MATHEMATICAL ANALYSIS 2

Worksheet 6.

Ordinary differential equations of the first order. Existence, uniqueness and extending solutions. Basic methods of solving of differential equations.

Theory outline and sample problems

Differential equations are the equations, where the unknown function is related to its derivatives. Such equations appear quite commonly in various mathematical and real life models.

Examples. 1. Let certain quantity grow with the speed proportional to the quantity itself. Then this quantity, as a function $f(t)$ of time t, will satisfy

$$f'(t) = af(t), \tag{1}$$

where $a \in \mathbb{R}$ is the coefficient of proportionality. Differential equation (1) with negative a used e.g. in physics to describe the process of radioactive decay; with positive a the same equation describes the basic financial and biological effects, such as inflation and a non-restricted growth of polpulation ('the Malthusian growth model').
2. Let the acceleration of a material point be proportional to the deviation of the point from the origin. Since the acceleration is the derivative of the velocity, and thus the second derivative of the deviation, the deviation as a function $x(t)$ of time t, will satisfy

$$x''(t) = ax(t). \tag{2}$$

With $a < 0$, this is the equation of small oscillations of a pendulum, known also as 'harmonic oscillator'.
General differential equation has the form

$$F(x, y, y', \dots, y^{(k)}) = 0, \tag{3}$$

where

- x is an independent variable;
- $y = y(x)$ is the unknown function;
- $F(x, y_0, \dots, y_n)$ is a given function.

Often, when the independent variable has a natural interpretation as time, it is denoted by t and then the unknown function can be taken $x = x(t)$. The highest order k of the derivative involved into equation (3) is called the order of the differential equation.
Examples. Equation (1) is of the form

$$F(t, f, f') = 0, \quad F(t, f, f') = f' - af,$$

and thus is the differential equation of the first order. Equation (2) is of the form

$$F(t, x, x', x'') = 0, \quad F(t, x, x', x'') = x'' - ax,$$

and thus is the equation of the second order. In this worksheet, we focus on the first order differential equations, only.

The simplest type of a differential equation you are already familiar with has the form

$$y'(x) = f(x),$$

where $f(x)$ is a given function which does not depend on unknown $y(x)$. The solution to this equation is the *anti-derivative*, or the *indefinite integral* for $f(x)$,

$$y(x) = \int f(x)\,dx + C.$$

For much more general differential equations, the structure of the set of solutions shows resemblance to the simple one above. In particular, there is typically a parametric set of the solutions rather than a fixed solution. Because of such analogy, terminology 'to integrate the differential equation' (too solve), 'integral curve' (one solution), etc. is frequently used.
Generally, the question how to integrate the differential equation (i.e., to determine all its solutions) is quite complicated. In our course we will focus on particular classes of equations which an algorithm for integration. The first, practically quite important class constitutes of *separable* equations, which by the definition are the differential equations of the form

$$y'(x) = \frac{M(x)}{N(y)} \tag{4}$$

Multiplying the both sides by $N(y)$ and changing the notation $y'(x) = \frac{dy}{dx}$, we write the separable equation in its canonic form

$$N(y)\frac{dy}{dx} = M(x). \tag{5}$$

Now it is easy to solve the equation (4). Integrate (5) w.r.t. x:

$$\int N(y)\frac{dy}{dx}\,dx = \int M(x)\,dx + C, \tag{6}$$

here C is an arbitrary constant (recall that indefinite integral is determined up to a constant). Changing the variable $u = y(x)$, we transform the left hand side to

$$\int N(u)\,du\Big|_{u=y(x)}.$$

Let F, G be anti-derivatives for M, N, then (6) can be written as

$$G(y(x)) = F(x) + C.$$

Taking the inverse function to G, we get finally the solution to (4):

$$y(x) = G^{-1}(F(x) + C).$$

This argument is mathematically rigorous, but somewhat too cumbersome, thus its shortened version is often used. Write (5) in the differential form

$$N(y)\,dy = M(x)\,dx \tag{7}$$

and integrate the left-and right-hand sides w.r.t. to y and x, respectively. This will give

$$\int N(y)\,dy = \int M(x)\,dx + C \Longleftrightarrow G(y) = F(x) + C \Longrightarrow y = G^{-1}(F(x) + C).$$

The identity (7) is not easy to justify, but it is quite convenient to use. It also helps one to memorise the main issue of the method, that we have to separate all the terms of the equation dependent on x, y and collect them at the left hand side (at dy) and the right hand side (at dx).

