 \\ -1 & 1 & -1 & 1 \\ -2 & 0 & 2 & 0 \\ 5 & -1 & -1 & 1 \end{pmatrix}, \quad \text{(c)}\ \begin{pmatrix} 1 & 0 & 2 & 3 \\ -2 & 0 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ -1 & 0 & 5 & 1 \end{pmatrix}, \quad \text{(d)}\ \begin{pmatrix} 4 & 1 & 3 & 0 \\ 4 & 1 & 0 & 2 \\ 4 & 0 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix},$$

$$\text{(e)}\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

32. Calculate the determinants for the matrices which depend on real parameters:

$$\text{(a)}\ \begin{pmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{pmatrix}, \quad \text{(b)}\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ b & 1 & 1 & 1 \\ b & b & 1 & 1 \\ b & b & b & 1 \end{pmatrix}, \quad \text{(c)}\ \begin{pmatrix} c & c & c & c & c \\ 1 & c & c & c & c \\ 1 & 1 & c & c & c \\ 1 & 1 & 1 & c & c \\ 1 & 1 & 1 & 1 & c \end{pmatrix},$$

$$\text{(d)}\ \begin{pmatrix} 1+a & b & c & d & e \\ a & 1+b & c & d & e \\ a & b & 1+c & d & e \\ a & b & c & 1+d & e \\ a & b & c & d & 1+e \end{pmatrix}.$$

33. Using the properties of the determinants, justify that the following matrices are not invertible

$$\text{(a)}\ \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ -1 & 0 & -1 \end{pmatrix}, \quad \text{(b)}\ \begin{pmatrix} 1 & 1 & -1 & -1 \\ 3 & 2 & 1 & 3 \\ 4 & 5 & 6 & 7 \\ 0 & 2 & 6 & 5 \end{pmatrix}.$$

34. For which values of the parameter c the following matrices are invertible?

$$\text{(a)} \begin{pmatrix} c & -1 \\ c & 1 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} c & 1 & 1 \\ 1 & c & -1 \\ 1 & 1 & -1 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 1 & 2 & c \\ -1 & 2 & -1 \\ 2 & 0 & 3 \end{pmatrix}, \quad \text{(d)} \begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & c \\ 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

35. Using the cofactor formula, calculate the inverse matrices to the given ones, if the inverse exists:

$$\text{(a)} \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad \text{(d)} \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix},$$

$$\text{(e)} \begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \quad \text{(f)} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

36. Solve the previous problem using the Gauss elimination method.

Systems of linear equations. Rank of a matrix

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37. Using Cramer's rule, find the specified unknown in the SLE

$$\text{(a)} \begin{cases} 3x + z = 1 \\ y + 3z - 2x = 2 \\ -x + y - z = -1 \end{cases}, \text{ find } x, \quad \text{(b)} \begin{cases} -x + y - z = 1 \\ 2x - y - z = 2 \\ x + y + z = -3 \end{cases}, \text{ find } z,$$

$$\text{(c)} \begin{cases} x + z + y + t = 1 \\ x - z + y - t = 2 \\ x - z - y + t = -1 \\ x - z - y - t = 2 \end{cases}, \text{ find } y, \quad \text{(d)} \begin{cases} x + y + z + t = 2 \\ x + 2y + 2z + 2t = 3 \\ x + 2y + 3z + 3t = 4 \\ x + 2y + 3z + 4t = 5 \end{cases}, \text{ find } z.$$

38. Solve the SLEs from the previous problem using the Gauss elimination method.

39. Solve the SLEs

$$\text{(a)} \begin{cases} x + 2y + z = 3 \\ 3x + 2y + z = 3 \\ x - 2y - 5z = 1 \end{cases}, \quad \text{(b)} \begin{cases} x + 2y + 3z - 4v = 0 \\ 2x - y + 3z - 2v = 2 \\ 3x + 4z + 2v = -1 \end{cases}, \quad \text{(c)} \begin{cases} x + y + 2z - v = -1 \\ 2x + y + 3z + v = 3 \\ 3x + y - z - 2v = -4 \end{cases}$$

40. (a) Show that the SLE

$$\begin{cases} x + 2y + z + 4v = 1 \\ 2x + y + 3v = 3 \\ -x + 2z + v = 1 \\ 2x + y + 3z + 6v = 6 \end{cases}$$

is inconsistent.
(b) Find all values of the parameters a, b, c, d such that the SLE

$$\begin{cases} x + 2y + z + 4v = a \\ 2x + y + 3v = b \\ -x + 2z + v = c \\ 2x + y + 3z + 6v = d \end{cases}$$

is consistent.
(c) For the SLE above, in case it is consistent, determine the dimension of the set of its solutions.

