MATHEMATICAL ANALYSIS 2

Worksheet 5.

Applications of double and triple integrals in geometry and physics. Cylindrical and spherical coordinates in $\mathbb{R}^3$

Theory outline and sample problems

Double integrals is a natural tool for calculation of a set of characteristics of planar and spatial figures. In what follows, D is a planar domain, i.e. a domain on the plane $\mathbb{R}^2$

Formula 1. *The area of a planar domain D is equal*

$$S(D) = \iint_D dxdy.$$

Formula 2. *Let U be a body in $\mathbb{R}^3$, bounded between the graphs of two functions $h_{lower}(x,y) \leqslant h_{upper}(x,y)$, defined on the same planar domain D. The volume of the body U is equal*

$$V(U) = \iint_D (h_{upper}(x,y) - h_{lower}(x,y))\, dxdy.$$

Formula 3. *Let Γ be the surface $z = f(x,y), (x,y) \in D$; that is, Γ is the graph of the function $f(x,y)$ with the domain D. The area of the surface Γ is equal*

$$S(\Gamma) = \iint_D \sqrt{1 + (\partial_x f(x,y))^2 + (\partial_y g(x,y))^2}\, dxdy.$$

Sample problem **1:** Calculate the volume of the ball B and the surface area of the sphere Γ of the radius R.

Solution. We can represent the ball as the body between the graphs of the functions

$$h_{upper}(x,y) = \sqrt{R^2 - x^2 - y^2}, \quad h_{lower}(x,y) = -\sqrt{R^2 - x^2 - y^2}$$

taken on the domain $D = \{(x,y) : x^2 + y^2 \leqslant R^2\}$. Thus

$$V(B) = 2 \iint_{x^2+y^2 \leqslant R^2} \sqrt{R^2 - x^2 - y^2}\, dxdy.$$

Changing the variables to polar, we calculate the volume of the ball:

$$\begin{aligned} V(B) &= 2\int_0^{2\pi} d\phi \int_0^R \sqrt{R^2 - \rho^2} \rho d\rho = \left| \begin{array}{c} \rho^2 = v \\ 2\rho d\rho = dv \end{array} \right| = 2\pi \int_0^{R^2} \sqrt{R^2 - v}\, dv \\ &= 2\pi \left(-\frac{2}{3}(R^2 - v)^{3/2} \right) \Big|_{v=0}^{v=R^2} = \frac{4\pi}{3} R^3. \end{aligned}$$

Next, the surface of the sphere consists of two equal parts which correspond to the upper and the lower semi-spheres, i.e. the graphs of the functions $h_{upper}(x,y)$, $h_{lower}(x,y)$. We have

$$\partial_x h_{upper}(x,y) = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}, \quad \partial_y h_{upper}(x,y) = -\frac{y}{\sqrt{R^2 - x^2 - y^2}},$$

and

$$1+(\partial_x h_{upper}(x,y))^2+(\partial_y h_{upper}(x,y))^2=\frac{R^2}{R^2-x^2-y^2},$$

thus

$$S(\Gamma)=2\iint_{x^2+y^2\leqslant R}\frac{R}{\sqrt{R^2-x^2-y^2}}\,dxdy.$$

Changing the variables to polar, we calculate the surface of the sphere:

$$S(\Gamma)=2\int_0^{2\pi}d\phi\int_0^R\frac{R}{\sqrt{R^2-\rho^2}}\rho\,d\rho=\left|\begin{array}{c}\rho^2=v\\ 2\rho d\rho=dv\end{array}\right|=2\pi\int_0^{R^2}\frac{R}{\sqrt{R^2-v}}\,dv$$
$$=2\pi\Big(-2R(R^2-v)^{1/2}\Big)\Big|_{v=0}^{v=R^2}=4\pi R^2.$$

The volume of a ball and the surface area of a sphere are well known and can be found in various handbooks. However, the method of the calculation explained above is quite general and allows one to consider much more sophisticated geometric shapes. Let us consider two more examples.

Sample problem **2:** Find the surface area obtained by cutting from the sphere with the radius R two cylinders $x^2+y^2\pm Rx\leqslant 0$.