Example. Let us solve the differential equation (1). Assume for a while that we know that $f(t) > 0$. Then, dividing (1) by f and (formally) multiplying by dt, we transform it to the form (7)

$$\frac{df}{f} = a\,dt.$$

Integrating both sides, we get

$$\ln f = at + C,$$

here we have used our knowledge that $f > 0$. Resolving this equation w.r.t. f (i.e., inverting the function $\ln f$) we get

$$f(t) = e^C e^{at}.$$

Changing the notation $\tilde{C} = e^C$, we get

$$f(t) = \tilde{C} e^{at}, \tag{8}$$

where $\tilde{C}$ is an arbitrary positive constant. Taking $\tilde{C}$ negative, we cover the case $f(t) < 0$. Finally, the identical solution $f(t) \equiv 0$ is covered by the above formula with $\tilde{C} = 0$. Thus all the solutions of (1) are given by (8) with $\tilde{C} \in \mathbb{R}$.
The above example well illustrates the fact that, for a differential equation, an entire family of solutions is available, depending on free parameter(s) which have the same spirit as the constant C in the indefinite integral. On the other hand, in various applications these parameters are uniquely specified by additional information available, e.g. about the *initial value* of the solution.

The Cauchy problem (the other name is *the initial value problem*) for the first order differential equation

$$F(x, y, y') = 0$$

it the pair of this equation and the identity $y(x_0) = x_0$, which fixes the initial value y_0 of the unknown function $y(x)$ at the given point x_0. The standard notation is

$$\begin{cases} F(x, y, y') = 0, \\ y(x_0) = x_0 \end{cases} \tag{9}$$

or

$$F(x, y, y') = 0, \quad y(x_0) = x_0.$$

Example. Let us find the solution to (1) which satisfies the initial condition $f(t_0) = f_0$. The general solution has the form

$$f(t) = Ce^{at},$$

thus by the initial condition

$$f(t_0) = Ce^{at_0} \Longleftarrow C = f_0 e^{-at_0}.$$

This gives finally
$$f(t_0) = f_0 e^{a(t-t_0)}.$$

Sample problem **1:** In a jar which contains 10 liters of water, a solution of salt is continuously added with the speed of 2 liters/min, and with the same speed the liquid from the jar is removed. One liter of the solution contains 0.3 kg of salt. How much salt will be contained in the jar in 5 minutes?

Solution: Let independent variable be time t, and the unknown function be $m(t)$ (the mass of salt in the jar at the time t). By a small time interval $\triangle t$, the jar receives $2 \times (0.3) \times \triangle t$ kg of salt from the input flow. At the same time interval, the jar looses $2 \times$ (current concentration) $\times \triangle t$ kg of salt from the output flow. The current concentration equals $\frac{m(t)}{10}$. Thus $m(t)$ should satisfy the differential equation
$$m'(t) = \lim_{\triangle t \to 0} \frac{\triangle m(t)}{\triangle t} = 2 \times (0.3) - 2 \times \frac{m(t)}{10} = 0.6 - 0.2m(t)$$

