MATHEMATICAL ANALYSIS 2

Worksheet 11.

Linear difference equations

Theory outline and sample problems

This worksheet collects basic facts and constructions related to *difference equations*. Such equations can be seen as analogues of differential equations, with the time variable changing discretely. Respectively, many results and algorithms for difference equations are similar to those we have learned for differential equations.
We will consider functions $f(n)$, with a discrete variable n changing within a part of the set of integer numbers $\mathbb{Z}$, e.g. $n \geqslant A, n \in \{A, \dots, B\}$. For a function $f(n), n \in \{A, \dots, B\}$ its *first (forward) difference* is defined by

$$\triangle_n f = f(n+1) - f(n), \quad n \in \{A, \dots, B-1\}.$$

The higher order differences are defined iteratively:

$$\triangle_n^2 f = \triangle_n(\triangle f), \dots, \triangle_n^k f = \underbrace{\triangle_n(\dots \triangle f)}_{k \text{ differences}}.$$

The following identities have some similarity with the Newton binomial formula:

$$\triangle_n^k f = \sum_{j=0}^{k} C_k^j (-1)^{k-j} f(n+j). \tag{1}$$

General difference equation of the order k has the form

$$\triangle_n^k f = F(\triangle_n^{k-1} f, \dots, \triangle_n f, f(n)). \tag{2}$$

On the other hand, by the formula (1) such an equation can be written in the form

$$f(n+k) = G(f(n+k-1), \dots, f(n)) \tag{3}$$

with a certain function G. The latter equation is often interpreted as an *auto-regression*, i.e. a relation where each value of a function is determined by several its preceding values. The number k of the values involved into the regression is called its *order*; for $k = 1$ the name *dynamical system* is frequently used.
A function with discrete parameter is actually a sequence. Hence in what follows we use the notation f_n rather than $f(n)$. If a sequence f_n satisfies (3) for $n \geqslant A$, then, knowing the values $f_A, f_{A+1}, \dots, f_{A+k-1}$ we can calculate one by one all the further values $f_{A+k}, f_{A+k+1}, \dots$. That is, the natural analogue of the Cauchy problem for a differential equation is the initial value problem for the difference equation (2) with k initial values of the sequence specified.

Definition 1. A linear difference equation of the order k is an equation of the form

$$a_{n,k} x_{n+k} + a_{n,k-1} x_{n+k-1} + \dots + a_{n,0} x_n = b_n,$$

$\{a_{n,j}\}, j = 0, \dots, k$ are the sequences of coefficients, $\{b_n\}$ is the sequence of free terms.

Likewise to linear differential equations, we have the following two basic properties:

- the set of all solutions to non-homogeneous linear difference equation can be obtained as a sum of a fixed solution and a set of solutions to corresponding homogeneous system (as usual, classification homogeneous/nonhomogeneous here relates for $b_n \equiv 0/b_n \not\equiv 0$);
- the set of all solutions to homogeneous linear equations is a vector space of dimension k.

A basis in the space of solutions to homogeneous linear difference equation is called its *fundamental system.* There is an algorithm for building fundamental systems for linear difference equations *with constant coefficients*, i.e. of the form

$$a_k x_{n+k} + a_{k-1} x_{n+k-1} + \cdots + a_0 x_n = 0.$$

Namely, consider the *characteristic polynomial* of the equation,

$$P(\lambda) = a_k \lambda^k + \cdots + a_0,$$

and determine its roots. Each real root λ_j of multiplicity m_j generates m_j solutions

$$(\lambda_j)^n, \ldots, n^{m_j-1}(\lambda_j)^n.$$

Each pair of conjugate complex roots $\lambda_j, \overline{\lambda_j}$ of multiplicity m_j generate $2m_j$ solutions

$$\mathrm{Re}\,(\lambda_j)^n, \mathrm{Im}\,(\lambda_j)^n, \ldots, n^{m_j-1}\mathrm{Re}\,(\lambda_j)^n, n^{m_j-1}\mathrm{Im}\,(\lambda_j)^n.$$

Collecting these solutions, we get the required fundamental system.