41. Decompose the rational functions into real partial fractions:

$$\text{(a)}\ \frac{x^3 + 2x^2 + 3x - 4}{(x^2 + x + 1)(x^2 + 2x + 5)}, \quad \text{(b)}\ \frac{x^5 + x^4 - 2x^2 + 1}{(x^2 + x + 1)(x^2 + x + 2)(x^2 + 2x + 5)},$$

$$\text{(c)}\ \frac{x^3 + 2x^2 + 3x - 4}{(x^2 + x + 1)^2(x^2 + 2x + 5)}.$$

42. Find the ranks of the matrices:

$$\text{(a)}\ \begin{pmatrix} 3 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}, \quad \text{(b)}\ \begin{pmatrix} -1 & 2 \\ 1 & -2 \\ -4 & 8 \end{pmatrix}, \quad \text{(c)}\ \begin{pmatrix} 2 & 0 & 2 & 4 & 6 \\ 1 & 2 & 0 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 4 & 1 & 2 & 1 & -2 \end{pmatrix}.$$

Compare the ranks with the dimensions of the matrices.

Analytic geometry in $\mathbb{R}^2$ and $\mathbb{R}^3$.

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43. A line ℓ and a point M on the plane are given. Write the parametric and the normal forms of equation for a line which contains M and is

(1) parallel to ℓ; (2) orthogonal to ℓ

in the following cases:

a) $M(2, -3)$, $\ell : 2x - 3y + 5 = 0$;

b) $M(1, -2)$, $\ell : 5x - y + 3 = 0$;

c) $M(4,-1), \quad \ell : -3x + y + 2 = 0.$

44. For the triangle ABC with $A(-2,3)$, $B(4,1)$, $C(6,-5)$, write the the parametric and the normal forms of equation for a line which contains

a) the median containing the vertex A;

b) the bisector containing the vertex A;

c) the altitude containing the vertex A.

45. The middle points of the sides of a triangle are $M_1(2,3)$, $M_2(-1,2)$ i $M_3(4,5)$. Find equations of the sides of the triangle.

46. Write an equation of the line such that the point $P(2,3)$ is the orthogonal projection of the origin on this line.

47. Calculate, if it is possible, $\vec{v} \cdot \vec{w}$ and $\vec{v} \times \vec{w}$ for

a) $\vec{v} = (1,2,3), \vec{w} = (-1,-2,3)$;

b) $\vec{v} = (2,0,1), \vec{w} = (1,2,0)$;

c) $\vec{v} = (2,1), \vec{w} = (1,2)$;

d) $\vec{v} = (2,0,1), \vec{w} = (1,2)$.

48. The lengths of vectors $\vec{v}$ and $\vec{w}$ are equal to 2 and 3, respectively. Knowing that $\vec{v} \cdot \vec{w} = -1$, calculate

a) $(\vec{v} + 2\vec{w}) \cdot (2\vec{v} - \vec{w})$;

b) the angle between $\vec{v} + \vec{w}$ and $\vec{v} - \vec{w}$.

49. Knowing that $\vec{v} \times \vec{w} = (-1,2,1)$, calculate

a) $\vec{w} \times \vec{v}$;

b) $(\vec{v} + 2\vec{w}) \times (2\vec{v} - \vec{w})$.

50. Knowing that $\vec{v} \cdot \vec{w} = -\sqrt{2}$ and $\vec{v} \times \vec{w} = (1,2,1)$, calculate the angle between $\vec{v}, \vec{w}$.

51. Find the values of the parameters t, s for which the vectors $\vec{v} = (2-2t, 2, -4)$ and $\vec{w} = (1, 3-s, 1)$ are parallel.

52. Find the values of the parameter t for which vectors $\vec{v} = (2-2t, 2, -4)$ and $\vec{w} = (1, 3-t, 1)$ are orthogonal.

53. Compute the area of the parallelogram spanned by vectors $\vec{v} = (2,2,-1)$ and $\vec{w} = (1,3,2)$.

54. Compute the area of the triangle with vertices $A = (1,0,1), B = (2,0,4)$ and $C = (0,1,1)$.

55. For the triangle from the previous problem calculate all the altitudes.

56. Compute the volume of the parallelepiped spanned by vectors $\vec{u} = (2, 2, -4)$ $\vec{v} = (1, 2, 0)$ and $\vec{w} = (1, 3, 1)$.

57. Compute the volume of the tetrahedron with vertices $A = (0, 1, 0), B = (1, 1, 2), C = (0, 2, 1)$ and $D = (3, 2, -1)$.