Answer: $8R^2$. This is a remarkable example of a figure on the sphere, whose surface area is a rational expression, i.e. does not involve irrational coefficients like 4π in the formula for the area of the entire sphere. This is *the Viviani problem*, named by Vincentio Viviani who posed it in 1692 and proposed the construction of the required figure, but did not give a proof. The calculation below easily verifies that Viviani's construction is correct.

Solution: The same calculation as in the previous problem gives us the formula for the surface area

$$S(\Gamma)=2\iint_D\frac{R}{\sqrt{R^2-x^2-y^2}}\,dxdy,$$

where the domain D now has the form of the circle $\{x^2+y^2\leqslant R\}$ with two circular holes cutted out:

$$\{x^2+y^2+Rx\leqslant 0\}\Longleftrightarrow\{(x+R/2)^2+y^2\leqslant(R/2)^2\},\quad\{x^2+y^2-Rx\leqslant 0\}\Longleftrightarrow\{(x-R/2)^2+y^2\leqslant(R/2)^2\}.$$

Let us divide D in two symmetric parts $D_+=D\cap\{x>0\}, D_-=D\cap\{x<0\}$. Since the function under the integral is even w.r.t. x, we have

$$S(\Gamma)=4\iint_{D_+}\frac{R}{\sqrt{R^2-x^2-y^2}}\,dxdy.$$

Now, let us change the variables to polar. The domain D_+ does not have the form of a rectangle in the polar coordinates, but still we can describe it efficiently. Namely, a point $x=\rho\cos\phi, y=\rho\sin\phi$ will belong to the positive semi-circle of the radius R if $\phi\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $\rho\leqslant R$. For this point to be outside the circle $\{x^2+y^2-Rx\leqslant 0\}$, we need that

$$\rho^2\cos^2\phi+\rho^2\sin^2\phi\geqslant R\rho\cos\phi\Longleftrightarrow\rho\geqslant R\cos\phi.$$

Then, changing coordinates to polar, we get

$$\begin{aligned}
S(\Gamma) &= 4\iint_{D_+} \frac{R}{\sqrt{R^2-x^2-y^2}}\,dxdy = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \int_{R\cos\phi}^{R} \frac{R}{\sqrt{R^2-\rho^2}}\rho\,d\rho \\
&= 8\int_0^{\frac{\pi}{2}} d\phi \int_{R\cos\phi}^{R} \frac{R}{\sqrt{R^2-\rho^2}}\rho\,d\rho = \left| \begin{array}{c} \rho^2 = v \\ 2\rho d\rho = dv \end{array} \right| \\
&= 4\int_0^{\frac{\pi}{2}} d\phi \int_{R^2\cos^2\phi}^{R^2} \frac{R}{\sqrt{R^2-v}}\,dv \\
&= 4\int_0^{\frac{\pi}{2}} d\phi \Big(-2R(R^2-v)^{1/2}\Big)\Big|_{v=R^2\cos^2\phi}^{v=R^2} \\
&= 8R^2\int_0^{\frac{\pi}{2}} \sqrt{1-\cos^2\phi}\,d\phi = 8R^2\int_0^{\frac{\pi}{2}} \sin\phi\,d\phi = 8R^2(-\cos\phi)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} = 8R^2.
\end{aligned}$$

Sample problem **3:** Find the area of the part of the surface $z^2 = 2xy$, cutted off by the planes $x+y=1, x=0, y=0$.

Comment and a control question: Equation $z^2 = 2xy$ defines a hyperbolic cone, you can imagine the shape of this surface by drawing its intersections with the planes $z=c$. What are the shapes of these intersections, considered as the curves in the (x,y)-plane?

Solution: The surface contains symmetric two parts, each of them being a graph of a function: $z=\sqrt{2xy}, z=-\sqrt{2xy}$. We have for the first function

$$\partial_x z = \sqrt{\frac{y}{2x}}, \quad \partial_y z = \sqrt{\frac{x}{2y}},$$

then

$$1+(\partial_x z)^2+(\partial_y z)^2 = 1+\frac{y}{2x}+\frac{x}{2y} = \frac{2xy+y^2+x^2}{2xy} = \frac{(x+y)^2}{2xy},$$

