# MATHEMATICAL ANALYSIS 2

## Worksheet 9.

*Geometric and physical problems using differential equations*

---

*Theory outline and sample problems*

This worksheet is devoted to applications of differential equations. One large field of applications appears in geometry, when a curve on the plane is described by certain relation involving the tangent line to the curve. Since the slope of the tangent line to a curve $y = f(x)$ at a point $x$ equals $f'(x)$, this naturally leads to a differential equation.

*Sample problem* **1:** Define the equation of a curve such that, for any point $(x_0, y_0)$ on the curve, in the triangle obtained by the intersection of the $Ox$ axis, the ordinate line $x = x_0$ and the tangent line, the sum of the lengths of the catheti is constant and equals $b$.

*Solution:* Let us look for the curves with $y > 0$; that is, located in the upper half-plane w.r.t. the $Ox$ asixe. The lengths of the vertical cathetus if $y_0 = f(x_0)$. Since the slope of the tangent line equals $|f'(x_0)|$, we have

$$f(x_0) = |f'(x_0)| \times (\text{the length of the horisontal cathetus}) \Longleftrightarrow$$

$$\Longleftrightarrow \text{the length of the horisontal cathetus} = \frac{f(x_0)}{|f'(x_0)|}.$$

Thus the relation specified in the problem can be written as

$$f(x_0) + \frac{f(x_0)}{|f'(x_0)|} = b. \tag{1}$$

Since this relation has to hold at every $x_0$, we get a differential equation

$$y + \frac{y}{|y'|} = b \Longleftrightarrow y \pm \frac{y)}{f'(x)} = b,$$

the choice of the signs is arbitrary. This equation is actually a separable differential equation:

$$y \pm \frac{y}{y'} = b \Longleftrightarrow \pm\frac{1}{y'} = -1 + \frac{b}{f(x)} \Longleftrightarrow y' = \pm\left(-1 + \frac{b}{f(x)}\right)^{-1},$$

and we have

$$dy\left(-1 + \frac{b}{y}\right) = \pm dx,$$

which after integration gives the equation

$$-y + b\ln y = \pm x + C.$$

Recall that we have assumed $y > 0$. On the other hand, it follows from the relation (1) that $y \leqslant b$. This gives the required description of the curve:

$$-y + b\ln y = \pm x + C, \quad 0 < y \leqslant b.$$

Another type of geometric problems which leads to differential equations is related to areas.

*Sample problem* **2:** Define the equation of a curve located in the 1st quadrant such that, for each point $(x_0, y_0)$ on this curve, the rectangle formed by the coordinate axes and the lines through the point $(x_0, y_0)$ parallel to these axes, is divided in two parts with the area of the upper part equal twice the area of the lower part.

*Solution:* Define the area of the lower part $S(x_0)$, then the area of the entire rectangle is $3S(x_0)$. On the other hand, the area of the entire rectangle is $x_0 y_0$; that is, $3S(x_0) = x_0 y_0$. We have that

$$S(x_0) = \int_0^{x_0} y(x)\, dx,$$

and by the Newton-Leibnitz formula $S'(x) = y(x)$. Thus the relation claimed in the problem can be written as

$$xS'(x) = 3S(x) \iff \frac{dS}{S} = \frac{3dx}{x} \iff \ln S = 3 \ln x + C \Longrightarrow S(x) = \tilde{C}x^3, \quad \tilde{C} = e^C > 0,$$

we have used our knowledge that $S > 0, x > 0$. Then $y(x) = S'(x) = \hat{C}x^2$, where $\hat{C} = 3e^C$ is arbitrary positive constant.
There is a plenty of physical settings which lead to differential equations; we have already met some of them in Worksheet 6. Here we discuss several types of such problems systematically.

**Flow Problems** These problems deal with a liquid flowing down from a tank. The main fact here is that the speed of the flow is given by the formula

$$0.6\sqrt{2gh}, \quad g = 9.8m/s^2, \tag{2}$$

where $h$ is the level of the liquid in the tank. We had one problem of such a type in Worksheet 6 (Problem 6). Let us consider one more such a problem, which will show that, for such problems, geometry plays a serious role.

