MATHEMATICAL ANALYSIS 2

Worksheet 10.

Elements of the operational calculus. The Laplace transform, its properties and applications

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Theory outline and sample problems

This worksheet is devoted to an alternative technique, well suited for solving linear differential equations, and based on the notion of the *Laplace transform.*

Definition 1. Let function $f(t)$ be defined on the half-line $[0,\infty)$, and satisfy the following properties:

- f is continuous on $[0,\infty)$ except maybe a finite set of points where the function has limits $f(t-), f(t+)$;
- there exist $M > 0$ and $C \in \mathbb{R}$ such that

$$|f(t)| \leqslant Me^{Ct}$$

The Laplace transform of the function f is the function $F(s), s > C$ given by the formula

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{T\to\infty}\int_0^T f(t)e^{-st}\,dt, \quad s > C.$$

Alternative notation:

$$F(s) = \mathcal{L}(f(t)).$$

Sample problem **1:** Find the Laplace transforms for the functions $f(t) = e^{at}, f(t) = \sin\alpha t$,

$$f(t) = 1_{[0,1]}(t) = \begin{cases} 1, & t \in [0,1]; \\ 0, & t > 1. \end{cases}$$

Solution: We have

$$\int_0^T e^{at}e^{-st}\,ds = \frac{1}{a-s}(e^{(a-s)t})|_{t=0}^T = \frac{1}{s-a}(1-e^{-(s-a)T}) \to \frac{1}{s-a}, \quad T\to\infty,$$

here we see that the condition $s > a$ is important because the later convergence will not be true otherwise. Next, applying integration-by-parts formula twice we get

$$\begin{aligned}\int_0^T \sin(\alpha t)e^{-st}\,ds &= \left(\sin(\alpha t)\frac{1}{-s}e^{-st}\right)\Big|_{t=0}^{t=T} + \frac{\alpha}{s}\int_0^T \cos(\alpha t)e^{-st}\,ds \\ &= \left(\sin(\alpha t)\frac{1}{-s}e^{-st} - \frac{\alpha}{s^2}\cos(\alpha t)\frac{1}{-s}e^{-st}\right)\Big|_{t=0}^{t=T} - \frac{\alpha^2}{s^2}\int_0^T \sin(\alpha t)e^{-st}\,ds,\end{aligned}$$

which gives

$$\left(1+\frac{\alpha^2}{s^2}\right)\int_0^T \sin(\alpha t)e^{-st}\,ds = -\sin(\alpha T)\frac{1}{-s}e^{-sT} - \frac{\alpha}{s^2}\cos(\alpha T)e^{-sT} + \frac{\alpha}{s^2} \to \frac{\alpha}{s^2}, \quad T\to\infty,$$

and thus

$$\mathcal{L}(\sin \alpha t) = \frac{\alpha}{s^2}\left(1+\frac{\alpha^2}{s^2}\right)^{-1} = \frac{\alpha}{s^2+\alpha^2}.$$

Finally,

$$\mathcal{L}(1_{[0,1]}(t)) = \int_0^1 e^{-st}\,dt = \frac{s}{1-e^{-s}}.$$

The following fact is the key property if the Laplace transform which makes it a convenient tool for a study of differential equations.

Proposition 1. *Let function $f(t)$ and all its derivatives $f^{(k)}, k = 1, \dots, n$ have the Laplace transform. Then*

$$\mathcal{L}(f^{(n)}(t)) = s^n\mathcal{L}(f(t)) - \left(f^{(n-1)}(0+) + \dots + s^{n-1}f(0+)\right),$$

where by $f(0+), \dots, f^{(n-1)}(0+)$ we denote the right limits at 0 of corresponding functions.

This fact yields that, after passing to Laplace transforms, a differential equation can be transformed to a (simpler) algebraic one. Together with the following *identification* property, this gives a tool for solving differential equations.

Proposition 2. *Let functions $f(t), g(t)$ have the same Laplace transforms. Then $f(t) = g(t)$ all the points of continuity t.*

Sample problem **2:** Solve the Cauchy problem $y' + y = e^t, y(0) = \frac{1}{2}$.