and

$$S(\Gamma) = 2\iint_D \sqrt{\frac{(x+y)^2}{2xy}}\,dxdy = \sqrt{2}\iint_D \left(\frac{\sqrt{x}}{\sqrt{y}}+\frac{\sqrt{y}}{\sqrt{x}}\right)dxdy,$$

where the domain D is bounded by the lines $x+y=1, x=0, y=0$. Representing this domain as x-normal

$$D = \{(x,y) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1-x\},$$

we write the above double integral as an iterated one:

$$\begin{aligned}
S(\Gamma) &= \sqrt{2}\int_0^1 dx \int_0^{1-x} \left(\frac{\sqrt{x}}{\sqrt{y}}+\frac{\sqrt{y}}{\sqrt{x}}\right)dy \\
&\sqrt{2}\int_0^1 \left(2\sqrt{x}\sqrt{1-x}+\frac{2}{3}(1-x)^{\frac{3}{2}}\frac{1}{\sqrt{x}}\right)dx.
\end{aligned}$$

The calculation of the integral in x requires some mastery. First, observe that

$$\left(\frac{2}{3}(1-x)^{\frac{3}{2}}\right)' = -\sqrt{1-x}, \quad (2\sqrt{x})' = \frac{1}{\sqrt{x}},$$

hence integrating by parts gives us

$$S(\Gamma) = 4\sqrt{2}\int_0^1 \sqrt{x}\sqrt{1-x}\,dx.$$

Second, making the change of variables $x = \sin^2 t$, we have $dx = 2\sin t\cos t$, $1 - x = \cos^2 t$, hence

$$\begin{aligned} S(\Gamma) &= 4\sqrt{2}\int_0^{\frac{\pi}{2}} \sin t\cos t(2\sin t\cos t)\,dt = 2\sqrt{2}\int_0^{\frac{\pi}{2}} \sin^2 2t\,dt \\ &= \Big|2t = s\Big| = \sqrt{2}\int_0^{\pi} \sin^2 s\,ds = \frac{\sqrt{2}}{2}\int_0^{\pi}(1-\cos 2s)\,ds = \frac{\sqrt{2}}{2}\Big(1 - \frac{1}{2}\cos 2s\Big)\Big|_{s=0}^{s=\pi} = \frac{\pi}{\sqrt{2}}. \end{aligned}$$

The area of the domain D can be physically interpreted as the mass of a thin material plate of the shape D, provided that the density of the material is constant and equals 1. The integral formula for the area can be extended to the situation, where the density of the material is non-constant, and is given by a continuous function $\gamma(x, y) \geqslant 0$.

Formula 4. *The mass of a planar figure D with the density $\gamma(x, y)$ equals*

$$M(D) = \iint_D \gamma(x, y)\,dxdy.$$

In the same spirit, the following mechanical characteristics can be calculated.

Formula 5. *The center of mass of a planar figure D with the density $\gamma(x, y)$ is the point $C(D)$ with the coordinates*

$$x_C = \frac{1}{M(D)}\iint_D x\gamma(x, y)\,dxdy, y_C = \frac{1}{M(D)}\iint_D y\gamma(x, y)\,dxdy.$$

The static moments w.r.t. axes Ox, Oy are given by

$$MS_x(D) = \iint_D y\gamma(x, y)\,dxdy, \quad MS_y(D) = \iint_D x\gamma(x, y)\,dxdy.$$

Formula 6. *The moments of inertia of a planar figure D with the density $\gamma(x, y)$ w.r.t. axes Ox, Oy are given by*

$$I_x(D) = \iint_D y^2\gamma(x, y)\,dxdy, \quad I_y(D) = \iint_D x^2\gamma(x, y)\,dxdy.$$

The moment of inertia w.r.t. the origin is given by

$$I_o(D) = \iint_D (x^2 + y^2)\gamma(x, y)\,dxdy.$$

Sample problem **4:** Let D be a right triangle with the right angle located at the origin and the legs located on the positive Ox, Oy semi-axes and having the lengths 1 and 2, respectively. The density function is $\gamma(x, y) = x$. Find the mass of D, its center of mass, its static moments and moments of inertia.

