MATHEMATICAL ANALYSIS 2

Worksheet 6.

Bernoulli differential equations, homogeneous differential equations. Higher order differential equations. Linear higher order differential equations.

Theory outline and sample problems

In the theory of differential equations, it is typical that some particular class of equations is considered separately, because the particular structure of an equation form that class admits a special strategy of its solution. We have already considered two examples of such classes: separable equations and linear differential equations of the 1st order. We begin this section considering two more classes of that type.

Definition 1. The Bernoulli differential equation is an equation of the form

$$y' + p(t)y = q(t)y^r,$$

where $r \neq 0, 1$ is a real number.

In the exclusive cases $r = 0, 1$ the above equation is a linear equation of the 1st order (homogeneous for $r = 1$, non-homogeneous for $r = 0$), and thus is easy to solve. In all the other cases, the Bernoulli equation can be reduced to a linear equation by the substitution $z = y^{1-r}$. Indeed, we have $z' = (1-r)y^{-r}y'$, which gives

$$y^r z' = (1-r)y' = -(1-r)p(t)y + (1-r)q(t)y^r \iff z' + (1-r)p(t)z = (1-r)q(t).$$

That is, the new function z satisfies a 1st order linear non-homogeneous equation

$$z' + (1-r)p(t)z = (1-r)q(t).$$

Sample Problem **1:** Solve the differential equation $y' - 2ty = 2ty^2$. *Solution:* This is a Bernoulli equation with $r = 2$. Hence $1 - r = -1$ and the function $z = y^{-1}$ solves the equation

$$z' + 2tz = -2t.$$

To find z, we use the method of variation of the unknown constant (the Lagrange method):

$$z = Ce^{-t^2} - e^{-t^2}\int 2te^{t^2}\,dt = Ce^{-t^2} - e^{-t^2}\int se^s\,ds\Big|_{s=t^2} = Ce^{-t^2} - 1.$$

Then the function y is obtained by solving the equation $z = y^{-1}$:

$$y = \frac{1}{z} = \frac{1}{Ce^{-t^2} - 1}.$$

Definition 2. Homogeneous differential equation is an equation of the form

$$y' = f\left(\frac{y}{t}\right).$$

Caution: Please mind that the word 'homogeneous' here is not to be mixed with the one in 'homogeneous linear (system of) equations'.
By the substitution $y = ut$, homogeneous equation is transformed to a separable differential equation

$$tu' = f(u) - u.$$

Sample Problem **2:** Solve the differential equation $y' = \dfrac{t+y}{t}$.

Solution: This is a homogeneous equation:

$$y' = 1 + \frac{y}{t},$$

hence $u = \frac{y}{t}$ satisfies

$$tu' = 1 + u - u = 1 \Longleftrightarrow du = \frac{dt}{t} \Longleftarrow u = C + \ln|t|.$$

Hence

$$y = tu = t(C + \ln|t|).$$

Let us proceed with the study of the higher order differential equations, which have the form

$$F(x, y, y', \ldots, y^{(k)}) = 0 \tag{1}$$

with $k > 1$. The general theory of such equations is quite similar to the theory of the 1st order equations ($k = 1$), with appropriate changes. One most notable change is that, for an equation of the k-th order the initial data, used in the Cauchy problem, should involve the values of the function itself and all the derivatives of the orders $< k$. With this natural modification, the Picard theorem (on local existence and uniqueness of the solution to the Cauchy problem) and the extension procedure under the linear growth condition are available for the equations of arbitrary order. Though, the problem to solve a particular differential equation is typically more sophisticated for equations of higher orders. Below, we will discuss in details one particularly important class of higher order differential equations, which admits algorithms of a solution.