Separating the coefficients, we get
$$\frac{dm}{3-m} = 0.2dt,$$
which gives
$$3 - m(t) = Ce^{-0.2t} \Longleftarrow m(t) = 3 - Ce^{-0.2t}.$$
Taking into account the initial condition $m(0) = 0$, we get $C = 3$ and thus
$$m(t) = 3 - 3e^{-0.2t}.$$
The final answer is
$$m(5) = 3 - 3e^{-1} \approx 1,9\,\text{kg}.$$
Some types of differential equations can be reduced to separable, and then solved. For instance, for an equation of the form
$$y' = F(ax + by),$$
changing $z = ax + by$ we get a separable differential equation for the new variable $z = z(x)$:
$$z' = a + bF(ax + by).$$

Sample problem **2:** Solve
$$y' = \sin^2(x - y)$$

Solution: Taking $z = x - y$, we get
$$z' = 1 - \sin^2 z.$$
For $\sin^2 z \neq 1$, this separable equation can be written as
$$\frac{dz}{1 - \sin^2 z} = dx,$$
which after integration gives
$$\int \frac{dz}{1 - \sin^2 z} = \int \frac{dz}{\cos^2 z} = x + C \Longleftrightarrow \operatorname{tg} z = x + C \Longleftrightarrow z(x) = \operatorname{arctg}(x + C) + k\pi, \quad k \in \mathbb{Z}.$$

The exceptional case $\sin^2 z = 1$ gives solutions

$$z(x) \equiv \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}.$$

For the original function $y(x)$, we have

$$y(x) = x - z(x) = \left[\begin{array}{c} x - \operatorname{arctg}(x + C) + k\pi \\ x - \frac{\pi}{2} + k\pi, \end{array} \right. \qquad k \in \mathbb{Z}.$$

An important subclass of differential equations is given by *linear* differential equations where function F in (3) depends on unknown function $y(x)$ and its derivatives linearly. The linear differential equations of the first order have the form

$$y'(x) = a(x)y(x) + b(x). \tag{10}$$

Linear differential equation (10) is called *homogeneous* if $b(x) \equiv 0$, and *non-homogeneous* otherwise. Homogeneous equation is separable and thus easily solvable. Indeed, we have

$$\frac{dy}{y} = a(x)\,dx \Longleftrightarrow y(x) = Ce^{A(x)},$$

where $A(x)$ a fixed antiderivative for $a(x)$ (i.e., any fixed version of the indefinite integral $\int a(x)\,dx$). To solve non-homogeneous linear equation, the following *method of variation of the unknown constant* is useful. Namely, write the unknown function in the form

$$y(x) = C(x)e^{A(x)},$$

where $C(x)$ is a new unknown function. That is, we write $y(x)$ as the solution of the associated homogeneous equation, but now C is not an unknown constant, but a new unknown function. Taking the derivative and recalling that $A(x)' = a(x)$, we get

$$y'(x) = C'(x)e^{A(x)} + C(x)a(x)e^{A(x)} = C'(x)e^{A(x)} + a(x)y(x),$$

i.e. equation (10) for $y(x)$ is equivalent to equation

$$C'(x)e^{A(x)} = b(x) \Longleftrightarrow C'(x) = b(x)e^{-A(x)}$$

for $C(x)$. The latter equation is solved simply by integration:

$$C(x) = \int b(x)e^{-A(x)}\,dx + C,$$

which gives finally

$$y(x) = Ce^{A(x)} + e^{A(x)} \int b(x)e^{-A(x)}\,dx.$$

Sample problem **3:** Find the solution of the following Cauchy problem:

$$ty' + 2y = t^2 - t + 1, \quad t > 0, \quad y(1) = \frac{1}{2}.$$

Solution: Write the equation in the form

$$y' = a(t) + b(t), \quad a(t) = -\frac{2}{t}, \quad b(t) = t - 1 - \frac{1}{t}.$$

We have

$$\int a(t)\,dt = -\int \frac{2}{t}\,dt = -2\ln|t| + C = -2\ln t + C,$$

because $t > 0$. Then we can take $A(t) = -2\ln t$, which gives

$$y(t) = C(t)e^{-2\ln t} = C(t)t^{-2}$$

with

$$C(t) = \int \left(t - 1 - \frac{1}{t}\right) e^{-A(t)}\,dt = \int \left(t - 1 - \frac{1}{t}\right) t^2\,dt = \int \left(t^3 - t^2 - t\right)\,dt = \frac{t^4}{4} - \frac{t^3}{3} - \frac{t^2}{2} + C.$$

