MATHEMATICAL ANALYSIS 2
Worksheet 8.
Systems of differential equations

Theory outline and sample problems

System of a differential equations relates derivatives of *several* unknown functions with the values of these functions:

$$\begin{cases} \frac{d}{dt}y_1(t) = F_1(t, y_1(t), \ldots, y_k(t)); \\ \frac{d}{dt}y_2(t) = F_2(t, y_1(t), \ldots, y_k(t)); \\ \qquad \vdots \\ \frac{d}{dt}y_k(t) = F_k(t, y_1(t), \ldots, y_k(t)). \end{cases} \tag{1}$$

System (1) can be seen as *one* equation for the *vector-valued* function

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_k(t) \end{pmatrix}.$$

Namely, in the vector notation (1) has the form

$$\mathbf{y}(t) = F(t, \mathbf{y}(t)), \tag{2}$$

where the function transforms a pair $t \in \mathbb{R}, \mathbf{y} \in \mathbb{R}^k$ into a vector in $\mathbb{R}^k$. From this perspective, systems of differential equations can be treated in the way quite familiar to differential equations of the first order. In particular, the Cauchy problem ($\Longleftrightarrow$ the initial value problem) for the system (1) is (1) combined with the conditions

$$y_1(t_0) = y_1^0, \ldots, y_k(t_0) = y_k^0,$$

where $t_0, y_1^0, \ldots, y_k^0$ are given numbers. In the vector notation, this Cauchy problem has a natural form

$$\mathbf{y}(t) = F(t, \mathbf{y}(t)), \quad \mathbf{y}(t) = \mathbf{y}^0 = \begin{pmatrix} y_1^0 \\ \vdots \\ y_k^0 \end{pmatrix}. \tag{3}$$

All the general theory of the first order differential equations extends, with minor modifications, to differential equations for vector-valued functions, and thus for systems of differential equations. In particular, the Picard theorem (on local existence and uniqueness of the solution to the Cauchy problem) and the extension procedure under the linear growth condition are available for a vector-valued Cauchy problem (3), and thus for a system of differential equations (1).
On the other hand, systems of differential equations have close relation to differential equations of higher order. A differential equation of the order k

$$y(k)(t) = F(t, y(t), \ldots, y^{(k-1)}(t)) \tag{4}$$

can be transformed to a system of (first order) differential equations if we introduce k unknown functions

$$y_1(t) = y(t), \quad y_2(t) = y'(t), \ldots, y_k(t) = y^{(k-1)}(t).$$

Then (4) is equivalent to the system

$$\begin{cases} \frac{d}{dt}y_1(t) = y_2(t); \\ \quad\vdots \\ \frac{d}{dt}y_{k-1}(t) = y_k(t); \\ \frac{d}{dt}y_k(t) = F(t, y_1(t), \ldots, y_k(t)). \end{cases} \tag{5}$$

On the other hand, it is often that a given *system* of differential equations can be reduced to *one* differential equation of higher order. The standard method here is the *elimination method*, where we eliminate some of the unknown functions from the equation by expressing them through other unknowns.

Sample Problem **1:** Using the elimination method, reduce the Cauchy problem

$$\begin{cases} x' = -y; \\ y' = -x - 2y; \end{cases} \qquad x(0) = 1, \quad y(0) = 1$$

to a Cauchy problem for a second order differential equation, and then solve it.

Solution: Using the first equation of the system, we express unknown y as $y = -x'$. Substituting this to the second equation, we get $-x'' = -x + 2x'$. From the initial condition, we have $x(0) = 1, x'(0) = -y(0) = -1$. That is, for the unknown x we have the second order Cauchy problem

$$x'' + 2x' - x = 0, \quad x(0) = 1, \quad x'(0) = -1.$$

The characteristic polynomial and its roots are

$$P(\lambda) = \lambda^2 + 2\lambda - 1, \quad \lambda_1 = -1 - \sqrt{2}, \quad \lambda_2 = -1 + \sqrt{2},$$

hence

$$x(t) = C_1 e^{-t-\sqrt{2}t} + C_2 e^{-t-\sqrt{2}t}.$$

From the initial conditions,

$$C_1 + C_2 = 1, \quad (-1 - \sqrt{2})C_1 + (-1 + \sqrt{2})C_2 = -1 \Longrightarrow C_1 = C_2 = \frac{1}{2},$$

and thus

$$x(t) = \frac{1}{2}e^{-t-\sqrt{2}t} + \frac{1}{2}e^{-t-\sqrt{2}t}.$$

Recalling that $y = -x'$, we get

$$y(t) = \frac{1+\sqrt{2}}{2}e^{-t-\sqrt{2}t} + \frac{1-\sqrt{2}}{2}e^{-t-\sqrt{2}t}.$$