*Sample problem* **3:** A cylindrical tank is placed on its side. Through a small hole at the bottom (which is now located on the side) the water is flowing down. One half of the water had flown in 5 minutes. How long will it take for the tank to become empty?

*Solution:* Unlike Problem 6 of Worksheet 6, the volume $V(t)$ of the liquid is not a good choice for the dependent variable to describe the model. The reason is that the level $h$, involved into (2), is badly expressed through $V(t)$, and the differential equation for $V(t)$ is difficult to solve. Much more convenient is to study $h(t)$, the level of the liquid. We have

$$\triangle h \approx \frac{\triangle V}{S},$$

where $S$ is area of the section of the tank at the level $h$. The area $S$ has the form

$$S = 2\sqrt{R^2 - (R-h)^2}H = 2\sqrt{2Rh - h^2}H,$$

where $R$ is the radius of the base of the cylinder, and $H$ is the height of its side. Since $\triangle V$ is negative and proportional to $\sqrt{h}$ by (2), we get

$$dh = -c\frac{\sqrt{h}}{\sqrt{2Rh - h^2}}dt.$$

Solving this separable differential equation, we get

$$\sqrt{2R-h}\,dh = ct \Longleftrightarrow (2R-h)^{3/2} = kt + C.$$

From the initial value $h(0) = 2R$, we get $C = 0$, and

$$(2R-h)^{3/2} = kt.$$

Condition $h(5) = R$ gives

$$R^{3/2} = 5k \Longleftrightarrow k = \frac{1}{5}R^{3/2}.$$

Then the time $T$ such that $h(T) = 0$ equals

$$T = k^{-1}(2R)^{3/2} = 5R^{-3/2}(2R)^{3/2} = 52^{3/2} = 10\sqrt{2} \approx 14min.$$

Comparing with the answer $\approx 17min.$ in Problem 6, Worksheet 6, where the cylinder was placed vertically, we see the impact of the geometry in such problems.

**Mixing Problems** These problems deal with a substance that is dissolved in a liquid. Liquid will be entering and leaving a holding tank. The liquid entering the tank may or may not contain more of the substance dissolved in it. Liquid leaving the tank will of course contain the substance dissolved in it. If $Q(t)$ is the amount of the substance dissolved in the liquid in the tank at any time $t$ we want to develop a differential equation that, when solved, will give us an expression for $Q(t)$. Note as well that in many situations we can think of air as a liquid for the purposes of these kinds of discussions and so we don't actually need to have an actual liquid but could instead use air as the "liquid".
The main assumption that we'll be using here is that the concentration of the substance in the liquid is uniform throughout the tank. Clearly this will not be the case, but if we allow the concentration to vary depending on the location in the tank the problem becomes very difficult and will involve partial differential equations, which is not the focus of this course.
The main "equation" that we'll be using to model this situation is:

$$\begin{aligned}\frac{dQ}{dt} = \text{ Rate of change of } Q(t) = &\text{ Rate of at which } Q(t) \text{ enters the tank} \\ &- \text{ Rate of at which } Q(t) \text{ exits the tank },\end{aligned} \tag{3}$$

where

$$\begin{aligned}\text{Rate of at which } Q(t) \text{ enters the tank } = &\text{ (flow rate of liquid entering)} \\ &\text{x (concentration of substance in liquid entering),}\end{aligned}$$

$$\begin{aligned}\text{Rate of at which } Q(t) \text{ enters the tank } = &\text{ (flow rate of liquid exiting)} \\ &\text{x (concentration of substance in liquid exiting).}\end{aligned}$$

For an example of typical mixing problem, see Sample problem 1 in Worksheet 6. This basic setting admits various extensions. One natural extension is related to cascades of tanks.

*Sample problem* **4:** To a first tank which contains 10 liters of water, a solution of salt is continuously added with the speed of 2 liters/min, and with the same speed the liquid from the first tank flows into the second tank, which at the beginning had 20 liters of water. The extra liquid from the second tank is removed. One liter of the solution contains 0.3 kg of salt. How much salt will be contained in the second tank in 5 minutes?

