MATHEMATICAL ANALYSIS 2

Worksheet 1.

Partial and directional derivatives. Gradient. Tangent plane. Approximate calculations

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Theory outline and sample problems

The notion of a *derivative* of a function, the central one in the real analysis, extends to the functions of two or more variables, leading to several related notions. First of them is the notion of the *partial derivative.*

Definition 1. For a function $f(x, y)$ the partial derivatives in variables x and y at the point x_0, y_0 are defined by

$$f'_x(x_0, y_0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(f(x_0 + \epsilon, y_0) - f(x_0, y_0)),$$

$$f'_y(x_0, y_0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(f(x_0, y_0 + \epsilon) - f(x_0, y_0)).$$

Alternative frequently used notations: $\frac{\partial}{\partial x} f, \partial_x f, (f)'_x$.

Simply speaking, to calculate partial derivative of $f(x, y)$ w.r.t. x we have to freeze the variable y and differentiate w.r.t. the variable x. The same definition applies to the function depending on more variables, we just have to freeze all the variables them except one, and differentiate w.r.t. this one.

Sample problem **1:** Calculate all the partial derivatives of the function $f(x, y, z) = 1 + x + yz^2$.
Solution: We have

$$f'_x(x, y, z) = 1 :$$

the term 1 differentiates to 0 since it is a constant, but now y, z are treated as constants, so yz^2 differentiates to 0, as well. Similarly,

$$f'_y(x, y, z) = (yz^2)'_y = z^2(y)'_y = z^2,$$

here we used the rule $(cF)' = cF'$ valid for a constant c. Finally,

$$f'_z(x, y, z) = (yz^2)'_z = 2yz.$$

Sample problem **2:** Calculate all the partial derivatives of the function $f(x, y) = x \ln(x + 2y)$.
Solution: By the product rule $(FG)' = F'G + FG'$ and the chain rule $[F(G)|' = F'(G)G'$,

$$f'_x(x, y) = \ln(x + 2y) + x\frac{1}{x + 2y}(x + 2y)'_x = \ln(x + y) + x\frac{1}{x + 2y}1 = \ln(x + 2y) + \frac{x}{x + 2y}.$$

When differentiating w.r.t. y, x is treated as a constant thus

$$f'_2(x, y) = x\frac{1}{x + 2y}(x + 2y)'_y = x\frac{1}{x + 2y}2 = \frac{2x}{x + 2y}.$$

Another related notion is the *directional derivative.*

Definition 2. The directional derivative of a function $f(x, y)$ at the point (x_0, y_0) in the direction $\mathbf{v} = (a, b)$ equals

$$D_{\mathbf{v}} f(x_0, y_0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(f(x_0 + a\epsilon, y_0 + b\epsilon) - f(x_0, y_0)).$$

This notion reflects the fact that, while the argument of the function $f(x, y)$ is a point (x, y) on a plane, there are many possibilities to perturb (that is, to change slightly) this argument. Simply speaking, when the argument is shifted by ϵ in the direction $\mathbf{v}$, the value of the function is changed by $\approx (D_{\mathbf{v}} f) \cdot \epsilon$.
The notions of partial and directional derivatives are closely related. The partial derivatives in x, y are the derivatives in basic directions $e_1 = (1, 0), e_2 = (0, 1)$ respectively. On the other hand, directional derivatives typically can be expressed as linear combinations of partial ones.

Theorem 1. *Let $f(x, y)$ have continuous partial derivatives at a point (x_0, y_0). Then for any $\mathbf{v} = (a, b)$ the directional derivative in this point equals*

$$D_{\mathbf{v}} f(x_0, y_0) = a f'_x(x_0, y_0) + b f'_y(x_0, y_0). \tag{1}$$

This leads to the definition of the *vector derivative*, or a *gradient*, which combines and unifies all the previous definitions.

Definition 3. The gradient of a function $f(x, y)$ at the point (x_0, y_0) is the vector

$$\nabla f(x_0, y_0) = (f'_x(x_0, y_0), f'_y(x_0, y_0)).$$

Equality (1) now can be written as

$$D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v},$$

where $\cdot$ stands for the scalar product. That is, arbitrary directional derivative equal to the scalar product of the direction with a given vector, *the gradient* of the function in the given point.