Definition 1. System of linear differential equations has the form

$$\begin{cases} \frac{d}{dt}y_1(t) = a_{11}(t)y_1(t) + \ldots a_{1k}(t)y_k(t) + b_1(t); \\ \frac{d}{dt}y_2(t) = a_{21}(t)y_1(t) + \ldots a_{2k}(t)y_k(t) + b_2(t); \\ \quad\vdots \\ \frac{d}{dt}y_k(t) = a_{k1}(t)y_1(t) + \ldots a_{kk}(t)y_k(t) + b_k(t). \end{cases} \tag{6}$$

In the vector notation, system (6) has the form

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{b}(t), \tag{7}$$

where

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1k}(t) \\ \vdots & \ddots & \vdots \\ a_{k1}(t) & \dots & a_{kk}(t) \end{pmatrix}, \quad \mathbf{b}(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_k(t) \end{pmatrix}$$

are the matrix of coefficients and vector of free terms, respectively.

The theory of systems of linear differential equations has many similarities with the theory of linear differential equations of higher orders; please compare the definitions and facts listed below with their analogues from the previous list.

System of linear equations (7) is called *homogeneous* if $\mathbf{b}(t) \equiv \mathbf{0}$, and *non-homogeneous* otherwise.

Proposition 1. *1. The family of solutions of a homogeneous system of k linear differential equations is a vector space w.r.t. the point-wise operations of addition and multiplication by a constant. This vector space has dimenstion k.*

2. Any solution to a non-homogeneous system of k linear differential equations can be obtained as a sum of a fixed solution to the non-homogeneous system and some solution to the associated homogeneous system.

A basis in the set of the solutions to a homogeneous system of linear differential equations is called a *fundamental system of solutions*.

Proposition 2. *Let $\mathbf{y}^1(t), \dots, \mathbf{y}^k(t)$ be solutions to a homogeneous system of k linear differential equations. For $\mathbf{y}^1(t), \dots, \mathbf{y}^k(t)$ to be a fundamental system of solutions it is necessary and sufficient that, at any given point t_0, the $k \times k$-matrix*

$$\begin{pmatrix} y_1^1(t_0) & \cdots & y_1^k(t_0) \\ \vdots & \ddots & \vdots \\ y_k^1(t_0) & \dots & y_k^k(t_0) \end{pmatrix} = \begin{pmatrix} \mathbf{y}^1(t_0) & \cdots & \mathbf{y}^k(t_0) \end{pmatrix}$$

is non-degenerate.

The above matrix is called the *fundamental matrix*, and its determinant is called the *Wronskian* of the system $y_1(t), \dots, y_k(t)$ at the point x_0; notation

$$W(t_0) = \begin{vmatrix} y_1^1(t_0) & \cdots & y_1^k(t_0) \\ \vdots & \ddots & \vdots \\ y_k^1(t_0) & \dots & y_k^k(t_0) \end{vmatrix} = \begin{vmatrix} \mathbf{y}^1(t_0) & \cdots & \mathbf{y}^k(t_0) \end{vmatrix}$$

System $\mathbf{y}^1(t), \dots, \mathbf{y}^k(t)$ is fundamental if, and only if, $W(t_0) \neq 0$ for some (and then for any) t_0. This gives a practical tool for choosing a fundamental system of solutions: consider a linearly independent system of vectors $\mathbf{v}^1, \dots, \mathbf{v}^k$ in $\mathbb{R}^k$, and take $\mathbf{y}^1(t), \dots, \mathbf{y}^k(t)$ as the solutions to the Cauchy problem with the initial values $\mathbf{v}^1, \dots, \mathbf{v}^k$:

$$\frac{d}{dt}\mathbf{y}^j(t) = \mathbf{A}(t)\mathbf{y}^j(t), \quad \mathbf{y}^j(t_0) = \mathbf{v}^j, \quad j = 1, \dots, k.$$

Given a fundamental system of solutions $\mathbf{y}^1(t), \dots, \mathbf{y}^k(t)$ to a homogeneous system of linear differential equations, we can express a general solution as

$$\mathbf{y}(t) = C_1\mathbf{y}^1(t) + \cdots + C_k\mathbf{y}^k(t),$$

where $C_1, \dots C_k$ are real parameters. Lets consider a typical example.