*Solution:* We have to compose two "main equations", each for its own tank. Let the amounts of salt in the 1st and 2nd tanks be $Q(t), R(t)$, then

$$Q'(t) = 2 \times (0.3) - 2 \times \frac{Q(t)}{10} = 0.6 - 0.2Q(t),$$

$$R'(t) = 2 \times \frac{Q(t)}{10} - 2 \times \frac{R(t)}{20} = 0.2Q(t) - 0.1R(t).$$

That is, for the pair $Q, R$ the system of differential equations holds:

$$\begin{cases} Q'(t) = 0.6 - 0.2Q(t) \\ R'(t) = 0.2Q(t) - 0.1R(t). \end{cases}$$

This system can be solved sequentially: first, we solve the first equation, which relates only $Q(t)$, and then we solve the second equation using that $Q(t)$ is already known. Namely, treating the first equation as a linear non-homogeneous equation, we write

$$Q(t) = e^{-0.2t} \int 0.6e^{0.2t} dt = 3 + Ce^{-0.2t}$$

and adding the initial condition $Q(0) = 0$ we get finally

$$Q(t) = 3 - 3e^{-0.2t}.$$

Then from the second equation we get

$$R(t) = e^{-0.1t} \int 0.2Q(t)e^{0.1t} dt = e^{-0.1t} \int (0.6e^{0.1t} - 0.6e^{-0.1t}) dt = 6 + 6e^{-0.2t} + Ce^{-0.1t},$$

and adding the initial condition $R(0) = 0$ we get finally

$$R(t) = 6 + 6e^{-0.2t} - 12e^{-0.1t}.$$

The final answer is

$$R(5) = 6 + 6e^{-1} - 12e^{-0.5} \approx 0.92$$

Unlike the previous problem, a system of differential equations may not be solvable on-by-one, if the tanks are connected not in a cascade, but in a circuit.

*Sample problem* **5:** Two tanks of the volumes 10 and 30 liters are linked by pumps, which move 2 liters/min of the liquid from the 1st tank to the 2nd, and independently from the 2nd to the 1st. Find the amounts of salt at each of the tanks at time $t$, if at the initial moment there is 1 kg of salt in the second tank, and no salt in the first one.

*Solution:* Like in the previous problem, let $Q(t), R(t)$ denote the amounts of salt in the 1st and 2nd tanks, respectively. The concentrations of salt are $\frac{Q(t)}{10}, \frac{R(t)}{30}$, and then combining the "main equations" for both tanks we get the system

$$\begin{cases} Q'(t) = -2\frac{Q(t)}{10} + 2\frac{R(t)}{30} \\ R'(t) = 2\frac{Q(t)}{10} - a\frac{R(t)}{30}. \end{cases}$$

The matrix

$$A = \begin{pmatrix} -\frac{1}{5} & \frac{1}{15} \\ \frac{1}{5} & -\frac{1}{15} \end{pmatrix}$$

has two eigenvalues $\lambda_1 = 0, \lambda_2 = -\frac{4}{15}$ with the eigenvectors

$$\mathbf{v}^1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \mathbf{v}^2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Hence the general solution is

$$\mathbf{v}(t) = C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 e^{-\frac{4}{15}t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

From the initial condition

$$\mathbf{v}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we get $C_1 = C_2 = \frac{1}{4}$, which gives

$$\mathbf{v}(t) = C_1 \begin{pmatrix} \frac{1}{4} + \frac{1}{4}e^{-\frac{4}{15}t} \\ \frac{3}{4} + \frac{1}{4}e^{-\frac{4}{15}t} \end{pmatrix}.$$

**Population problems**. These are somewhat easier than the mixing problems although, in some ways, they are very similar to mixing problems. So, if $P(t)$ represents a population in a given region at any time $t$ the basic equation that we'll use is identical to the one that we used for mixing. Namely, the "main equation" is

$$\begin{aligned} \frac{dP}{dt} = \text{ Rate of change of } P(t) = & \text{ Rate at which } P(t) \text{ enters the region} \\ & - \text{ Rate at which } P(t) \text{ exits the region} \end{aligned} \tag{4}$$

The enter rate typically combines immigration (with a constant rate) and birth (with the rate of the form $aP(t)$). The exit rate may combine emigration and death, which typically is described as a polynomial of $P(t)$. The typical model of population without immigration/emigration and with quadratic death rate (which corresponds to a realistic model with a bounded amount of resources available) is the *logistic model*

$$P'(t) = kP(t)(a - P(t))$$

The constant $a$ is called the *saturation state*, since the solution to the logistic equation with $P(0) < a$ has the form

$$P(t) = \frac{a}{1 + (\frac{a}{P(0)} - 1)e^{-akt}}$$

and tends to $a$ when $t \to \infty$.