Solution: Parametrise the triangle D as an x-normal domain: $D = \{(x, y), 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 2 - 2x\}$. Then

$$M(D) = \int_0^1 dx \int_0^{2-2x} x\, dy = \int_0^1 x(2-2x)\, dx = (x^2 - \frac{2}{3}x^3)\Big|_0^1 = \frac{1}{3},$$

$$MS_x(D) = \int_0^1 dx \int_0^{2-2x} xy\, dy = \frac{1}{2}\int_0^1 x(2-2x)^2\, dx = 2\Big(\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\Big)\Big|_0^1 = \frac{1}{6},$$

$$MS_y(D) = \int_0^1 dx \int_0^{2-2x} x^2\, dy = \int_0^1 x^2(2-2x)\, dx = 2\Big(\frac{x^3}{3} - \frac{x^4}{4}\Big)\Big|_0^1 = \frac{1}{6},$$

and thus

$$x_C = \frac{MS_y(D)}{M(D)} = \frac{1}{2}, \quad y_C = \frac{MS_x(D)}{M(D)} = \frac{1}{2}.$$

The moments of inertia w.r.t. axes Ox, Oy are equal

$$\begin{aligned} I_x(D) &= \int_0^1 dx \int_0^{2-2x} xy^2\, dy = \frac{1}{3}\int_0^1 x(2-2x)^3\, dx \\ &= \frac{8}{3}\int_0^1 x(1 - 3x + 3x^2 - x^3)\, dx \\ &= \frac{8}{3}\Big(\frac{x^2}{2} - \frac{3x^3}{3} + \frac{3x^4}{4} - \frac{x^5}{5}\Big)\Big|_0^1 = \frac{8}{3}\frac{1}{20} = \frac{2}{15}, \\ I_y(D) &= \int_0^1 dx \int_0^{2-2x} x^3\, dy = 2\int_0^1 x^3(1-x)\, dx \\ &= 2\Big(\frac{x^4}{4} - \frac{x^5}{5}\Big)\Big|_0^1 = \frac{1}{10}. \end{aligned}$$

Then the moment of inertia w.r.t. the origin is

$$I_o(D) = I_x(D) + I_y(D) = \frac{2}{15} + \frac{1}{10} = \frac{7}{30}$$

The above formulae have natural extension from the planar (thin) figures D to spatial bodies U. Such extensions will involve triple integrals over U, which are treated in the same fashion with the double integrals we have studied so far. Let us briefly outline corresponding definitions and main facts. The triple integral

$$\iiint_U f(x, y, z)\, dxdydz$$

is defined as a limit of integral sums, and is typically calculated by transforming it to an iterated integral. The body U is called xy-normal, if it can be represented in the form

$$U = \{(x, y, z) : (x, y) \in D, h_{lower}(x, y) \leqslant z \leqslant h_{upper}(x, y)\}$$

with some domain D and functions $h_{lower}(x, y), h_{upper}(x, y)$ (That is, the body U is bounded between the graphs of two functions, likewise to the one in Formula 2). For an xy-normal body U and continuous function $f(x, y, z)$, the triple integral can be written as

$$\iiint_U f(x, y, z)\, dxdydz = \iint_D \left[\int_{h_{lower}(x,y)}^{h_{upper}(x,y)} f(x, y, z)\, dz\right] dxdy = \iint_D dxdy \int_{h_{lower}(x,y)}^{h_{upper}(x,y)} f(x, y, z)\, dz. \tag{1}$$

The following formula for the volume of a body is analogous to Formula 1; for xy-normal domains, it is obviously equivalent to Formula 2.

Formula 7. *The volume of a body U is equal*

$$V(U) = \iiint_U dxdydz.$$

The following formulae give spatial versions of the Formulae 4 – 6.

Formula 8. *The mass of a body U with the density $\gamma(x,y,z)$ equals*

$$M(U) = \iiint_U \gamma(x,y,z)\, dxdydz.$$

Formula 9. *The center of mass of a body U with the density $\gamma(x,y,z)$ is the point $C(U)$ with the coordinates*

$$x_C = \frac{1}{M(U)} \iiint_U x\gamma(x,y,z)\, dxdydz, \quad y_C = \frac{1}{M(U)} \iiint_U y\gamma(x,y,z)\, dxdydz,$$