Solution: Denote by $Y(s)$ the Laplace transform of the unknown function $f(t)$. Taking the Laplace transforms from the both sides of the differential equation, we get

$$\underbrace{sY(s) - \frac{1}{2}}_{\mathcal{L}(y'(t))} + \underbrace{Y(s)}_{\mathcal{L}(y'(t))} = \underbrace{\frac{1}{s-1}}_{\mathcal{L}(e^t)}.$$

This gives the relation

$$(s+1)Y(s) = \frac{1}{s-1} + \frac{1}{2} = \frac{s+1}{2(s-1)} \Longleftrightarrow Y(s) = \frac{1}{2(s-1)}.$$

Since $\frac{1}{2(s-1)}$ is the Laplace transform of the function $\frac{1}{2}e^t$, we get $y(t) = \frac{1}{2}e^t$.
The algorithm of solving a linear differential equation using the Laplace transform can be outlined as follows:

- I: using Proposition 1, transform the original differential equation to an algebraic equation for the Laplace transform for the unknown function;
- II: solve the algebraic equation and identify the Laplace transform for the unknown function;
- III: identify the unknown function knowing its Laplace transform.

Practically, the third step is most tricky because the general formula for inverting the Laplace transform, though available, is hard to apply in practice. Thus this step is frequently made by composing the required function $f(t)$ for the given transform $F(s)$ from the previously calculated 'blocks'. For that, one has to remember a table of the basic Laplace transforms, and to be able to combine them. The following list of basic properties enables such combinations.

Proposition 3. *(Properties of the Laplace transform)*

I: *(Linearity)* $\mathcal{L}(f(t)+g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t)), \mathcal{L}(cf(t)) = c\mathcal{L}(f(t))$;

II: *(Derivative of the transformation)* $\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(s)$;

III: *(Scale change)* $\mathcal{L}(f(ct)) = \frac{1}{c}F(\frac{s}{c})$;

(Shift of the argument of the transformation) $\mathcal{L}(e^{at}) = F(s-a)$.

The following table contains few most important Laplace transforms.

Function $f(t)$	Laplace transform $F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin \alpha t$	$\frac{\alpha}{s^2+\alpha^2}$
$\cos \alpha t$	$\frac{s}{s^2+\alpha^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin \alpha t$	$\frac{\alpha}{(s-a)^2+\alpha^2}$
$e^{at}\cos \alpha t$	$\frac{s-a}{(s-a)^2+\alpha^2}$

You can see here how the properties of the Laplace transform can be used: knowing $\mathcal{L}(\sin \alpha t)$, we can get $\mathcal{L}(e^{at}\sin \alpha t)$ by shifting the argument s by a.

Sample problem **3:** Find the functions $f(t)$ knowing their Laplace transforms $F(s)$: $\frac{1}{(s-2)^4}$, $\frac{4s}{(s^2+4)^2}$; $\frac{s+2}{s^2+4s+5}$.

Solution: Since $\mathcal{L}(e^{2t}) = \frac{1}{s-2}$ and

$$\left(\frac{1}{(s-2)}\right)' = -\frac{1}{(s-2)^2}, \quad \left(\frac{1}{(s-2)}\right)'' = 2\frac{1}{(s-2)^3}, \left(\frac{1}{(s-2)}\right)''' = 6\frac{1}{(s-2)^4},$$

$$\mathcal{L}(t^3 e^{2t}) = -6\frac{1}{(s-2)^4},$$

and the required function is $f(t) = -\frac{t^3}{6}e^{2t}$. Next, since $\mathcal{L}(\sin t) = \frac{2}{s^2+4}$ and

$$\frac{4s}{(s^2+4)^2} = -\left(\frac{2}{(s^2+4)}\right)',$$

we have

$$\mathcal{L}(t\sin t) = \frac{4s}{(s^2+4)^2},$$

and the required function is $f(t) = t\sin t$. Finally,

$$\frac{s+1}{s^2+4s+5} = \frac{s+2}{s^2+4s+4+1} = \frac{s+1}{(s+2)^2+1} = \frac{s+2}{(s+2)^2+1} - \frac{1}{(s+2)^2+1},$$

and the required function is $e^{2t}\cos t - e^{2t}\sin t$.
Let us consider two typical examples.