*Sample problem* **6:** Without any resource limits, population of rabbits will double its size in one month. What will be the size of the population of rabbits in two months, if the population follows the logistic model due to resource limitations, initial size is 1 mln and the saturation level is 2 mln.

*Solution:* Without any resource limits, the population is described by

$$P'(t) = \lambda P(t), \quad \lambda = ak,$$

and we know that $e^{\lambda} = 2$; that is $\lambda = \ln 2$. Then

$$k = \frac{\ln 2}{2},$$

and in the logistic model we have

$$P(2) = \frac{2}{1 + e^{-\ln 2}} = \frac{2}{\frac{3}{2}} = \frac{4}{3}.$$

**Falling Objects** The basic equation that we'll use here is the Newton Second Law of Motion:

$$mv'(t) = F(t, v), \tag{5}$$

where $m$ is the mass of the object and $F(t, v)$ is the force acting on the object. In the 'falling objects' framework, the two forces that are combined in $F(t, v)$ are gravity and air resistance. The gravity force is given by

$$F = mg, \quad g = 9.8m/s^2.$$

The air resistance typically has the value proportional to the velocity or its square, and the direction, opposite to the velocity.

*Sample problem* **7:** W A 60 kg sky diver jumps out of a plane with no initial velocity and an air resistance of $0.8v$. Find his velocity in 25 seconds.

*Solution:* The equation of the motion is

$$60v'(t) = 60g - 0.8v(t) \Longleftrightarrow v'(t) = 9.8 - \frac{v(t)}{75}.$$

The solution is

$$v(t) = 735 + Ce^{-\frac{1}{75}t},$$

and from $v(0) = 0$ we finally

$$v(t) = 735 - 735e^{-\frac{1}{75}t}.$$

That is,

$$v(25) = 735(1 - e^{-1/3}) \approx 208m/s.$$

**Electrical circuits** The dynamic of the *voltage* and the *current* in an electrical circuit is naturally described by differential equations. There are three principal types of the elements in a circuit, each of them having their own way the voltage $V$ across the element is depending on the current $I$:

- *resistor*, usually denoted the same letter $R$ as its *resistance*, with the voltage over it given by
$$V_R = RI;$$
- *inductor*, usually denoted the same letter $L$ as its *inductance*, with the voltage over it given by
$$V_L = \frac{d}{dt}I;$$

- *capacitor*, usually denoted the same letter $C$ as its *capacity*, with the voltage over it satisfying

$$V_C = \frac{1}{C} \int I \, dt.$$

The Kirchhoff voltage law says that the directed sum of the voltages around a circuit must be zero. In the following three problems, three basic types of electric circuits are discussed.

*Sample problem* **8:** Find the current in the *series RL-circuit*, which is contained of a resistor $R$, inductor $L$, and a source of constant voltage $V$ connected into a circle. The circuit also contains a switch which is turned from 'off' to 'on' at the time $t = 0$.

*Solution:* By the Kirchhoff voltage law,

$$V_R + V_L = V,$$

which gives the equation

$$RI + L\frac{d}{dt}I = V.$$

This is a linear non-homogeneous differential equation for $I$, which has the general solution

$$I(t) = e^{-\frac{R}{L}t} \int \frac{V}{L} e^{\frac{R}{L}t} \, dt = \frac{V}{R} + Ce^{-\frac{R}{L}t}.$$

Substituting the initial condition $I(0) = 0$, we get

$$I(t) = \frac{V}{R}(1 - e^{-\frac{R}{L}t}).$$

*Sample problem* **9:** Find the current in the *series RC-circuit*, which is contained of a resistor $R$, capacitor $C$, and a source of constant voltage $V$ connected into a circle. The circuit also contains a switch which is turned from 'off' to 'on' at the time $t = 0$.