Sample problem **1:** Find general solutions to the following difference equations:

$$x_{n+2} = 2x_{n+1} + 3x_n, \quad x_{n+2} = x_{n+1} - x_n.$$

Solution: For the first equation, the characteristic polynomial and its roots are

$$P(\lambda) = \lambda^2 + 2\lambda - 3, \quad \lambda_1 = -3, \quad \lambda_2 = 1.$$

Thus the fundamental system of solutions is $\{(-3)^n, 1^n = 1\}$. Respectively, the general solution is

$$x_n = C_1(-3)^n + C_2.$$

For the second equation, the characteristic polynomial and its roots are

$$P(\lambda) = \lambda^2 - \lambda + 1, \quad \lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \lambda_1 = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

We have by the de Moivre formula

$$(\lambda_1)^n = \rho^n(\cos n\phi + i \sin n\phi),$$

where ρ, ϕ are the modulus and the argument of λ_1. Straightforward computation gives $\rho = 1, \phi = \frac{\pi}{3}$, hence the general solution has the form

$$x_n = C_1 \cos(\frac{\pi n}{3}) + C_2 \sin(\frac{\pi n}{3}).$$

In particular, any solution to $x_{n+2} = x_{n+1} - x_n$ is periodic with period 6.
To find a solution to a non-homogeneous linear difference equation, one can use the method of unknown coefficients. Namely, to get the free term of the form

$$b_n = n^l \lambda^n \tag{4}$$

with $l = 0, 1, \dots$ and $\lambda \in \mathbb{R}$ being NOT a root of the characteristic polynomial of the homogeneous equation (the *non-resonance case*), one can take

$$x_n = A\lambda^n + \dots A_l n^l \lambda^n$$

and find $A_1, \dots, A_l$ from a system of linear equations. If λ is a root of multiplicity r (the *resonance case*), then one has to take

$$x_n = An^r \lambda^n + \dots A_l n^{l+r} \lambda^n$$

instead. Similar algorithm applies to the free terms of the form

$$b_n = n^l \lambda^n \sin(\phi n), \quad b_n = n^l \lambda^n \cos(\phi n), \tag{5}$$

here one has to separate the resonance/non-resonance cases depending on the *control variable* $\sigma = \lambda(\cos\phi + i\sin\phi)$, which should/should not be a root of the characteristic polynomial.

Sample problem **2:** Find general solution to the following difference equations:

$$x_{n+2} = x_{n+1} - x_n + \sin(\frac{\pi n}{3}).$$

Solution: This is a non-homogeneous equation with the free term of the form (5), where the control constant $\sigma = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a simple root of the characteristic polynomial of the associated homogeneous equation. Since $l = 0$ in (5), we have to look for the solution in the form

$$x_n = An\sin(\frac{\pi n}{3}) + Bn\cos(\frac{\pi n}{3}).$$

Then

$$\begin{aligned} x_{n+1} &= An\sin(\frac{\pi(n+1)}{3}) + Bn\cos(\frac{\pi(n+1)}{3}) + A\sin(\frac{\pi(n+1)}{3}) + B\cos(\frac{\pi(n+1)}{3}), \\ x_{n+2} &= An\sin(\frac{\pi(n+2)}{3}) + Bn\cos(\frac{\pi(n+2)}{3}) + 2A\sin(\frac{\pi(n+2)}{3}) + 2B\cos(\frac{\pi(n+1)}{3}), \end{aligned}$$

and

$$\begin{aligned} x_{n+2} - x_{n+1} + x_n = An &\underbrace{\left(\sin(\frac{\pi(n+2)}{3}) - \sin(\frac{\pi(n+1)}{3}) + \sin(\frac{\pi n}{3})\right)}_{=0} \\ + Bn &\underbrace{\left(\cos(\frac{\pi(n+2)}{3}) - \cos(\frac{\pi(n+1)}{3}) + \cos(\frac{\pi n}{3})\right)}_{=0} \\ &+ A\sin(\frac{\pi(n+1)}{3}) + 2A\sin(\frac{\pi(n+2)}{3}) \\ &+ B\cos(\frac{\pi(n+1)}{3}) + 2B\cos(\frac{\pi(n+2)}{3}). \end{aligned}$$