Sample problem **4:** Solve the following Cauchy problem for a system of differential equations:

$$\begin{cases} x'(t) = y(t) \\ y'(t) = -x(t) - y(t) \end{cases}, \quad x(0) = 1, y(0) = -1.$$

Solution: For the Laplace transforms $X(s), Y(s)$ of the unknown functions, we get the system of equations

$$\begin{cases} sX(s) - 1 = Y(s) \\ sY(s) + 1 = -X(s) - Y(s) \end{cases},$$

and solving this system we get

$$X(s) = \frac{s}{s^2+s+1}, \quad Y(s) = -\frac{s+1}{s^2+s+1}.$$

Since

$$\begin{aligned} X(s) &= \frac{s}{(s+1/2)^2+3/4} = \frac{s+1/2}{(s+1/2)^2+3/4} - \frac{1/2}{(s+1/2)^2+3/4} \\ &= \frac{s+1/2}{(s+1/2)^2+3/4} - \frac{1}{2}\frac{2}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}, \\ Y(s) &= -\frac{s+1/2}{(s+1/2)^2+3/4} - \frac{1}{2}\frac{2}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}, \end{aligned}$$

we get

$$x(t) = e^{-t/2}\cos(t\sqrt{3}/2) - \frac{1}{\sqrt{3}}e^{-t/2}\sin(t\sqrt{3}/2), \quad y(t) = -e^{-t/2}\cos(t\sqrt{3}/2) - \frac{1}{\sqrt{3}}e^{-t/2}\sin(t\sqrt{3}/2).$$

Sample problem **5:** Solve the following Cauchy problem:

$$y''' - 6y'' + 11y' - 6y = e^{4t}, \quad y(0) = y'(0) = y''(0) = 0.$$

Solution: Passing to the Laplace transforms, we get the following equation for the transform $Y(s)$ of the unknown function:

$$(s^3 - 6s^2 + 11s - 6)Y(s) = \frac{1}{s-4}.$$

The characteristic polynomial $s^3 - 6s^2 + 11s - 6$ has the roots $1, 2, 3$, hence solving this equation gives

$$Y(s) = \frac{1}{(s-1)(s-2)(s-3)(s-4)}.$$

Decomposing this rational functions into simple fractions, we get

$$Y(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} + \frac{D}{s-4},$$

where A, B, C, D are unknown constants such that

$A(s-2)(s-3)(s-4)+B(s-1)(s-3)(s-4)+C(s-1)(s-2)(s-4)+D(s-1)(s-2)(s-3)=1.$

Substituting in the above identity $s=1,2,3,4$ we get $-6A=1, 2B=1, -2C=1, 6D=1$. Hence

$$Y(s)=-\frac{1}{6}\frac{1}{s-1}+\frac{1}{2}\frac{1}{s-2}-\frac{1}{2}\frac{1}{s-3}+\frac{1}{6}\frac{1}{s-4},$$

and therefore

$$y(t)=-\frac{1}{6}e^{t}+\frac{1}{2}e^{2t}-\frac{1}{2}e^{3t}+\frac{1}{6}e^{4t}.$$

We have seen that, while solving differential equations using the Laplace transform we often obtain products of Laplace transforms (because we actually have to solve algebraic equations). This leads to the natural question: what function is transformed to a product by the Laplace transform? Below we answer this question.

Definition 2. For two functions $f(t), g(t)$ defined on $[0,\infty)$ their *convolution* $f*g$ is defined as

$$(f*g)(t)=\int_0^t f(s)g(t-s)\,ds, \quad t\geqslant 0.$$

Proposition 4. *(Borel's formula) If f, g are two functions with the Laplace transforms $F(s), G(s)$, then the Laplace transform of their convolution equals $F(s)G(s)$.*

Sample problem **6:** Find the function $f(t)$ which has the Laplace transform $F(s)=\frac{1}{s^2(s+2)}$.

Solution: Here we have two different possibilities. First: decompose the rational function $F(s)$ into simple fractions:

$$\frac{1}{s^2(s+2)}=\frac{A}{s^2}+\frac{B}{s}+\frac{C}{s+2}.$$

Then

$$A(s+2)+Bs(s+2)+Cs^2=1,$$

and substituting $s=-2$ we get $C=\frac{1}{4}$. Equating the coefficients at s^2 and s^0, we get $B=-\frac{1}{4}$ and $A=\frac{1}{2}$. Then

$$F(s)=\frac{1}{2}\frac{1}{s^2}-\frac{1}{4}\frac{1}{s}+\frac{1}{4}\frac{1}{s+2},$$

and

$$f(t)=\frac{1}{2}t-\frac{1}{4}+\frac{1}{4}e^{-2t}.$$

Second: by the Borel formula, $f(t)=t*e^{-2t}$. We can calculate the convolution directly:

$$`f(t)=t*e^{-2t}=\int_0^t se^{-2t+2s}\,ds=e^{-2t}\int_0^t se^{2s}\,ds=e^{-2t}\left[\frac{1}{2}se^{2s}\Big|_{s=0}^{s=t}-\frac{1}{2}\int_0^t e^{2s}\,ds\right]$$
$$=e^{-2t}\left[\frac{1}{2}te^{2t}-\frac{1}{4}e^{2t}+\frac{1}{4}\right]=\frac{1}{2}t-\frac{1}{4}+\frac{1}{4}e^{-2t}.$$