Definition 3. Linear differential equation of the order $k \geqslant 1$ is an equation of the form

$$a_k(t)y^{(k)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = b(t).$$

Likewise to the linear equations of the 1st order, or systems of systems of linear equations in the Linear Algebra, there is a sense in separating *homogeneous* and *non-homogeneous* higher order linear equations (mind the word 'linear'!), which correspond to the cases $b(t) \equiv 0$ and $b(t) \not\equiv 0$, respectively.
The following statements are quite analogous to those from the theory of systems of linear equations in the Linear Algebra.

Proposition 1.

1. *The family of solutions of a homogeneous linear differential equation is a vector space w.r.t. the point-wise operations of addition and multiplication by a constant.*

2. *Any solution to a non-homogeneous linear differential equation can be obtained as a sum of a fixed solution to the non-homogeneous equation and some solution to the associated homogeneous equation.*

A basis in the set of the solutions to a homogeneous linear differential equation is called the *fundamental system of solutions*. It is known that, provided that $a_k(x) \neq 0$, the length of any such basis coincides with the order k of the equation; that is, the dimension of the space of the solutions to a linear homogeneous differential equation equals to the order of the equation.

Proposition 2. *Let $y_1(t), \dots, y_k(t)$ be solutions to a homogeneous linear differential equations of the order k with $a_k(t) \neq 0$. For $y_1(t), \dots, y_k(t)$ to be a fundamental system of solutions it is necessary and sufficient that, at any given point t_0, the $k \times k$-matrix*

$$\begin{pmatrix} y_1(t_0) & \cdots & y_k(t_0) \\ \vdots & \ddots & \vdots \\ y_1^{(k-1)}(t_0) & \dots & y_1^{(k-1)}(t_0) \end{pmatrix}$$

is non-degenerate.

The above matrix is called the *fundamental matrix*, and its determinant is called the *Wronskian* of the system $y_1(t), \dots, y_k(t)$ at the point x_0; notation

$$W(t_0) = \begin{vmatrix} y_1(t_0) & \cdots & y_k(t_0) \\ \vdots & \ddots & \vdots \\ y_1^{(k-1)}(t_0) & \dots & y_1^{(k-1)}(t_0) \end{vmatrix}.$$

Clearly, the system $y_1(t), \dots, y_k(t)$ is fundamental if, and only if, $W(t_0) \neq 0$ for any t_0. This gives a practical tool for describing the set of solutions to a given linear differential equation. Lets consider a typical example.

Sample Problem **3:** Solve the differential equation $y'' + y = t$.

Solution: This is a non-homogeneous linear equation of the 2nd order. First, let us guess a fixed solution: taking $y_*(t)$ we get $y_*'' \equiv 0$ and thus $y_*'' + y_* = t$. Then, let us describe all the solutions to associated homogeneous equation $y'' + y = 0$. Its a direct calculation to check that $y_1(t) = \sin t$, $y_2(t) = \cos t$ solve the homogeneous equation. Their Wronskian at a point t equals

$$W(0) = \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -1 \neq 0,$$

hence $\{\sin t, \cos t$ is a fundamental system of solutions. Thus it is a basis in the set of the solutions, and any solution to the homogeneous equation has the form

$$y(t) = C_1 \sin t + C_2 \cos t, \quad C_1, C_2 \in \mathbb{R}.$$

Thus all the solutions to the original equation are given by the formula

$$y(t) = t + C_1 \sin t + C_2 \cos t, \quad C_1, C_2 \in \mathbb{R}.$$

The general solution to a linear equation of the order k depends on k parameters; see the above example where there are two free constants (parameters) C_1, C_2. While solving the Cauchy problem, we will typically find these constants using the initial conditions.

Sample Problem **4:** Solve the Cauchy problem $y'' + y = t, y(0) = 1, y'(0) = 1$.