Thus

$$\sin(\frac{\pi n}{3}) = A(\sin(\frac{\pi n}{3})\frac{1}{2} + \cos(\frac{\pi n}{3})\frac{\sqrt{3}}{2}) + 2A(\sin(\frac{\pi n}{3})(-\frac{1}{2}) + \cos(\frac{\pi n}{3})\frac{\sqrt{3}}{2})$$
$$+ B(\cos(\frac{\pi n}{3})\frac{1}{2} - \sin(\frac{\pi n}{3})\frac{\sqrt{3}}{2}) + 2B(\cos(\frac{\pi n}{3})(-\frac{1}{2}) - \sin(\frac{\pi n}{3})\frac{\sqrt{3}}{2}),$$

which gives the system of equations for the unknown coefficients A, B:

$$\begin{cases} -\frac{1}{2}A - \frac{3\sqrt{3}}{2}B = 1 \\ \frac{3\sqrt{3}}{2}A - \frac{1}{2}B = 0 \end{cases} \iff A = -\frac{1}{14}, B = -\frac{3\sqrt{3}}{14}.$$

Thus the general solution to the equation is

$$x_n = C_1\cos(\frac{\pi n}{3}) + C_2\sin(\frac{\pi n}{3}) - \frac{1}{14}n\sin(\frac{\pi n}{3}) - \frac{3\sqrt{3}}{14}n\cos(\frac{\pi n}{3}).$$

Likewise we have it for the differential equations, there is a tight connection between linear difference equations of higher order and systems of difference equations. Systems of linear difference equations, in vector notation, have general form

$$\mathbf{x}_n = \mathbf{A}(n)\mathbf{x}_{n-1} + \mathbf{b}_n.$$

Again,

- the set of all solutions to non-homogeneous linear difference equation can be obtained as a sum of a fixed solution and a set of solutions to corresponding homogeneous system;
- the set of all solutions to homogeneous linear equations is a vector space of dimension k.

To construct a fundamental system of solutions in the case of constant matrix of coefficients, one can use a version of the Euler method. Namely, if $\mathbf{v}^j$ is an eigenvector for $\mathbf{A}$ with a real eigenvalue λ_j, then

$$\mathbf{v}^j_n = (\lambda_j)^n\mathbf{v}^j$$

is a solution to the homogeneous equation. If there are two complex conjugate eigenvalues $\lambda_j, \overline{\lambda_j}$, then

$$\mathbf{v}^j_n = \mathrm{Re}\,(\lambda_j)^n\mathbf{v}^j, \quad \tilde{\mathbf{v}}^j_n = \mathrm{Im}\,(\lambda_j)^n\mathbf{v}^j$$

are two linearly independent solutions to the homogeneous equation. If the matrix $\mathbf{A}$ has an eigenbasis, collecting these solutions we get the fundamental system of solutions to the homogeneous equation.

A solution to non-homogeneous system can be found by the method of variation of unknown constant. Namely, let $\mathbf{V}(n)$ be the *fundamental matrix* of the system, i.e. the matrix which consists of the solutions from the fundamental system, taken as columns. Then $\mathbf{V}(n)$ satisfies

$$\mathbf{V}(n+1) = \mathbf{A}\mathbf{V}(n),$$

and any solution to a homogeneous system has the form

$$\mathbf{V}(n)\mathbf{c},$$

where $\mathbf{c}$ is a constant vector. To find a solution to a non-homogeneous equation, consider

$$\mathbf{x}_n = \mathbf{V}(n)\mathbf{c}_n,$$

then

$$\mathbf{x}_{n+1} = \mathbf{V}(n+1)\mathbf{c}_{n+1} = \mathbf{A}\mathbf{V}(n)\mathbf{c}_{n+1} = \mathbf{A}\mathbf{x}_{n+1} + \mathbf{V}(n+1)(\mathbf{c}_{n+1} - \mathbf{c}_n),$$

and the required equation verifies if

$$\mathbf{V}(n+1)(\mathbf{c}_{n+1} - \mathbf{c}_n) = \mathbf{b}_n \Longleftrightarrow \mathbf{c}_{n+1} - \mathbf{c}_n = \mathbf{V}(n+1)^{-1}\mathbf{b}_n.$$