$$z_C = \frac{1}{M(U)} \iiint_U z\gamma(x,y,z)\, dxdydz.$$

The static moments w.r.t. planes Oxy, Oxz, Oyz are given by

$$MS_{xy}(U) = \iiint_U z\gamma(x,y,z)\, dxdydz, \quad MS_{xz}(U) = \iiint_U y\gamma(x,y,z)\, dxdydz,$$

$$MS_{yz}(U) = \iiint_U x\gamma(x,y,z)\, dxdydz$$

Formula 10. *The moments of inertia of a body U with the density $\gamma(x,yz)$ w.r.t. axes Ox, Oy, Oz are given by*

$$I_x(U) = \iiint_U (y^2+z^2)\gamma(x,y,z)\, dxdydz, \quad I_y(U) = \iiint_U (x^2+z^2)\gamma(x,y,z)\, dxdydz,$$

$$I_z(U) = \iiint_U (x^2+y^2)\gamma(x,y,z)\, dxdydz.$$

The moment of inertia w.r.t. the origin is given by

$$I_o(U) = \iiint_U (x^2+y^2+z^2)\gamma(x,y,z)\, dxdydz.$$

Sample problem **5:** Find the mass of the sphere U with the radius R and the density function $\gamma(x,y,z) = x^2+y^2+z^2$.

Solution: We can represent the body U as xy-normal with $D = \{x^2+y^2 \leqslant R\}$ and

$$h_{lower}(x,y) = -\sqrt{R^2-x^2-y^2}, \quad h_{upper}(x,y) = \sqrt{R^2-x^2-y^2}.$$

Then by (1) we have

$$M(U) = \iiint_U (x^2+y^2+z^2)\, dxdydz = \iint_D \left(2(x^2+y^2)\sqrt{R^2-x^2-y^2} + \frac{2}{3}(\sqrt{R^2-x^2-y^2})^3\right) dxdy$$

Changing the variables to polar we transform the disk D into a rectangle, which gives

$$M(U) = \int_0^{2\pi} d\phi \int_0^R \Big(2\rho^2 + \frac{2}{3}(R^2-\rho^2)\Big)\sqrt{R^2-\rho^2}\rho\, d\rho = 2\pi \int_0^R \Big(\frac{2}{3}R^2 + \frac{4}{3}\rho^2\Big)\sqrt{R^2-\rho^2}\rho\, d\rho$$

Change the variables $\rho = R\sin t$, then $d\rho = R\cos t$ and

$$\begin{aligned} M(U) &= 2\pi R^5 \int_0^{\frac{\pi}{2}} \Big(\frac{2}{3} + \frac{4}{3}\sin^2 t\Big)\sin t\cos^2 t\, dt = 2\pi R^5 \int_0^{\frac{\pi}{2}} \Big(2 - \frac{4}{3}\cos^2 t\Big)\sin t\cos^2 t\, dt \\ &= 2\pi R^5 \int_0^{\frac{\pi}{2}} \Big(2\cos^2 t - \frac{4}{3}\cos^4 t\Big)\sin t\, dt = 2\pi R^5 \Big(-\frac{2}{3}\cos^3 t + \frac{4}{15}\cos^5 t\Big)\Big|_{t=0}^{t=\frac{\pi}{2}} = \frac{4\pi}{5}R^5 \end{aligned}$$

In many cases, calculation of a triple integral can be simplified if a proper change of variables is performed. The change of variables formula for triple integrals have the form

$$\iiint_U f(x,y,z)\, dxdydz = \iiint_\Omega f(F_1(u,v,w), F_2(u,v,w))\, |J_F(u,v,w)|\, dudvdw, \qquad (2)$$

where U is obtained as a one-to-one image of a body Ω under the mapping $F(u,v,w)$, which has the Jacobian matrix

$$D_F(u,v,w) = \begin{pmatrix} \partial_u F_1(u,v,w) & \partial_v F_1(u,v,w) & \partial_w F_1(u,v,w) \\ \partial_u F_2(u,v,w) & \partial_v F_2(u,v,w) & \partial_w F_2(u,v,w) \\ \partial_u F_3(u,v,w) & \partial_v F_3(u,v,w) & \partial_w F_3(u,v,w) \end{pmatrix}$$

and the Jacobian determinant

$$J_F(u,v,w) = \det \begin{pmatrix} \partial_u F_1(u,v,w) & \partial_v F_1(u,v,w) & \partial_w F_1(u,v,w) \\ \partial_u F_2(u,v,w) & \partial_v F_2(u,v,w) & \partial_w F_2(u,v,w) \\ \partial_u F_3(u,v,w) & \partial_v F_3(u,v,w) & \partial_w F_3(u,v,w) \end{pmatrix}$$