Solution: We already know the formula for the general solution:

$$y(t) = t + C_1 \sin t + C_2 \cos t,$$

which yields

$$y'(t) = 1 + C_1 \cos t - C_2 \sin t.$$

Substituting the initial conditions, we get

$$\begin{cases} 1 = 0 + C_1 0 + C_2 1 \\ 1 = 1 + C_1 1 - C_2 0 \end{cases} \Longrightarrow C_1 = 0, C_2 = 1,$$

which gives the answer $y(t) = t + \cos t$.
In the above examples, two principal questions are hidden:

- how to find the fundamental system of solutions to a homogeneous system?
- how to find a fixed solution to a hon-homogeneous system?

In what follows, we discuss some ways to resolve these questions. The first one has a simple answer in the important partial case of a homogeneous linear equation with *constant coefficients*:

$$a_k y^{(k)} + \cdots + a_1 y' + a_0 y = 0, \tag{2}$$

where $a_0, \ldots, a_k$ are fixed numbers and $a_k \neq 0$. For the equation (2), the following *characteristic polynomial* is very useful:

$$P(\lambda) = a_k \lambda^k + \cdots + a_1 \lambda + a_0. \tag{3}$$

The polynomial (3) keeps enough information to construct a fundamental system of solutions for the original equation (2); before we discuss the general case let us give such a construction for an equation of the second order.
For $k = 2$, equation (2) and the characteristic polynomial (3) have the forms

$$ay'' + by' + cy = 0, \tag{4}$$

and

$$P(\lambda) = a\lambda^2 + b\lambda + c\lambda \tag{5}$$

respectively. Quadratic equation $P(\lambda) = 0$, also called the characteristic equation, reveals three different cases according to the value of $\Delta = b^2 - 4ac$.

Case I: $\Delta > 0$, the characteristic equation has two different real roots $\lambda_1 \neq \lambda_2$. In this case, the pair of functions

$$y_1(t) = e^{\lambda_1 t}, \quad y_2(t) = e^{\lambda_2 t}$$

gives a fundamental system of solutions to the original equation.

Case II: $\Delta < 0$, the characteristic equation has two mutually conjugate complex roots $\lambda_1 = \lambda_0 + i\omega, \lambda_2 = \lambda_0 - i\omega$. In this case, the pair of functions

$$y_1(t) = e^{\lambda_0 t} \sin \omega t, \quad y_2(t) = e^{\lambda_0 t} \cos \omega t$$

gives a fundamental system of solutions to the original equation.

Case III: $\Delta = 0$, the characteristic equation has one root λ_0 of multiplicity 2. In this case, the pair of functions

$$y_1(t) = e^{\lambda_0 t}, \quad y_2(t) = te^{\lambda_0 t}$$

gives a fundamental system of solutions to the original equation.

Note that the cases I and II are quite analogous, if we recall the complex variable notation for the trigonometric functions:

$$\sin \omega t = \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t}), \quad \cos \omega t = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}).$$

That is, in both these cases the fundamental system of solutions can be chosen in the form

$$y_1(t) = e^{\lambda_1 t}, \quad y_2(t) = e^{\lambda_2 t},$$

where λ_1, λ_2 are two different roots of the characteristic equation.

Sample Problem **5:** Solve the Cauchy problem $y'' + y' - 2y = 0, y(0) = 1, y'(0) = 1$.

Solution: The characteristic equation and the roots are

$$\lambda^2 + \lambda - 2\lambda = 0, \quad \lambda_1 = -2, \quad \lambda = 1.$$

Then the general solution to the equation has the form

$$y(t) = C_1 e^{-2t} + C_2 e^t.$$

Substituting the initial values we get

$$\begin{cases} 1 = C_1 + C_2 \\ 1 = (-2)C_1 + C_2 \end{cases} \Longrightarrow C_1 = 0, C_2 = 1,$$

hence the answer is $y(t) = e^t$.
Let us consider one more problem of that type, now with two complex roots for the characteristic polynomial.

Sample Problem **6:** Solve the Cauchy problem $y'' + y' + y = 0, y(0) = -1, y'(0) = 1$.