Sample problem **3:** Find general solution to the following system of difference equations:

$$\begin{cases} x_{n+1} = \frac{5}{4}x_n + \frac{3}{4}y_n + 1 \\ y_{n+1} = \frac{3}{4}x_n + \frac{5}{4}y_n - 1 \end{cases}$$

Solution: The matrix of coefficients of the homogeneous equation is

$$\mathbf{A} = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix},$$

and its eigenvalues/vectors are

$$\lambda_1 = 2, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = \frac{1}{2}, \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

thus the fundamental matrix and its inverse are

$$\mathbf{V}(n) = \begin{pmatrix} 2^n & 2^{-n} \\ 2^n & -2^{-n} \end{pmatrix}, \quad \mathbf{V}(n)^{-1} = \begin{pmatrix} 2^{-n-1} & 2^{-n-1} \\ 2^{n-1} & -2^{n-1} \end{pmatrix},$$

and

$$\mathbf{c}_{n+1} - \mathbf{c}_n = \begin{pmatrix} 2^{-n-2} & 2^{-n-2} \\ 2^n & -2^n \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2^{n+1} \end{pmatrix}.$$

That is,

$$\begin{aligned} \mathbf{c}_{n+1} &= \begin{pmatrix} 0 \\ 2^{n+1} \end{pmatrix} + \mathbf{c}_n = \begin{pmatrix} 0 \\ 2^{n+1} \end{pmatrix} + \begin{pmatrix} 0 \\ 2^n \end{pmatrix} + \mathbf{c}_{n-1} = \ldots \\ &= \begin{pmatrix} 0 \\ 2^{n+1} \end{pmatrix} + \ldots \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \mathbf{c}_0 = \begin{pmatrix} 0 \\ 2^{n+2} - 2 \end{pmatrix} + \mathbf{c}_0 = \begin{pmatrix} 0 \\ 2^{n+2} \end{pmatrix} + \mathbf{c}. \end{aligned}$$

Then the general solution to the non-homogeneous equation is given by

$$\mathbf{x}_n = \mathbf{V}(n)\left[\begin{pmatrix} 0 \\ 2^{n+1} - 2 \end{pmatrix} + \mathbf{c}_0\right] = \begin{pmatrix} 2 \\ -2 \end{pmatrix} + C_1 \begin{pmatrix} 2^n \\ 2^n \end{pmatrix} + C_2 \begin{pmatrix} 2^{-n} \\ -2^{-n} \end{pmatrix}.$$

Let us consider several situations, in which the general technique developed above is perfectly applicable.

Sample problem **4:** *(Fibonacci numbers).* A farmer has 1 pair of a newly born breeding pair of rabbits; each breeding pair mates at the age of one month, and at the end of their second month they always produce another pair of rabbits; and rabbits never die, but continue breeding forever. What will be the population size of rabbits in n months?

Solution: The population at the n-th month satisfies

$$x_n = x_{n-1} + x_{n-2},$$

because the increment $x_n - x_{n-1}$ will be equal to the number of the mated pairs at the month $n-1$, which is exactly x_{n-2}. This is the homogeneous difference equation, the characteristic polynomial and its roots are

$$P(\lambda) = \lambda^2 - \lambda - 1, \quad \lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

Thus the general solution is

$$x_n = C_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + C_2 \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The initial condition $x_1 = x_2 = 1$ yields the system

$$\begin{cases} C_1 \frac{1+\sqrt{5}}{2} + C_2 \frac{1-\sqrt{5}}{2} = 1 \\ C_1 \frac{3+\sqrt{5}}{2} + C_2 \frac{3-\sqrt{5}}{2} = 1 \end{cases} \iff C_1 = -C_2 = \frac{1}{\sqrt{5}}.$$

This gives classical *Binet's formula*:

$$x_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Comment: The number $\phi = \frac{1+\sqrt{5}}{2}$ which appears Binet's formula is the famous *golden ratio* number.

Sample problem **5:** *(Ruin probability).* Two players are playing a series of games with 50% chance for each player to win in a game. The one who lost the game pays 1zł to the winner. The overall play finishes when some player looses all his money. Knowing the initial amounts of money for each player, find the chance for each of them to win/loose in the overall play.