58. For the tetrahedron from the previous problem compute the altitude through the vertex D.

59. Find normal and parametric equations of the plane

(a) through the points $P = (1, 2, 1), Q = (2, 1, 5)$ and $C = (3, 0, 1)$;

(b) through the point $P = (-2, 3, 2)$ and including the Ox axis;

(c) through the point $P = (1, 0, 1)$ and orthogonal to the Oy axis.

60. Explain why the parametric equations

$$\begin{cases} x = 2 + t \\ y = 1 + t \\ z = -1 + 3t \end{cases} \quad \text{and} \quad \begin{cases} x = 2t \\ y = -1 + 2t \\ z = -7 + 6t \end{cases}$$

describe the same line.

61. Do the parameteric equations

$$\begin{cases} x = 2 + 3t + s \\ y = 1 + t + 2s \\ z = -1 + t - s \end{cases} \quad \text{and} \quad \begin{cases} x = 5 + 4t + 2s \\ y = 2 + 3t + 4s \\ z = -2s \end{cases}$$

describe the same plane? Justify your answer.

62. Find a parametric equation of the plane given by the equation $x + 2y - z + 5 = 0$.

63. Find a normal equation of the plane given by the parametric equation

$$\begin{cases} x = 2 + t + 2s \\ y = 1 + 2t + s \\ z = 3 + t - s \end{cases}$$

64. Find a parametric equation of the line in which two planes

$$\begin{cases} x + 2y + z + 3 = 0 \\ 2x - y + z + 5 = 0 \end{cases}$$

intersect each other.

65. Find the intersection point of the line $l : x = t, y = 1 + 2t, z = 3 + t$ and the plane $\pi : x + 2y - z - 3 = 0$.

66. For the point $P = (1, 0, 1)$ and the plane $\pi : x + 2y - z + 3 = 0$, find

(a) the projection of P on π;

(b) the distance from P to π;

(c) the point, symmetric to P with respect to π.

67. For the point $P = (1, 2, 3)$ and the line $l : x = 2t, y = 1 - t, z = -2 + 3t$, find

(a) the projection of P on l;

(b) the distance from P to l;

(c) the point, symmetric to P with respect to l.

68. Find the distance between two parallel lines

$$\begin{cases} x + y + z + 2 = 0 \\ 2x - y + z + 5 = 0 \end{cases} \quad \text{and} \quad \begin{cases} x + y + z + 2 = 0 \\ 2x - y + z + 7 = 0 \end{cases}$$

69. A line ℓ and a point P on the plane are given. Find the point Q, which is the projection of P on ℓ, and the point R, which is symmetric to P w.r.t. ℓ

a) $P(-6, 4)$, $\quad \ell : 4x - 5y + 3 = 0$;

b) $P(-5, 13)$, $\quad \ell : 2x - 3y - 3 = 0$;

c) $P(-8, 12)$, $\quad \ell$ contains $M_1(2, -3), M_2(-5, 1)$;

d) $P(8, -9)$, $\quad \ell$ contains $M_1(3, -4), M_2(-1, -2)$.

70. Check if the lines ℓ_1 and ℓ_2 are parallel. For parallel lines find the distance between them. For non-parallel lines find the acute angle between them.

a) $\ell_1 : x + y + 3 = 0, \ell_2 : \begin{cases} x = 1 - t, \\ y = 2 + t, \end{cases}$;

b) $\ell_1 : 2x - y + 1 = 0, \ell_2 : \begin{cases} x = 1 + t, \\ y = 2 - t, \end{cases}$.

Change of a basis. Linear transformations

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71. Is $(1, -1, 2), (2, 1, 0), (2, 0, -1)$ a basis in $\mathbb{R}^3$? If yes, give the coordinates of the vector $(2, 3, 4)$ in this basis.

72. Is $(1, -1, 1), (1, 1, 0), (0, 1, 1)$ a basis in $\mathbb{R}^3$? If yes, give the coordinates of the vector $(-2, 0, 3)$ in this basis.

73. Is $(1,-1,1),(1,1,0),(1,0,1)$ a basis in $\mathbb{R}^3$? If yes, give the coordinates of the vector $(-3,0,2)$ in this basis.

74. Is $(1,-1,1),(1,1,0)$ a basis in $\mathbb{R}^3$? If yes, give the coordinates of the vector $(-2,0,2)$ in this basis.

75. Is $(-2,0,2)$ a linear combination of $(1,-1,1),(1,1,0)$? If yes, with which coefficients?

76. Is $(-2,0,1)$ a linear combination of $(1,-1,1),(1,1,0)$? If yes, with which coefficients?

77. Let the linear mapping of $\mathbb{R}^2$ be given by $T(x,y)=(2x+y,x-y)$. Find its matrices in the standard basis $B=\{e_1,e_2\}$ and in the basis $B'=\{v_1,v_2\}$ given by $v_1=(1,1),v_2=(1,-1)$.