Solution: The characteristic equation and the roots are

$$\lambda^2 + \lambda + 1 = 0, \quad \lambda_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \lambda_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

To illustrate the 'complex exponents' technique, let us consider two different ways to complete the solution. First, write the general solution in the form

$$y(t) = C_1 e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2}t + C_2 e^{-\frac{t}{2}} \cos \frac{\sqrt{3}}{2}t,$$

then

$$y'(t) = \left(-\frac{1}{2}\right)C_1 e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2}t + \left(\frac{\sqrt{3}}{2}\right)C_1 e^{-\frac{t}{2}} \cos \frac{\sqrt{3}}{2}t + \left(-\frac{1}{2}\right)C_2 e^{-\frac{t}{2}} \cos \frac{\sqrt{3}}{2}t - \left(\frac{\sqrt{3}}{2}\right)C_1 e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2}t.$$

Substituting the initial values, we get

$$0C_1 + C_2 = -1 \Longrightarrow C_2 = -1,$$

$$\Big(\frac{\sqrt{3}}{2}\Big)C_1 + \Big(-\frac{1}{2}\Big)C_2 = 0 \Longrightarrow C_1 = \frac{C_2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}.$$

This gives the answer

$$y(t) = -\frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t - e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t.$$

Alternatively, we can write the general equation in the form

$$y(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t},$$

note that now the constants are different from those calculated above. We have

$$y'(t) = \lambda_1C_1e^{\lambda_1 t} + \lambda_2C_2e^{\lambda_2 t},$$

which gives the system

$$\begin{cases} -1 = C_1 + C_2 \\ 1 = \lambda_1C_1 + \lambda_2C_2 \end{cases} \Longrightarrow C_1 = -\frac{\lambda_2+1}{\lambda_2-\lambda_1}, C_2 = \frac{\lambda_1+1}{\lambda_2-\lambda_1}.$$

This gives the answer

$$y(t) = -\frac{\lambda_2+1}{\lambda_2-\lambda_1}e^{\lambda_1 t} + \frac{\lambda_1+1}{\lambda_2-\lambda_1}e^{\lambda_2 t},$$

which yet has to be transformed from the complex-valued notation to the real-valued one. We have

$$\frac{\lambda_2+1}{\lambda_2-\lambda_1} = \frac{\frac{1}{2}-\frac{\sqrt{3}}{2}i}{-\sqrt{3}i} = \frac{1}{2} + \frac{1}{2\sqrt{3}}i,$$

$$\frac{\lambda_1+1}{\lambda_2-\lambda_1} = \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{-\sqrt{3}i} = -\frac{1}{2} + \frac{1}{2\sqrt{3}}i,$$

$$\begin{aligned} y(t) &= -\left(\frac{1}{2} + \frac{1}{2\sqrt{3}}i\right)\left(e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t + ie^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right) + \left(-\frac{1}{2} + \frac{1}{2\sqrt{3}}i\right)\left(e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t - ie^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\right) \\ &= -\frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t - e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t. \end{aligned}$$

Now, let us proceed to the general case $k \in \mathbb{N}$. Let the characteristic polynomial to have the form

$$P(\lambda) = a_k(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_r)^{m_r},$$

where $\lambda_1, \dots, \lambda_r$ are different roots of $P(\lambda)$ which have multiplicities $m_1, \dots, m_r$, respectively. Then a fundamental system of solutions to the equation (2) can be obtained as a sets of functions obtained by the collection of:

- for each real root λ_j, the functions

$$e^{\lambda_j t}, \dots, t^{m_j-1}e^{\lambda_j t};$$

- for each pair of complex conjugate imaginary roots $\lambda_j = a_j + i\omega_j, \overline{\lambda_j} = a_j - i\omega_j$, the functions

$$e^{\lambda_j t} \sin \omega_j t,\ e^{\lambda_j t} \cos \omega_j t, \dots, t^{m_j - 1} e^{\lambda_j t} \sin \omega_j t,\ t^{m_j - 1} e^{\lambda_j t} \cos \omega_j t.$$

In the complex exponent notation, each root λ_j (probably, complex) of multiplicity m_j generates the set of solutions

$$e^{\lambda_j t}, \dots, t^{m_j - 1} e^{\lambda_j t}.$$

Sample Problem **7:** Solve the Cauchy problem $y''' - y = 0, y(0) = -1, y'(0) = 1, y''(0) = 0$.