Solution: Let the first player have A zł and the second player have B zł at the beginning. During the overall play, the money won by the 1st player can take values from $-A$ to B, and $-A$ means his overall loss, while B means his overall win. Let us denote by x_n the chance for the 1st player to win, if its current win is n, $n = -A, \ldots, B$. We actually need to calculate x_0, but instead of doing that we embed this one quantity in a series of quantities $x_n, n = -A, \ldots, B$. If $n \neq -A, B$ (the play is not yet finished), then after one game the money won by the first player can become either $n+1$ or $n-1$ with same chance 50%. Then x_n, the chance to win with the starting position n, equals

$$\underbrace{x_n}_{\text{chance to win}} = \underbrace{\frac{1}{2}x_{n-1} + \frac{1}{2}x_{n+1}}_{\text{chance to win after one play}}.$$

This is a difference equation, and the characteristic polynomial $P(\lambda) = \frac{1}{2}\lambda^2 - \lambda + \frac{1}{2}$ has multiple root $\lambda = 1$. That is, the sequence x_n should have the form

$$x_n = C_1 1^n + C_2 n 1^n = C_1 + C_2 n.$$

To find the constants, we use the identities $x_{-A} = 0, x_B = 1$ to get

$$x_n = \frac{n+A}{A+B}.$$

In particular, the chance to win in the overall play for the first player is $x_0 = \frac{A}{A+B}$, and for the second player is $\frac{B}{A+B}$.
Comment: In this example we did not have initial condition at two neighbouring points, and had instead two conditions at the ends $-A, B$ of the interval where the equation was given. Such type of problems are called *boundary value problems*, and appear quite often in difference and differential equations.

Sample problem **6:** *(Ruin time).* In the previous setting, find the mean time for the overall play to finish.

Solution: Similarly to the above solution, we introduce the quantity v_n which is the mean time, starting from the current moment, for the play to finish, if the current win of the 1st player equals 1. The boundary conditions are $v_{-A} = v_B = 0$. The mean times $v_n, n = -A+1, \dots, B-1$ satisfy

$$\underbrace{v_n}_{\text{mean time}} = \underbrace{1}_{\text{one game spent}} + \underbrace{\frac{1}{2}v_{n-1} + \frac{1}{2}v_{n+1}}_{\text{mean time, calculated after the game spent}}$$

This is a non-homogeneous equation with the fundamental system for the homogeneous part $1, n$. The free term has the form (4) with $l = 0$ and $\lambda = 1$ being a root of characteristic polynomial of multiplicity 2. Then the solution to non-homogeneous equation should be taken in the form $v_n = An^2$. We have

$$-1 = \frac{1}{2}A(n+1)^2 - An^2 + \frac{1}{2}A(n-1)^2 = A,$$

hence $A = -1$ and the general solution to the non-homogeneous equation is

$$v_n = -n^2 + C_1 + nC_2.$$

Substituting the boundary values, we get

$$\begin{cases} A^2 = C_1 - AC_2 \\ B^2 = C_1 + BC_2 \end{cases} \iff C_1 = AB, \quad C_2 = B - A,$$

and

$$v_n = -n^2 + AB + n(B-A).$$

In particular, the mean time we are looking for is

$$v_0 = AB.$$

Sample problem **7:** Using a difference equation, calculate the sum

$$S_n = 1 + 2^2 + \dots + n^2.$$

Solution: Taking a difference, we get

$$\triangle_n S = n^2.$$

The following property is quite analogous to differentiation of a power function: if $f_n = P(n)$ is a polynomial of the order k, then $\triangle_n^k f = a_k k!$, where a_k is the highest order coefficient in the polynomial $P(n)$. In particular,

$$\triangle_n^3 S = \triangle_n^2(\triangle S) = 2.$$

Thus S_n satisfies the non-homogeneous linear difference equation

$$S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n = 2.$$

The free term has the form (4) with $l = 0$ and $\lambda = 1$ being a root of characteristic polynomial of multiplicity 3. Then the solution to non-homogeneous equation should be taken in the form $s_n = An^3$. We have