*Solution:* By the Kirchhoff voltage law,

$$V_R + V_C = V,$$

which gives the equation

$$RI + \frac{1}{C} \int I \, dt = V.$$

Differentiating it, we get the first order homogeneous differential equation

$$RI' + \frac{1}{C}I = 0.$$

The initial condition comes from the voltage law: since $V_R = RI$,

$$I(0) = \frac{1}{R}V_R(0) = \frac{V - V_C(0)}{R},$$

where $V_C(0)$ denotes the initial voltage at the capacitor. Then

$$I(t) = \frac{V - V_C(0)}{R} e^{-RCt}.$$

This is a linear non-homogeneous differential equation for $I$, which has the general solution

$$I(t) = e^{-\frac{R}{L}t} \int \frac{V}{L} e^{\frac{R}{L}t}\, dt = \frac{V}{L} + Ce^{-\frac{R}{L}t}.$$

Substituting the initial condition $I(0) = 0$, we get

$$I(t) = \frac{V}{L}(1 - e^{-\frac{R}{L}t}).$$

*Sample problem* **10:** Find the current in the *series RLC-circuit*, which is contained of a resistor $R$, inductor $L$, capacitor $C$, and a source of constant voltage $V$ connected into a circle. The circuit also contains a switch which is turned from 'off' to 'on' at the time $t = 0$.

*Solution:* By the Kirchhoff voltage law,

$$V_R + V_L + V_C = V,$$

which gives the equation

$$RI + L\frac{d}{dt}I + \frac{1}{C}\int I\, dt = V.$$

Differentiating it, we get the second order homogeneous differential equation

$$RI' + LI'' + \frac{1}{C}I = 0. \tag{6}$$

The general solution has different structure in the following three cases:

- (over-damped case): if $R^2 > \frac{4L}{C}$, then the characteristic equation for (4) has two negative real roots $\lambda_1, \lambda_2$ and
$$I(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t};$$
- (critically damped case): if $R^2 = \frac{4L}{C}$, then the characteristic equation for (4) has one negative real root $\lambda = -\frac{R}{2L}$ of multiplicity 2 and
$$I(t) = C_1 e^{-\frac{R}{2L}t} + C_2 t e^{-\frac{R}{2L}t};$$
- (under-damped case): if $R^2 < \frac{4L}{C}$, then the characteristic equation for (4) has two complex roots and
$$I(t) = e^{-\alpha t}(C_1 \cos\omega t + C_2 \sin\omega t),$$
where
$$\alpha = \frac{R}{2L}, \quad \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

The choice of the constants $C_1, C_2$ depends on the initial values of $I(0), I'(0)$.
Let us consider one problem where the circuit is more sophisticated than just a circle.

*Sample problem* **11:** Find the total current in the *"two-mesh"-circuit*, which is obtained by adding to a series $RL$-circuit, in parallel to the $RL$ part, another resistor $R_2$. The circuit also contains a switch which is turned from 'off' to 'on' at the time $t = 0$.

*Solution:* The circuit consist of two smaller circuits, the first containing the original $RL$ part and the second containing $R_2$. By the Kirchhoff voltage law applied to both of them, we get two equations

$$V_R + V_L = V, \quad V_{R_2} = V.$$

Then the currents $I_1$, $I_2$ at these circuits satisfy

$$RI_1(t) + L\frac{d}{dt}I_1 = V, \quad R_2 I_2(t) = V,$$

and thus the total current equals

$$I(t) = I_1(t) + I_2(t) = \frac{V}{R}(1 - e^{-\frac{R}{L}t}) + \frac{V}{R_2} = (\frac{V}{R} + \frac{V}{R_2}) - \frac{V}{R}e^{-\frac{R}{L}t}.$$

**Mechanical vibrations** The second Newton law (5) also applies naturally when a motion of a body nuder the deformation forces is considered. In this case the 'main part' of the force $F$ is the deformation force, which according to Hook's law of elasticity is proportional to the size of the deformation of the body. Damping forces (caused by friction, and similar to the air resistance) and external forces can be also presented. If the body is hanging vertically, the gravity force is also naturally included.