Solution: The characteristic equation and the roots are

$$\lambda^3 - 1 = 0, \quad \lambda_1 = 1, \quad \lambda_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i, \quad \lambda_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i.$$

The general solution has the form

$$y(t) = C_1 e^{-t} + C_2 e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2} t + C_2 e^{-\frac{t}{2}} \cos \frac{\sqrt{3}}{2} t,$$

and

$$y(0) = C_1 + C_3, \quad y'(0) = -C_1 + C_2 \frac{\sqrt{3}}{2} + C_3 \left(-\frac{1}{2}\right), \quad y''(0) = C_1 - C_2 \frac{\sqrt{3}}{2} - C_3 \frac{1}{2}.$$

This gives

$$C_1 = 0, \quad C_2 = \sqrt{3}, \quad C_3 = -1,$$

and

$$y(t) = \sqrt{3} e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2} t - e^{-\frac{t}{2}} \cos \frac{\sqrt{3}}{2} t.$$

The above considerations show that, for homogeneous linear differential equations the fundamental system of solutions can be found efficiently, provided we are able to find the roots of the characteristic polynomial. This practically answer the first question posed above. For the second question, about a construction of a fixed solution to a non-homogeneous equation, there exist several possible approaches. Here we will discuss one of them, the *method of unknown coefficients*. The method of unknown coefficients applies in the case where

- the equation has the form

$$a_k y^{(k)} + \cdots + a_1 y' + a_0 y = b(t); \tag{6}$$

  that is, its homogeneous part has constant coefficients;

- the right hand side term $b(t)$ has a special form, namely, it is a linear combination of the functions of the form

$$t^j e^{at} \sin \omega t, \quad t^l e^{at} \cos \omega t, \quad j, l \leqslant m. \tag{7}$$

In this case, the candidate for being a solution is written in a prescribed form with certain known functions multiplied by unknown coefficients. Then the unknown coefficients are found from a system of linear equations. To formulate the recipe more explicitly, we have to perform a case study. There are two cases to be treated separately; this classification depends on the *control constant* $\sigma = a + i\omega$ (here a, ω are same as in (7)).

I. *(Non-resonance case).* Let $P(\sigma) \neq 0$, i.e. σ is not a root of the characteristic polynomial for the homogeneous part of the equation. Then the solution to (6) can be found in the form

$$[A_m t^m + \dots A_0]e^{at} \sin \omega t + [B_m t^m + \dots B_0]e^{at} \cos \omega t.$$

II. *(Resonance case).* Let σ be a root of multiplicity r of the characteristic polynomial for the homogeneous part of the equation. Then the solution to (6) can be found in the form

$$[A_m t^{m+r} + \dots A_0 t^r]e^{at} \sin \omega t + [B_m t^{m+r} + \dots B_0 t^r]e^{at} \cos \omega t.$$

Let us consider several examples.

Sample Problem **8:** Solve the differential equation $y'' + y = \sin 2t$.