Sample Problem **2:** Find the fundamental system and the general solution for $t > 0$ of the system of linear differential equations

$$\begin{cases} tx' = -x + ty; \\ t^2y' = -2x + ty. \end{cases}$$

Solution: We use the elimination method: from the 1st equation, $y = x' + \frac{x}{t}$.Then

$$y' = x'' + \frac{x'}{t} - \frac{x}{t^2},$$

and from the 2nd equation we get

$$t^2x'' + tx' - x = -2x + tx' + x \Longleftrightarrow t^2x'' = 0.$$

This is the Euler equation, which by the change of variables $t = e^\tau$ transforms to a 2nd order linear differential equation with constant coefficients:

$$x''_{\tau\tau} - x'_\tau = 0.$$

The characteristic polynomial of this equation and its roots are

$$P(\lambda) = \lambda^2 - \lambda, \quad \lambda_1 = 0, \lambda_2 = 1,$$

hence the general solution to this equation has the form

$$x(\tau) = C_1 + C_2e^\tau.$$

Changing the variables back, we get

$$x(t) = C_1 + C_2t,$$

and recalling that $y = x' + \frac{x}{t}$, we get

$$y(t) = C_2 + C_1t^{-1} + C_2 = C_1t^{-1} + 2C_2.$$

Thus the general solution has the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ t^{-1} \end{pmatrix} + C_2 \begin{pmatrix} t \\ 2 \end{pmatrix}$$

Fundamental system of solutions can be taken in the form

$$\begin{pmatrix} 1 \\ t^{-1} \end{pmatrix}, \quad \begin{pmatrix} t \\ 2 \end{pmatrix}.$$

The method of elimination reduces a system of linear differential equations to one linear equation of higher order. However, there are often good reasons to keep the system in the vector notation. One of such reasons is that, in the vector notation, the *method of variation of unknown constants* for solving a non-homogeneous equation takes a particulary simple form.
Consider a non-homogeneous system (7), and assume that a fundamental system of solutions $\mathbf{y}^1(t), \dots, \mathbf{y}^k(t)$ to the associated homogeneous system is fixed. Denote

$$\mathbf{V}(t) = \begin{pmatrix} y_1^1(t) & \cdots & y_1^k(t) \\ \vdots & \ddots & \vdots \\ y_k^1(t) & \dots & y_k^k(t) \end{pmatrix} = \begin{pmatrix} \mathbf{y}^1(t) & \cdots & \mathbf{y}^k(t) \end{pmatrix},$$

the fundamental matrix of the system at the point t.

Proposition 3. *The function*

$$\mathbf{y}(t) = \mathbf{V}(t)\mathbf{c}(t)$$

solves the non-homogeneous system (7) *if, and only if, the vector-valued function* $\mathbf{c}(t)$ *solves*

$$\mathbf{V}(t)\mathbf{c}'(t) = \mathbf{b}(t) \Longleftrightarrow \mathbf{c}'(t) = \mathbf{V}(t)^{-1}\mathbf{b}(t), \tag{8}$$

where the derivative is understood component-wise.

Note that

$$\det \mathbf{V}(t) = W(t) \neq 0,$$

and thus $\mathbf{V}(t)^{-1}$ is well defined. The of the above relation uses essentialy the same argument as for usual linear differential equations of the 1st order. Namely, because each column of the matrix $\mathbf{V}(t)$ is a solution to the homogeneous system, we have

$$\frac{d}{dt}\mathbf{V}(t) = \mathbf{A}(t)\mathbf{V}(t),$$

again, the derivative is understood component-wise. Then for $\mathbf{y}(t) = \mathbf{V}(t)\mathbf{c}(t)$,

$$\mathbf{y}'(t) = (\mathbf{V}(t)\mathbf{c}(t))' = \mathbf{A}(t)\mathbf{V}(t)\mathbf{c}(t) + \mathbf{V}(t)\mathbf{c}'(t) = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{V}(t)\mathbf{c}'(t),$$

and for (7) to hold true it is necessary and sufficient that (8) holds.