*Sample problem* **12:** A point of the mass $m$ is attached to one end of spring, the other end is rigidly fixed at a wall. The Hook elastic force is $F_H = -kx$, where $x$ is the increment of the length of the spring w.r.t. its natural length. The point is subject to a damping (friction) force $F_D = -bv$, where $v$ is the velocity of the point. Find the law of the motion of the point if the initial increment of the length of the spring is 0 and the initial velocity is $v_0$.

*Solution:* By the Newton law,

$$mv' = F_H + F_D = -kx - bv.$$

Recalling that $x' = v$, we get the second order linear differential equation on $x$:

$$x'' + \frac{b}{m}x' + \frac{k}{m}x = 0$$

Depending on the sign of

$$b^2 - 4km,$$

we have three different cases, similar to the over-, critical, and under-dumped cases in Sample problem 10. Let us consider one of them, say, the under-damped case $b^2 - 4km < 0$. Then the general solution to the homogeneous equation is

$$e^{-\frac{b}{2m}t}(C_1 \cos \omega_0 t + C_2 \sin \omega_0 t), \quad \omega_0 = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$

Since $x(0) = 0$, we have $C_1 = 0$, and

$$x(t) = Ce^{-\frac{b}{2m}t} \sin \omega_0 t.$$

Then

$$x'(0) = C\omega_0,$$

and from the condition $x'(0) = v_0$ we have

$$x(t) = \frac{v_0}{\omega_0}e^{-\frac{b}{2m}t} \sin \omega_0 t.$$

*Problems to solve*

**1.** Define the equation of a curve such that, for any point $(x_0, y_0)$ on the curve, the triangle obtained by the intersection of the $Ox$ axis, the ordinate line $x = x_0$ and the tangent line, has the (constant) area $a^2$.

**2.** Define the equation of a curve such that, for any point $(x_0, y_0)$ on the curve, the segment cutted from the $Ox$ axis by the tangent line and the normal line to the curve at this point has the constant length $2a$. (The normal line is orthogonal to the tangent line and hits $(x_0, y_0)$).

**3.** A spherical tank has a small hole at the bottom, and the water is flowing down from the tank. One half of the water had flown in 5 minutes. How long will it take for the tank to become empty?

**4.** A package of milk has the form of a regular tetrahedron. A small hole is made at its vertex, and then the package is placed with the hole looking straight down. Half of the milk had flown in 5 min. How long will it take for the package to become empty?

**5.** Two tanks contain at the beginning 10 and 20 liters of pure water, respectively. A solution of salt is continuously added with the speed of 2 liters/min to the first tank, and then a pump moves the liquid from the first tank into the second tank with the speed of 3 liters/min. Another pump moves the liquid from the second tank to the first with the speed of 1 liter/min. The extra liquid from the second tank is removed. One liter of the solution contains 0.3 kg of salt. How much salt will be contained in the tanks in 5 minutes?

**6.** For a population, which follows the logistic model, the following amounts have been observed (in mln.):

$$P(0) = 0.25, \quad N(1) = 0.4, \quad N(2) = 0.5.$$

Find the saturation level of the model.

**7.** A man jumps out from a plane at the altitude 2km and opens a parachute at the altitude 0.5 km. Determine the time spent before opening a parachute, if its known that the air resistance force if proportional to the square of the velocity, and the limiting speed of free fall is 50 m/s.

**8.** Find the current in the series RL, RC, and RLC circuits if the source of voltage produces periodic input $V(t) = A \sin \omega t$.

**9.** In the circuit, an inductor $L$ and capacitor $C$ are placed in parallel, and then linked into a circle with a resistor $R$ and a source of voltage with periodic input $V(t) = A \sin \omega t$. Find the current trough the resistor.

**10.** * Two points of the equal masses $m$ are attached by three identical springs: the first springs binds a rigid wall an the 1st point, the second binds the points together, the third binds the 2nd point with another rigid wall. Find the law of the motion of the points if the initial increments of the length of the springs are 0 and the initial velocities are $v_0, w_0$.