Solution: this is a non-homogeneous linear differential equation with the characteristic polynomial of the homogeneous part

$$P(\lambda) = \lambda^2 + 1$$

and corresponding roots $\lambda_1 = i, \lambda_2 = -i$. The free term has the special form (7) with $m = 0$ and the control constant $\sigma = 2i$, which is not a root of $P(\lambda)$. Thus we should look for a solution to the non-homogeneous equation in the form

$$y(t) = A \sin 2t + B \cos 2t.$$

We have

$$y''(t) = -4A \sin 2t - 4B \cos 2t \Longleftrightarrow y''(t) + y(t) = -3A \sin 2t - 3B \cos t,$$

and from the equation on $y(t)$ we have the system of equations on the coefficients A, B :

$$\begin{cases} -3A = 1 \\ -3B = 0 \end{cases} \Longrightarrow A = -\frac{1}{3}, \quad B = 0,$$

and one solution to the non-homogeneous equation is

$$y(t) = -\frac{1}{3} \sin 2t.$$

The general solution then is given by

$$y(t) = -\frac{1}{3} \sin 2t + C_1 \sin t + C_2 \cos t.$$

The previous problem may mislead you, because the answer contains the function $\sin 2t$ only, likewise to the right hand side in the original equation. The next example shows that in general this is not the case, and we have to take into account the terms with (say) $\cos \omega t$ even if they are not present in the original equation.

Sample Problem **9:** Solve the differential equation $y'' + y = e^t \sin t$.

Solution: The characteristic polynomial is the same as above, and the control constant $\sigma = 1 + i$ is not a root. Thus we should look for a solution to the non-homogeneous equation in the form

$$y(t) = Ae^t \sin t + Be^t \cos t.$$

Since

$$(e^t \sin t)' = e^t \sin t + e^t \cos t, \quad (e^t \cos t)' = -e^t \sin t + e^t \cos t,$$
$$(e^t \sin t)'' = (e^t \sin t + e^t \cos t)' = 2e^t \cos t, \quad (e^t \cos t)'' = (-e^t \sin t + e^t \cos t)' = -2e^t \sin t,$$

we have

$$y''(t) + y(t) = 2Ae^t \cos t - 2Be^t \sin t + Ae^t \sin t + Be^t \cos t.$$

Thus from the equation on $y(t)$ we have the system of equations on the coefficients A, B :

$$\begin{cases} A - 2B = 1 \\ 2A + B = 0 \end{cases} \Longrightarrow A = \frac{1}{5}, \quad B = -\frac{2}{5},$$

and one solution to the non-homogeneous equation is

$$y(t) = \frac{1}{5}e^t \sin t - \frac{2}{5}e^t \cos t$$

The general solution then is given by

$$y(t) = \frac{1}{5}e^t \sin t - \frac{2}{5}e^t \cos t + C_1 \sin t + C_2 \cos t.$$

Finally, let us consider one example of the resonance case.

Sample Problem **10:** Solve the differential equation $y'' + y = \sin t$.

Solution: The characteristic polynomial is the same as above, and the control constant $\sigma = i$ is its root of multiplicity 1. Thus we should look for a solution to the non-homogeneous equation in the form

$$y(t) = At \sin t + Bt \cos t.$$

We have

$$(t \sin t)' = t \cos t + \sin t, \quad (t \cos t)' = -t \sin t + \cos t,$$
$$(t \sin t)'' = -t \sin t + 2 \cos t, \quad (t \cos t)' = -t \cos t - 2 \sin t.$$

Then

$$y''(t) + y(t) = A(-t \sin t + 2 \cos t) + B(-t \cos t - 2 \sin t) + At \sin t + Bt \cos t = 2A \cos t - 2B \sin t$$

Thus from the equation on $y(t)$ we have the system of equations on the coefficients A, B :

$$\begin{cases} 2A = 0 \\ -2B = 1 \end{cases} \Longrightarrow A = 0, \quad B = -\frac{1}{2},$$

and one solution to the non-homogeneous equation is

$$y(t) = -\frac{t}{2} \cos t.$$

The general solution then is given by

$$y(t) = -\frac{t}{2} \cos t + C_1 \sin t + C_2 \cos t.$$

At the end of this chapter, let us discuss shortly one type of linear differential equations with varying coefficients which, by proper change of variables, can be transformed to linear differential equations with constant coefficients.