Sample Problem **3:** Find the general solution for $t > 0$ of the system of linear differential equations

$$\begin{cases} tx' = -x + ty + \sqrt{t}; \\ t^2y' = -2x + ty + \sqrt[3]{t}; \end{cases} \qquad x(1) = 1, \quad y(1) = 0.$$

Solution: The fundamental system for the associated homogeneous system is found in the previous problem, and the corresponding fundamental matrix is

$$\mathbf{V}(t) = \begin{pmatrix} 1 & t \\ t^{-1} & 2 \end{pmatrix}.$$

The Wronskian and the inverse matrix are

$$W(t) = 1, \quad \mathbf{V}(t) = \begin{pmatrix} 2 & -t \\ -t^{-1} & 1 \end{pmatrix}.$$

The free term is

$$\mathbf{b}(t) = \begin{pmatrix} \sqrt{t} \\ \sqrt[3]{t} \end{pmatrix},$$

hence the function $\mathbf{c}(t)$ satisfies

$$\mathbf{c}'(t) = \begin{pmatrix} 2 & -t \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} \sqrt{t} \\ \sqrt[3]{t} \end{pmatrix} = \begin{pmatrix} 2t^{1/2} - t^{4/3} \\ -t^{-1/2} + t^{1/3} \end{pmatrix}$$

Integrating each component of the vector we get

$$\mathbf{c}(t) = \begin{pmatrix} \int(2t^{1/2} - t^{4/3})\,dt \\ \int(-t^{-1/2} + t^{1/3})\,dt \end{pmatrix} = \begin{pmatrix} \frac{4}{3}t^{3/2} - \frac{3}{7}t^{7/3} + C_1 \\ -2t^{1/2} + \frac{3}{4}t^{4/3} + C_2 \end{pmatrix}.$$

Thus the general solution is

$$\begin{aligned} \mathbf{V}(t)\mathbf{c}(t) &= \begin{pmatrix} 1 & t \\ t^{-1} & 2 \end{pmatrix} \begin{pmatrix} \frac{4}{3}t^{3/2} - \frac{3}{7}t^{7/3} + C_1 \\ -2t^{1/2} + \frac{3}{4}t^{4/3} + C_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & t \\ t^{-1} & 2 \end{pmatrix} \begin{pmatrix} \frac{4}{3}t^{3/2} - \frac{3}{7}t^{7/3} \\ -2t^{1/2} + \frac{3}{4}t^{4/3} \end{pmatrix} + C_1 \begin{pmatrix} 1 \\ t^{-1} \end{pmatrix} + C_2 \begin{pmatrix} t \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{2}{3}t^{3/2} + \frac{9}{28}t^{7/3} \\ -\frac{8}{3}t^{3/2} + \frac{15}{14}t^{7/3} \end{pmatrix} + C_1 \begin{pmatrix} 1 \\ t^{-1} \end{pmatrix} + C_2 \begin{pmatrix} t \\ 2 \end{pmatrix} \end{aligned}$$

Let us consider separately systems of linear equations with *constant coefficients*. In the vector notation, such systems have the form

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{b}(t), \tag{9}$$

where the matrix of coefficients $\mathbf{A}$ does not depend on t. For such systems, the fundamental system of solutions often can be found easily using the *Euler method*, which we will now describe. The Euler method is based on the following simple observation. Let $\mathbf{v}$ be an eigenvector for the matrix $\mathbf{A}$ with the eigenvalue λ. Then, by definition,

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$$

Hence, if we consider a vector-valued function $\mathbf{v}(t) = e^{\lambda t}\mathbf{v}$, then

$$\mathbf{v}'(t) = (e^{\lambda t})'\mathbf{v} = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}\mathbf{A}\mathbf{v} = \mathbf{A}(e^{\lambda t}\mathbf{v}) = \mathbf{A}\mathbf{v}(t);$$

that is, this function solves the homogeneous system of linear equations with the matrix of coefficients $\mathbf{A}$. Thus, in the ideal situation where the matrix $\mathbf{A}$ admits an eigenbasis $\mathbf{v}_1, \ldots, \mathbf{v}_k$, the corresponding system of functions

$$\mathbf{v}_1(t) = e^{\lambda_1}\mathbf{v}_1, \ldots, \mathbf{v}_k(t) = e^{\lambda_k}\mathbf{v}_k$$

forms a fundamental system of equations to the homogeneous system above.