Then

$$y(t) = C(t)t^{-2} = \frac{t^2}{4} - \frac{t}{3} - \frac{1}{2} + \frac{C}{t^2}.$$

Substituting the initial value condition $y(1) = \frac{1}{2}$, we get $C = \frac{1}{12}$, which gives finally

$$y(t) = \frac{t^2}{4} - \frac{t}{3} - \frac{1}{2} + \frac{1}{12t^2}.$$

The question whether a given differential equation uniquely defines the unknown function, is answered by the following fundamental theorem.

Theorem 1. *(Picard's theorem on existence and uniqueness of the solution to a Cauchy problem) Let function $f(x,y)$ be a function defined on the rectangle $R = \{|x - x_0| \leqslant a, |y - y_0| \leqslant b\}$ and such that $f(x,y)$ is continuous and $f'_y(x,y)$ is bounded on R. Denote $m = \max_{(x,y)\imath R}|f(x,y)|$, $d = \min(a, \frac{b}{m}$. Then on the interval $[x_0 - d, x_0 + d]$ there exists a unique solution to the Cauchy problem*

$$y' = f(x,y), \quad y(x_0) = y_0. \tag{11}$$

The solution to the Cauchy problem in the Picard theorem is given constructively, as a limit of the *Picard approximations*

$$y_k(x)' = f(x, y_{k-1}(x)), \quad y_k(x_0) = y_0, \quad k \geqslant 1, \quad y_0(x) \equiv 0.$$

Thus each Cauchy problem, under proper conditions on the coefficients, can be approximatively solved by successive iteration, where each step requires an integration.

Sample problem **4:** Find two successive approximations (apart of the initial one) for the solution of the following Cauchy problem:

$$y'(x) = x - y^2, \quad y(0) = 0.$$

Solution: We have $y_0 \equiv 0$,

$$y_1(x) = \int x dx = \frac{x^2}{2} + C, \quad y_1(0) = 0 \Longrightarrow y_1(x) = \frac{x^2}{2},$$

$$y_1(x) = \int \left(x - \frac{x^2}{2}\right) dx = \frac{x^2}{2} - \frac{x^3}{6} + C, \qquad y_2(0) = 0 \Longrightarrow y_2(x) = \frac{x^2}{2} - \frac{x^3}{6}.$$

The interval, provided by Theorem (1) where the solution is uniquely defined is not the maximal possible one. While reaching the endpoints $x_0 \pm d$, one can recalculate the rectangle R using the endpoint and respective value of the solution as the new points $\tilde{x}_0, \tilde{y}_0$ and extend the solution on the larger interval. Such a procedure is called the *extension* of the solution. By means of such a procedure, the following statement can be achieved.

Theorem 2. *Let function $f(x, y)$ be continuous and have bounded derivative $f'_y(x, y)$ on each bounded subset of $\mathbb{R} \times \mathbb{R}$. Assume also that, for a given segment $[\alpha, \beta]$ the following* linear growth condition *holds true: for some $K > 0$,*

$$|f(x, y)| \leqslant K(1 + |y|), \quad x \in [\alpha, \beta], \quad y \in \mathbb{R}.$$

Then for any $x_0 \in [\alpha, \beta]$, $y_0 \in \mathbb{R}$ there exists unique solution to (11)*, defined for all $x \in [\alpha, \beta]$.*

The following two sample problems show two typical troubles caused by the violation of the conditions of the above theorems.