Definition 4. The Euler differential equation is a linear differential equation of the form

$$t^k y^{(k)} + a_{k-1}t^{k-1}y^{(k-1)} + \cdots + a_1 t y'(t) + a_0 y(t) = 0, \tag{8}$$

where $a_0, \dots, a_n$ are given constants.

It is known that by a substitution $t = e^\tau$ (for $t > 0$) or $t = -e^\tau$ (for $t < 0$) equation (8) can be transformed to a linear differential equations with constant coefficients. In particular, the second order equation

$$t^2 y'' + pty' + qy = 0 \tag{9}$$

after such a change is transformed to

$$\frac{d^2y}{d\tau^2} + (p-1)\frac{dy}{d\tau} + qy = 0. \tag{10}$$

This makes it possible to construct fundamental systems of solutions for such equations.

Sample Problem **11:** Solve the Cauchy problem $t^2y'' + ty' + y = t, y(1) = 0, y'(1) = 1$.

Solution: This is a linear non-homogeneous differential equation with the homogeneous part being the Euler equation. Change the variables $t = e^\tau$ (we will look for the solution for $t > 0$), then the equation transforms to

$$y''_{\tau\tau} + y = e^\tau.$$

The characteristic polynomial is $P(\lambda) = \lambda^2 + 1$, and the fundamental system of solutions is $\sin\tau, \cos\tau$. The right hand side term has a special form (7) with $m = 0$ and the control variable $\sigma = 1$ not being a root for $P(\lambda)$. Thus a fixed solution to the equation can be found in the form

$$y(\tau) = Be^\tau,$$

(note that $e^\tau = e^\tau \cos 0t$, and the function $e^t \sin 0\tau \equiv 0$ does not appear) and we easily find that $B = \frac{1}{2}$. Hence we have the general solution

$$y(\tau) = \frac{1}{2}e^\tau + C_1 \sin\tau + C_2\cos\tau.$$

Changing the variables back $\tau = \ln t$, we get the general solution

$$y(t) = \frac{1}{2}t + C_1 \sin\ln t + C_2 \cos\ln t.$$

Then

$$y'(t) = \frac{1}{2} + C_1\frac{\cos\ln t}{t} - C_2\frac{\sin\ln t}{t},$$

and by the initial condition

$$\begin{cases} \frac{1}{2} + C_2 = 0 \\ \frac{1}{2} + C_1 = 1 \end{cases} \Longrightarrow C_1 = \frac{1}{2}, \quad C_2 = -\frac{1}{2},$$

and the answer is

$$y(t) = \frac{1}{2}t + \frac{1}{2}\sin\ln t - \frac{1}{2}\cos\ln t, \quad t > 0.$$

At the end, let us give an observation which simplifies a lot calculations in the problems related to the linear differential equations with constant coefficients. We have seen that it is often needed to calculate the higher order derivatives of the functions of the form $e^{at}\sin\omega t, e^{at}\cos\omega t$, or the same functions multiplied by t^r. Such calculations appear either when the solution to the Cauchy problem is found (we have then to calculate the derivatives up to the order $k-1$ of the functions from the fundamental system at the given point), or the non-homogeneous equation is solved by the method of unknown coefficients. Each of the functions e^{at}, $\sin\omega t$, $\cos\omega t$, t^r can be easily differentiated to arbitrary order:

$$(e^{at})^{(k)} = a^k e^{at}, \quad (t^r)^{(k)} = k(k-1)\dots(k-r-1)t^{k-r},$$

$$(\sin\omega t)^{(k)} = \begin{cases} (-\omega^2)^l \sin\omega t, & k = 2l \\ (-\omega^2)^l \omega\cos\omega t, & k = 2l+1, \end{cases} \qquad (\cos\omega t)^{(k)} = \begin{cases} (-\omega^2)^l \cos\omega t, & k = 2l \\ -(-\omega^2)^l \omega\sin\omega t, & k = 2l+1. \end{cases}$$