The difference between two possibilities we have seen above becomes visible in the next problem.

Sample problem **7:** Find the function $f(t)$ which has the Laplace transform $F(s) = \frac{s^2}{s^4+2s^2+1}$.

Solution: In this case, there is no visible sense in decomposing $F(s)$ into simple fractions, because among these fractions there will be $\frac{1}{s^4+2s^2+1}$, $\frac{s}{s^4+2s^2+1}$, and the problem remains essentially the same. On the other hand, since $F(s) = (\frac{s}{s^2+1})^2$, we have $f(t) = \cos * \cos t$, which we can calculate directly:

$$
\begin{aligned}
f(t) = \cos * \cos(t) &= \int_0^t \cos s \cos(t-s)\, ds = \int_0^t \cos s(\cos t \cos s + \sin t \sin s)\, ds \\
&= \cos t \int_0^t \cos^2 s\, ds + \sin t \int_0^t \sin s \cos s\, ds \\
&= \frac{\cos t}{2} \int_0^t (1 + \cos 2s)\, ds + \frac{\sin t}{2} \int_0^t \sin 2s\, ds \\
&= \frac{\cos t}{2} \left(t + \frac{1}{2} \sin 2t\right) + \frac{\sin t}{4} (1 - \cos 2t) \\
&= \frac{t \cos t}{2} + \frac{\cos t \sin t \cos t}{2} + \frac{\sin^3 t}{2} \\
&= \frac{t \cos t + \sin t}{2}.
\end{aligned}
$$

Problems to solve

1. Using the definition, calculate the Laplace transforms of the functions:

a) te^{-t};

b) $\cos(2t + \frac{\pi}{4})$;

c) $f(t) = t, t \in [0, 1], f(t) = 1 - t, t \in [1, 2], f(t) = 0, t > 2$.

d) $f(t) = |t - 5|$.

2. Find the function $f(t)$ knowing its Laplace transform:

a) $\frac{1}{16+s^2}$;

b) $\frac{s-4}{s^2-4}$;

c) $\frac{9}{s^2+3s}$;

d) $\frac{s+2}{s^2-4s+5}$;

e) $\frac{6}{s^2-4s-5}$;

f) $\frac{1}{s^5-4s^4+5s^3}$.

3. Using the Laplace transform, solve the following Cauchy problems:

a) $\begin{cases} x' = 12x + 5y \\ y' = -6x + y \end{cases}, x(0) = 0, y(0) = 1;$

b) $\begin{cases} x' = x - y - e^{-t} \\ y' = 2x + 3y + e^{-t} \end{cases}, x(0) = 0, y(0) = 0;$

c) $\begin{cases} x' = -x - y \\ y' = -y \\ z' = -2z \end{cases}, x(0) = 4, y(0) = 1, z(0) = 1;$

d) $\begin{cases} x' + y' = x \\ y' + z' = xz' + x' = x \end{cases}, x(0) = 1, y(0) = 1, z(0) = 1;$

e) $y'' - 2y' - 2y = \sin t, y(0) = 0, y'(0) = 1;$

f) $y^{(4)} = 16y, y(0) = 7, y'(0) = 20, y''(0) = -44, y'''(0) = 58, y(0) = 1;$

g) $y^{(6)} + 6y^{(5)} + 15y^{(4)} + 20y''' + 15y'' + 6y' + y = 0, y(0) = y'(0) = \cdots = y^{(4)}(0) = 0, y^{(5)} = 1.$

4. Calculate the convolutions:

a) $t * t^2$;

b) $t^2 * e^t$;

c) $\cos t * \sin 2t$;

d) $e^t * e^{2t}$.

5. Find the function $f(t)$ which has the Laplace transform $F(s) = \frac{s+1}{s^4+2s^2+1}$.