Sample problem **5:** Find two different solutions to the Cauchy problem

$$y' = \sqrt{|y|}, \quad y(0) = 0.$$

The function $f(x, y) = \sqrt{|y|}$ is not differentiable at the point $y_0 = 0$, hence the Picard theorem is not applicable. One obvious solution of the above Cauchy problem is $y(0) \equiv 0$. To get another one, solve the differential equation separating the variables:

$$dy = \sqrt{|y|} dx \Longleftrightarrow \frac{dy}{\sqrt{|y|}} = dx \Longleftrightarrow 2\sqrt{|y|}\mathrm{sgn}(y) = x + C \Longleftrightarrow y = \pm\frac{1}{2}(x + C)^2,$$

and substituting $C = 0$ we get another solution $y(x) = \frac{x^2}{2}$.

Sample problem **6:** Find the maximal interval where the solution to the the Cauchy problem

$$y' = y^2, \quad y(0) = y_0$$

is well defined.

The function $f(x, y) = y^2$ does not have a linear growth bound, thus it may happen that, in finite time, the solution will reach ∞; such an effect is called an 'explosion'. In the particular case, we have by separation of variables

$$dy = y^2 dx \Longleftrightarrow \frac{dy}{y^2} = dx \Longleftrightarrow \frac{1}{y} = -x + C,$$

From the initial condition,

$$\frac{1}{y_0} = C,$$

which gives $C = \frac{1}{y_0}$. Hence we have three cases:

1. $y_0 > 0$, then the solution $y(t) = (y_0^{-1} - x)^{-1}$ is well defined on the interval $(-\infty, y_0^{-1})$;
2. $y_0 < 0$, then the solution $y(t) = (y_0^{-1} - x)^{-1}$ is well defined on the interval (y_0^{-1}, ∞)
3. $y_0 = 0$, then the solution $y(t) \equiv= 0$ is well defined on entire $(-\infty, \infty)$.

Problems to solve

1. Solve the following differential equations:

- $xy\,dx + (x+1)\,dy = 0$;
- $\sqrt{y^2+1}\,dx = xy\,dy$;
- $y' = 2^{x+y}$;
- $y' - y = 2x - 3$;
- $y' = \sqrt{4x + 2y - 1}$.

2. Solve the following Cauchy problems:

- $(x^2 - 1)y' + 2xy^2 = 0, y(0) = 1$;
- $xy' + y = y^2, y(1) = \frac{1}{2}$;
- $(x + 2y)y' = 1, y(0) = -1$.

3. Find two successive approximations (apart of the initial one) for the solution of the following Cauchy problems:

(a)
$$y'(x) = y^2 + 3x^2 - 1, \quad y(1) = 1;$$

(b)
$$y'(x) = y + e^{y-1}, \quad y(0) = 1;$$

(c)
$$y'(x) = 1 + x \sin y, \quad y(\pi) = 2\pi.$$

4. A body is cooling with the velocity proportional to the difference between its temperature and the external temperature. With external temperature 20°, a body have been cooled from 100° to 60° in 10 minutes. How long will it take to cool it down to 25°?

5. A boat is resisted by the water, with the force (and thus the negative acceleration) proportional to the velocity of the boat. The boat started its motion with the velocity 1.5 meters/sec., and after 4 seconds had the velocity 1 meter/sec. When the velocity will be 1 cm/sec.? What is the total length of the path the boat can go?

6. A cylindrical tank has a hole in the bottom, where the liquid flows off with the velocity proportional to the square root of the remaining volume of the liquid. At the beginning the tank was full, and after 5 minutes it is half empty. How long will it take until the tank become completely empty?

7. Solve the Cauchy problem

$$2y' - y = 4\sin(3t), \quad y(0) = y_0.$$

Determine for which values y_0 the solution $y(t)$

(a) tends to $+\infty$ when $t \to +\infty$;

(b) tends to $-\infty$ when $t \to +\infty$;

(c) remains bounded when $t \to +\infty$.

8. Find two different solutions to the Cauchy problem

$$y' = \sqrt[3]{y}, \quad y(0) = 0.$$

9. Find the maximal interval where the solution to the following Cauchy problem is well defined:

$$y' = y^3, \quad y(0) = 1.$$