Let us introduce two coordinate systems frequently used to perform the change of variables in the triple integrals.
The *cylindrical coordinates* are $\rho \geqslant 0, \phi \in [0, 2\pi], z \in \mathbb{R}$, with the change of variables formula

$$\begin{cases} x = \rho\cos\phi \\ y = \rho\sin\phi \\ z = z \end{cases},$$

which actually means that the coordinates (x,y) are changed to the polar form, while the coordinate z remains unchanged. The Jacobian matrix and the Jacobian determinant for cylindrical change of variables are

$$D_C(\rho,\phi,z) = \begin{pmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J_C(\rho,\phi,z) = \rho.$$

That is, the particular version of the change of variables formula for the cylindrical coordinates is

$$\iiint_U f(x,y,z)\, dxdydz = \iiint_\Omega f(\rho\cos\phi, \rho\sin\phi, z)\rho\, d\rho d\phi dz. \qquad (3)$$

The *spherical coordinates* $\rho \geqslant 0, \phi \in [0, 2\pi], \psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, with the change of variables formula

$$\begin{cases} x = \rho \cos\phi \cos\psi \\ y = \rho \sin\phi \cos\psi \\ z = \rho \sin\psi \end{cases},$$

Here $\rho = \sqrt{x^2 + y^2 + z^2}$ is the distance form the point (x, y, z) to the origin. The angles ϕ, ψ have the following geometric meaning: ψ is the angle between the vector (x, y, z) and the plane Oxy, which is taken positive when $z > 0$ and negative otherwise. Taking the projection of the vector (x, y, z), we can determine then the polar coordinates of the projection in this plane; the angular coordinate of the projection is exactly the angle ϕ.
The Jacobian matrix and the Jacobian determinant for spherical change of variables are

$$D_S(\rho, \phi, \psi) = \begin{pmatrix} \cos\phi\cos\psi & -\rho\sin\phi\cos\psi & -\rho\cos\phi\sin\psi \\ \sin\phi\cos\psi & \rho\cos\phi\cos\psi & -\rho\sin\phi\sin\psi \\ \sin\psi & 0 & \rho\cos\psi \end{pmatrix},$$

$$\begin{aligned} J_S(\rho, \phi, \psi) &= \sin\psi(\rho^2\sin^2\phi\cos\psi\sin\psi + \rho^2\cos^2\phi\cos\psi\sin\psi) \\ &+ \rho\cos\psi(\rho\cos^2\phi\cos^2\psi + \rho\sin^2\phi\cos^2\psi) \\ &= \rho^2\cos\phi. \end{aligned}$$

That is, the particular version of the change of variables formula for the spherical coordinates is

$$\iiint_U f(x, y, z)\, dxdydz = \iiint_\Omega f(\rho\cos\phi\cos\psi, \rho\sin\phi\cos\psi, \rho\sin\psi)\rho^2\cos\psi\, d\rho d\phi d\psi. \quad (4)$$

Spherical coordinates are particularly convenient when the domain of integration has the shape of a part of a sphere. To illustrate the benefits, let us recalculate the Sample Problem 4.

Sample problem 4': Solve Sample Problem 4 using spherical coordinates.

Solution: In the spherical coordinates, the sphere U has the simple form $\Omega = \{(\rho, \phi, \psi) : \rho \leqslant R, \phi \in [0, 2\pi], \psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$; that is, only the radial variable is assumed to be bounded by R, and the angular variables are not restricted. Then, performing the change of variables, we get

$$\begin{aligned} M(U) &= \iiint_U (x^2 + y^2 + z^2)\, dxdydz = \iiint_{[0,R]\times[0,2\pi]\times[-\frac{\pi}{2},\frac{\pi}{2}]} \rho^2\rho^2\cos\psi\, d\rho d\phi d\psi \\ &= \frac{R^5}{5}(2\pi)\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos\psi\, d\psi = \frac{R^5}{5}(2\pi)(\sin\psi)\Big|_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} = \frac{4\pi}{5}R^5. \end{aligned}$$

We see that the calculation of the integral after the change to the spherical coordinates is much simpler than the direct computation based on representation of the sphere as a normal body.
Let us give one more example of such kind.