Sample Problem **4:** Find the fundamental solution for the system

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t), \quad \mathbf{A} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}.$$

Solution: The characteristic polynomial of the matrix $\mathbf{A}$ equals

$$P_{\mathbf{A}}(\lambda) = \begin{vmatrix} 3-\lambda & -1 & -1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = -(\lambda^3 - 11\lambda^2 + 36\lambda - 36),$$

and has three real roots

$$\lambda_1 = 2, \quad \lambda_2 = 3, \quad \lambda_3 = 6.$$

These roots are the eigenvalues of the matrix $\mathbf{A}$; the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Hence a fundamental system can be chosen in the form

$$\mathbf{v}^1(t) = \begin{pmatrix} -e^{2t} \\ 0 \\ e^{2t} \end{pmatrix}, \quad \mathbf{v}^2(t) = \begin{pmatrix} e^{3t} \\ e^{3t} \\ e^{3t} \end{pmatrix}, \quad \mathbf{v}^3(t) = \begin{pmatrix} e^{6t} \\ -e^{6t} \\ e^{6t} \end{pmatrix}.$$

It may happen that some roots of the characteristic polynomial $P_{\mathbf{A}}(\lambda)$ are complex; in that case, we actually have pairs of mutually conjugate complex roots. In that case, the Euler method should be modified as follows: if $\lambda_1, \lambda_2 = \overline{\lambda_1}$ is a pair of complex roots, then there exists an eigenvector $\mathbf{v}^1$ (possibly, with complex coordinates) with the eigenvalue λ_1, and the following pair of functions should be added to the fundamental system:

$$\mathbf{v}^1(t) = \mathrm{Re}(e^{\lambda_1 t}\mathbf{v}^1), \quad \mathbf{v}^2(t) = \mathrm{Im}(e^{\lambda_1 t}\mathbf{v}^1).$$

Sample Problem **5:** Find the fundamental solution for the system

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t), \quad \mathbf{A} = \begin{pmatrix} 0 & 8 & 0 \\ 0 & 0 & -2 \\ 2 & 8 & -2 \end{pmatrix}.$$

Solution: The characteristic polynomial of the matrix $\mathbf{A}$ equals

$$P_{\mathbf{A}}(\lambda) = \begin{vmatrix} -\lambda & 8 & 0 \\ 0 & -\lambda & -2 \\ 2 & 8 & -2-\lambda \end{vmatrix} = -(\lambda + 2)(\lambda^2 + 16),$$

and has three roots

$$\lambda_1 = -2, \quad \lambda_2 = 4i, \quad \lambda_3 = -4i.$$

The first eigenvalue corresponds to eigenvector

$$\mathbf{v}^1 = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix},$$

which gives one element of the fundamental system

$$\mathbf{v}^1(t) = \begin{pmatrix} -4e^{-2t} \\ e^{-2t} \\ e^{-2t} \end{pmatrix}.$$

Two other elements are obtained from the eigenvector

$$\mathbf{v}^2 = \begin{pmatrix} 2 \\ i \\ 2 \end{pmatrix},$$

which corresponds to the eigenvalue $\lambda_2 = 4i$. Namely, we get

$$\mathbf{v}^2(t) = \mathrm{Re}\begin{pmatrix} 2e^{4it} \\ ie^{4it} \\ 2e^{4it} \end{pmatrix} = \mathrm{Re}\begin{pmatrix} 2(\cos 4t + i\sin 4t) \\ i(\cos 4t + i\sin 4t) \\ 2(\cos 4t + i\sin 4t) \end{pmatrix} = \begin{pmatrix} 2\cos 4t \\ -\sin 4t \\ 2\cos 4t \end{pmatrix},$$

$$\mathbf{v}^3(t) = \mathrm{Im}\begin{pmatrix} 2(\cos 4t + i\sin 4t) \\ i(\cos 4t + i\sin 4t) \\ 2(\cos 4t + i\sin 4t) \end{pmatrix} = \begin{pmatrix} 2\sin 4t \\ \cos 4t \\ 2\sin 4t \end{pmatrix}.$$