To calculate the higher order derivatives of a product, the following *Leibniz formula* is quite useful:

$$(fg)^k = \sum_{j=0}^{k} C_k^j f^{(j)} g^{(k-j)}, \tag{11}$$

where $C_k^j = \frac{k!}{j!(k-j)!}$ is the binomial coefficient. For $k = 2, 3$ this formula gives

$$(fg)'' = f''g + 2f'g' + g'', \quad (fg)''' = f'''g + 3f''g' + 3f'g'' + g'''.$$

For a product of more terms the following extension of the Leibniz formula is available:

$$(f_1 \dots {}_p)^{(k)} = \sum_{j_1+\dots j_p=k} \frac{k!}{j_1! \dots j_p!} (f_1)^{(j-1)} \dots (f_p)^{j_p}.$$

For $k = 2, p = 3$ this formula gives

$$(fgh)'' = f''gh + fg''h + fgh'' + 2f'g'h + 2f'gh' + 2fg'h'.$$

Example.

$$\begin{aligned}
(e^{2t}\sin 3t)'' &= 4e^{2t}\sin 3t + 2(2e^{2t})(3\cos 3t) + e^{2t}(-9\sin 3t) \\
&= -5e^{2t}\sin 3t + 6e^{2t}\cos 3t, \\
(e^{2t}\sin 3t)'' &= 4e^{2t}\sin 3t + 3(4e^{2t})(3\cos 3t) + 3(2e^{2t})(-9\sin 3t) + e^{2t}(-27\cos 3t) \\
&= -50e^{2t}\sin 3t - 3e^{2t}\cos 3t, \\
(te^{2t}\sin 3t)'' &= t(e^{2t}\sin 3t)'' + 2(e^{2t}\sin 3t)' \\
&= -5te^{2t}\sin 3t + 6te^{2t}\cos 3t + 4e^{2t}\sin 3t + 6e^{2t}\cos 3t.
\end{aligned}$$

Problems to solve

1. Find the general solution to the following differential equations:

a) $y'' + y' - 2y = 0$;

b) $y'' + 2y' + y = 0$;

c) $y''' + 3y'' - 4y' = 0$;

d) $y''' + y'' + y' + y = 0$;

e) $y^{(4)} - 5y'' + 4y = 0$;

f) $y^{(4)} + 8y'' + 16y = 0$;

g) $y^{(7)} + 2y^{(5)} - y''' - 2y' = 0$.

2. Solve the following Cauchy problems:

a) $y'' - 4y' + 3y = 0, y(0) = 7, y'(0) = 16$;

b) $2y'' + 4y' - 6 = 0, y(0) = 4, y'(0) = 0$;

c) $y''' - y'' + y' - y = 0, y(0) = 0, y'(0) = 0, y''(0) = 1$.

3. Find the general solution to the following differential equations:

a) $y'' - 2y' - 2y = e^t + t\cos t$;

b) $y'' - 8y' + 20y = 5te^{4t}\sin x$;

c) $y'' - 2y' + 5y = 2te^t + e^t \sin 2t$;

4. Solve the Cauchy problem $y''' + y' = \sin t + t\sin t, y(0) = 1, y'(0) = 0, y''(0) = 0$.

5. Specify the particular class of differential equation and use proper substitution in order either to find a general solution or to solve the Cauchy problem:

a) $ty' - y = t\,\mathrm{tg}\,\frac{y}{t}$;

b) $y' + 2y = y^2 e^t$;

c) $ty' = \sqrt{t^2 - y^2} + y, y(1) = 0$;

d) $ty' + 2y + t^5 y^3 e^t = 0, y(1) = 1$;

e) $ty' = y + t(1 + e^{\frac{y}{t}}), y(1) = 0$;

f) $y - y' = y^2 + ty'$;

g) $t^2 y'' + ty' - 9y = t$;

h)* $t^3 y''' - t^2 y'' - 2ty' + 6y = \sqrt{t}$.