Sample problem 5: For the ellipsoid

$$U = \{(x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\}$$

with the density function $\gamma(x, y, z) = 1$ find the mass and the moment of inertia w.r.t. the axis Ox.

Solution: Change the variables $x = au, y = bv, z = cw$, then the Jacobian of the transformation is

$$J_F(u, v, w) = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc,$$

and

$$M(U) = \iiint_U dxdydz = abc \iiint_{u^2+v^2+w^2 \leqslant 1} dudvdw.$$

Performing the change to spherical coordinates, we get further

$$\begin{aligned} M(U) &= abc \iiint_{[0,1]\times[0,2\pi]\times[-\frac{\pi}{2},\frac{\pi}{2}]} \rho^2 \cos\psi \, d\rho d\phi d\psi \\ &= abc \left[\int_0^1 \rho^2\right] (2\pi) \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\psi \, d\psi\right] \\ &= abc\frac{1}{3}(2\pi)2 = \frac{4\pi}{3}abc. \end{aligned}$$

Similarly,

$$\begin{aligned} I_x(U) &= \iiint_U (y^2 + z^2) \, dxdydz = abc \iiint_{u^2+v^2+w^2 \leqslant 1} (b^2v^2 + c^2w^2) \, dudvdw \\ &= abc \iiint_{[0,1]\times[0,2\pi]\times[-\frac{\pi}{2},\frac{\pi}{2}]} (b^2\rho^2 \sin^2\phi \cos^2\psi + c^2\rho^2 \sin^2\psi)\rho^2 \cos\psi \, d\rho d\phi d\psi \\ &= \frac{abc}{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{2\pi} (b^2 \sin^2\phi \cos^3\psi + c^2 \sin^2\psi \cos\psi) \, d\phi \\ &= \frac{abc}{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\pi b^2 \cos^3\psi + 2\pi c^2 \sin^2\psi \cos\psi) \, d\psi \\ &= \frac{4\pi}{15} abc(b^2 + c^2). \end{aligned}$$

Problems to solve

In the problems below, $a, b, c, A, B, C, \ldots$ are fixed positive numbers.

1. Find the area of the part of the surface $z = axy$, cutted off by the cylinder $x^2 + y^2 = b$.

2. Find the area of the part of the sphere $x^2 + y^2 + z^2 = R^2$, cutted off by the cylinder $x^2 + y^2 = r^2$ $(r < R)$.

3. Find the area of the part of the cone $z^2 = x^2 + y^2$, cutted off by the cylinder $x^2 + y^2 = 2x$.

4. Find the area of the part of the surface $z^2 = \frac{1}{2}(x^2 - y^2)$, cutted off by the planes $x + y = \pm 1, x - y = \pm 1$.

5. For a given plate D and density function $\gamma(x, y)$, find the mass of D, its center of mass, its static moments and moments of inertia:

(a) D is a rectangle with the sides A, B from which a smaller rectangle with the sides a, b is cutted in the middle. All the sides are parallel to the axes, and the centers of the rectangles coincide and are located in the origin. The density function $\gamma(x, y) = 1$.

(b) D is the quarter of a circle of the radius R, located symmetrically w.r.t. the Ox axis with the center placed at the origin; the density function $\gamma(x, y) = 1$.

(c) D is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leqslant 1$, the density function $\gamma(x, y) = 1 + c(x^2 + y^2)$.

6. For a given body D with the density function $\gamma(x, y, z) = 1$, find the mass of U and its center of mass:

(a) U is a right parallelepiped with the sides A, B, C from which a smaller parallelepiped with the sides a, b, c is cutted in the middle. All the sides are parallel to the coordinate planes, and the centers of the rectangles coincide and are located at the origin.

(b) U is a cutted straight circular cone with the height h and the radii of the bases r, R. The larger base is located on the Oxy plane, with the center placed at the origin.

(c) U is the half of the sphere of the radius R centered at the origin, located at the upper half-space $\{z \geqslant 0\}$.