Thus the required fundamental system is

$$\mathbf{v}^1(t) = \begin{pmatrix} -4e^{-2t} \\ e^{-2t} \\ e^{-2t} \end{pmatrix}, \quad \mathbf{v}^2(t) = \begin{pmatrix} 2\cos 4t \\ -\sin 4t \\ 2\cos 4t \end{pmatrix}, \quad \mathbf{v}^3(t) = \begin{pmatrix} 2\sin 4t \\ \cos 4t \\ 2\sin 4t \end{pmatrix}.$$

Problems to solve

1. Using the elimination method, solve the Cauchy problems for the system of linear differential equations $\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}(t)\mathbf{y}(t)$:

a) $\mathbf{A}(t) = \mathbf{A} = \begin{pmatrix} 1 & 3 \\ -1 & 5 \end{pmatrix}, \mathbf{y}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$;

b) $\mathbf{A}(t) = \mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & 7 \end{pmatrix}, \mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$;

c) $\mathbf{A}(t) = \mathbf{A} = \begin{pmatrix} -1 & 2 \\ -2 & -5 \end{pmatrix}, \mathbf{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;

d)* $\mathbf{A}(t) = \begin{pmatrix} -t^{-1} & t \\ -2t^{-3} & t^{-1} \end{pmatrix}, \mathbf{y}(0) = \begin{pmatrix} 1 \\ 2+\sqrt{2} \end{pmatrix}$.

2. Using the Euler method, find the general solutions to the following homogeneous systems of linear differential equations ($\dot{x}$ denotes $\frac{dx}{dt}$ etc.):

a) $\begin{cases} \dot{x} = 2x + y \\ \dot{y} = 3x + 4y \end{cases}$;

b) $\begin{cases} \dot{x} = x + y \\ \dot{y} = 3y - 2x \end{cases}$;

c) $\begin{cases} \dot{x} = x - 2y - z \\ \dot{y} = y - x + z \\ \dot{z} = x - z \end{cases}$;

d) $\begin{cases} \dot{x} = 3x - y + z \\ \dot{y} = x + y + z \\ \dot{z} = 4x - y + 4z \end{cases}$;

e) $\begin{cases} \dot{x} = x - y - z \\ \dot{y} = x + y \\ \dot{z} = 3x + z \end{cases}$;

f) $\begin{cases} \dot{x} = x - y - z \\ \dot{y} = x + y \\ \dot{z} = 3x + z \end{cases}$;

g) $\begin{cases} \dot{x} = 4x - y - z \\ \dot{y} = x + 2y - z \\ \dot{z} = x - y + 2z \end{cases}$.

3. Find the general solution to the following non-homogeneous systems of linear differential equations:

a) $\begin{cases} \dot{x} = y + 2e^t \\ \dot{y} = x + t^2 \end{cases}$;

b) $\begin{cases} \dot{x} = 2x + y + 2e^t \\ \dot{y} = x + 2y - 3e^{4t} \end{cases}$;

c) $\begin{cases} \dot{x} = y + \operatorname{tg}^2 t - 1 \\ \dot{y} = x + \operatorname{tg} t \end{cases}$;

d) $\begin{cases} \dot{x} = 2y - x \\ \dot{y} = 4y - 3x + \frac{e^{3t}}{e^{2t}+1} \end{cases}$;

e) $\begin{cases} \dot{x} = -4x - 2y + \frac{2}{e^t-1} \\ \dot{y} = 6x + 3y - \frac{3}{e^t-1} \end{cases}$;

f) $\begin{cases} \dot{x} = x - y + \frac{1}{\cos t} \\ \dot{y} = 2x - y \end{cases}$;

g) $\begin{cases} \dot{x} = 3x - 2y \\ \dot{y} = 2x - y + 15e^t\sqrt{t} \end{cases}$.