78. The linear mapping of $\mathbb{R}^2$ transforms the vector $(1,2)$ to $(-1,1)$, and the vector $(2,1)$ to $(3,1)$. Write the matrix of this mapping in the standard basis in $\mathbb{R}^2$.

79. For the linear mapping of $\mathbb{R}^2$ which corresponds to rotation clockwise around the origin by the angle α composed with the reflection with respect to Ox axis, write the matrix of this mapping in the standard basis in $\mathbb{R}^2$.

80. For the linear mapping of $\mathbb{R}^2$ which corresponds to reflection with respect to

(a) the Oy axis;

(b) the line $y+x=0$;

(c) the line $3y-4x=0$,

write the matrices of these mappings in the standard basis in $\mathbb{R}^2$.

81. For the linear mapping of $\mathbb{R}^3$ which corresponds to reflection with respect to

(a) the Oz axis;

(b) the Oyz plane;

(c) the plane $x+2y-3z=0$,

write the matrices of these mappings in the standard basis in $\mathbb{R}^3$.

82. For the linear mappings of $\mathbb{R}^3$ which corresponds to rotation counter-clockwise around the Oy and Oz axes by the angle α, write the matrices of these mappings in the standard basis in $\mathbb{R}^3$. For which values of α these mappings commute?

83. Write the matrices in the standard basis in $\mathbb{R}^3$ of the rotation counter-clock wise by angle $\frac{2\pi}{3}$ around the line $x=y=z$.

Eigenvalues and eigenvectors. Diagonalization and the Jordan normal form of a matrix

84. Determine the real eigenvalues and eigenvectors of the following matrices:

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & -2 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

85. Find the complex eigenvalues and eigenvectors of the following matrices:

$$\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

86. Find the eigenvalues and eigenvectors of the following linear mappings:

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T(x, y) = (x - 2y, x + y)$;

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T(x, y, z) = (2z, x, y)$;

(c) $T : \mathbb{C}^3 \to \mathbb{C}^3$, where $T(x, y, z) = (2z, x, y)$;

(d) $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T(x, y, z) = (2x + y, y - z, z)$.

In each of the above cases, answer whether the mapping has an eigenbasis.

87. For the matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix},$$

(a) determine its eigenvalues and eigenvectors;

(b) check if the matrix is diagonalizable;

(c) give geometric interpretation for the linear transformation given by this matrix

(d) calculate A^{10}, e^A.

88. For the matrix

$$A = \begin{pmatrix} -4 & 6 & -10 \\ -1 & 1 & -2 \\ 1 & -2 & 3 \end{pmatrix},$$

(a) determine its eigenvalues and eigenvectors;

(b) check if the matrix is diagonalizable;

(c) give geometric interpretation for the linear transformation given by this matrix

(d) calculate A^{10}, e^A.

89. For the matrix

$$A = \begin{pmatrix} 3 & 1 & -1 \\ -4 & -2 & 1 \\ 8 & 2 & -3 \end{pmatrix},$$

(a) determine its eigenvalues and eigenvectors;

(b) check if the matrix is diagonalizable;

(c) give geometric interpretation for the linear transformation given by this matrix

(d) calculate A^{10}, e^A.

90. Diagonalize the real matrices

$$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & -4 \\ 0 & -1 & 0 \\ 2 & -4 & 5 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} -5 & 0 & -2 \\ 4 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix},$$

91. For the matrix

$$A = \begin{pmatrix} 5 & 3 & -3 \\ -8 & -6 & 2 \\ 4 & 4 & 1 \end{pmatrix},$$

(a) determine its eigenvalues and eigenvectors;

(b) for each eigenvalue determine its algebraic multiplicity and geometric multiplicity;

(c) explain why the matrix is not diagonizable;

(d) determine its generalized eigenvectors;

(e) write the decomposition of the matrix to the Jordan normal form.

92. For the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 1 & -2 \\ 3 & -1 & -1 \end{pmatrix},$$

(a) determine its eigenvalues and eigenvectors;

(b) for each eigenvalue determine its algebraic multiplicity and geometric multiplicity;

(d) determine its generalized eigenvectors;

(e) write the decomposition of the matrix to the Jordan normal form.

93. For the matrix

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 1 & 0 \\ 2 & -2 & 2 \end{pmatrix},$$

(a) determine its eigenvalues and eigenvectors;

(b) for each eigenvalue determine its algebraic multiplicity and geometric multiplicity;

(d) determine its generalized eigenvectors

(e) write the decomposition of the matrix to the Jordan normal form.