$$s_{n+3} - 3s_{n+2} + 3s_{n+1} - s_n = \triangle_n^3 s = A3! = 6A,$$

hence $A = \frac{1}{3}$. Then the general solution to the hon-homogeneous equation is

$$S_n = \frac{1}{3}n^3 + C_1 + C_2 n + C_3 n^2.$$

Substituting the initial conditions $S_0 = 0, S_1 = 1, S_2 = 5$, we get $C_1 = 0$ and

$$\begin{cases} 1 = \frac{1}{3} + C_2 + C_3 \\ 5 = \frac{8}{3} + 2C_2 + 4C_3 \end{cases} \Longleftrightarrow C_2 = \frac{1}{6}, \quad C_3 = \frac{1}{2}.$$

Hence

$$S_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{(2n+1)(n+1)n}{6}.$$

Problems to solve

1. Find the general solution to the following difference equations:

a) $x_{n+2} + x_{n+1} - 2x_n = 0$;

b) $x_{n+2} + 2x_{n+1} + x_n = 0$;

c) $x_{n+3} + 3x_{n+2} - 4x_{n+1} = 0$;

d) $x_{n+3} + x_{n+2} + x_{n+1} + x_n = 0$;

e) $x_{n+4} - 5x_{n+2} + 4x_n = 0$;

f) $x_{n+4} + 8x_{n+2} + 16x_n = 0$;

g) $x_{n+7} + 2x_{n+5} - x_{n+3} - 2x_{n+1} = 0$.

2. Solve the following initial value problems:

a) $x_{n+2} - 2x_{n+1} - 2x_n = 2^n \sin n, x_0 = 0, x_1 = 1$;

b) $x_{n+2} - 8x_{n+1} + 20x_n = 20^{n/2} \sin \phi n, x_0 = 0, x_1 = 0$, where $\phi = \operatorname{arctg} \frac{1}{2}$.

3. Find the general solution to the following difference equations:

a) $\begin{cases} x_{n+1} = x_n + y_n \\ y_{n+1} = 3y_n - 2x_n \end{cases}$;

b) $\begin{cases} x_{n+1} = 2x_n + y_n + 1 \\ y_{n+1} = 3x_n + 4y_n + 2^n \end{cases}$;

c) $\begin{cases} x_{n+1} = x_n - 2y_n - z_n + 1 \\ y_{n+1} = y_n - x_n + z_n - 1 \\ z_{n+1} = x_n - z_n \end{cases}$;

d) $\begin{cases} x_{n+1} = 3x_n - y_n + z_n - 1 \; y_{n+1} = x_n + y_n + z_n \\ z_{n+1} = 4x_n - y_n + 4z_n + 1 \end{cases}$.

4. Using a proper difference equation, calculate the sum

$$S_n = 1 + 2^3 + \cdots + n^3.$$

5. A farmer has 1 pair of a newly born breeding pair of rabbits; each breeding pair mates at the age of one month, and at the end of their second month they always produce three pairs of rabbits; and rabbits never die, but continue breeding forever. What will be the population size of rabbits in n months?

6. Two players are playing a series of games with the chances for win in one game $p\%$ for the 1st player and $q\%$ for the second player. The one who lost the game pays 1zł to the winner. The overall play finishes when some player looses all his money. Knowing the initial amounts of money for each player, find

a) the chance for each of them to win/loose in the overall play;

b) the mean duration the of the overall play.

7. Two players are playing a series of games with the chances for win in one game 40% for each player, and the chance of a draw 20%. The one who lost the game pays 1zł to the winner, in the case of draw no money is paid. The overall play finishes when some player looses all his money. Knowing the initial amounts of money for each player, find

a) the chance for each of them to win/loose in the overall play;

b) the mean duration of the of the overall play.

8. *(Escape from Las Vegas)* A player staying at Las Vegas needs 100$ to book a cheapest flight home. He has 50$, and he can play a roulette, betting any amount of money and either losing it or doubling it with chances 55% and 45%, respectively. Compare the chances for success in the following two strategies: (I) make one bet 50$; (II) continue betting 1$ until either he gets 100$ or loses all the money.