Sample problem **3:** For the function $f(x, y) = (x + y)2^x$ calculate the gradient and find the direction $\mathbf{v}$ with $|\mathbf{v}| = 1$ such that the derivative in this direction at the point $(0, 1)$

- equals 0;
- is maximal possible;
- is minimal possible.

Solution: Calculating the partial derivatives, we get

$$\nabla f(x, y) = (2^x + (x + y)2^x \ln 2, 2^x),$$

thus in particular

$$\nabla f(0, 1) = (1 + \ln 2, 1).$$

The since the directional derivative is the scalar product with the vector $\nabla f(0, 1)$, to get the required properties we have to take the unit length $\mathbf{v}$, respectively, orthogonal to, having the same direction with, and having the opposite direction with $\nabla f(0, 1)$. This gives the answers

- $\mathbf{v} = \frac{1}{(1+\ln 2)^2+1}(1 + \ln 2, -1)$;
- $\mathbf{v} = \frac{1}{(1+\ln 2)^2+1}(1 + \ln 2, 1)$;
- $\mathbf{v} = \frac{1}{(1+\ln 2)^2+1}(-1 - \ln 2, -1)$.

A convenient geometric way to understand better the notion of derivative and the properties of a function of two (or more) variables is provided by the notion of a *section* (or *cross-section*, or *trace*) of a function.

Definition 4. For the function of two variables $f(x, y)$ its section in the direction $\mathbf{v} = (a, b)$ at the point (x_0, y_0) is the function $F_{(x_0,y_0),\mathbf{v}}(t) = f(x_0 + at, y_0 + bt), t \in \mathbb{R}$.

In simple words, to build a section we pick a line $\ell = \{(x,y) = (x_0,y_0) + t\mathbf{v}\}$, restrict the function $f(x,y)$ to this line, and then consider this restriction as a function of one variable t used to parametrize the line. While the graph of the entire function is a surface and may have a complicated geometric shape, for each section the graph is a curve, which is easier to visualize and analyse. On the other hand, the entire graph of the function is a collection of the graphs of its sections.
By the definition, the directional derivative $D_{\mathbf{v}}f(x_0,y_0)$ is the derivative of the corresponding section $F_{(x_0,y_0),\mathbf{v}}(t)$ at the point $t=0$. In other words, the directional derivative $D_{\mathbf{v}}f(x_0,y_0)$ is the slope of the tangent line to the corresponding section. Collecting the tangent lines for various sections, we get the *tangent plane* for the graph $z = f(x,y)$ at the point (x_0,y_0,z_0), $z_0 = f(x_0,y_0)$. This tangent plane is given by the equation

$$z = z_0 + f'_0(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0). \tag{2}$$

Sample problem **4:** Write the general and the directional forms of equation of the tangent plane to the graph of the function $f(x,y) = x\sqrt{y^3+1}$ at the point $(1,2,z_0)$.
Solution: We have $z_0 = 1\sqrt{2^3+1} = 3$,

$$f'_x(1,2) = \sqrt{2^3+1} = 3, \quad f'_y(1,2) = -\frac{x}{2\sqrt{y^3+1}}3y^2|_{(x,y)=(1,2)} = -2.$$

Then the equation of the tangent line is

$$z = 3 + 3(x-1) - 2(y-2),$$

which leads to the normal form

$$3x - 2y - z + 7 = 0.$$

To write the directional form, observe that two following vectors, corresponding to the sections in the basic directions, span the plane:

$$\mathbf{u} = (1,0,f'_x(1,2)) = (1,0,3), \quad \mathbf{w} = (0,1,f'_y(1,2)) = (0,1,-2).$$

Then the parametric form of the equation of the tangent plane is

$$\begin{cases} x = 1+t \\ y = 2+s \\ z = 3+3t-2s \end{cases}$$

The tangent plane (2) is the graph of a linear function; for small values of the increments $(x - x_0), (y-y_0)$ this function serves as a natural simple approximation for (typically) complicated function $f(x,y)$. Namely, the following approximate formula holds:

$$f(x,y) \approx f(x_0,y_0) + f'_0(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0), \tag{3}$$

with the sign $\approx$ used in the sense that the difference $\delta(x,y)$ between the left- and right-hand sides is negligible when compared with the length of the increment; that is,

$$\lim_{x\to x_0, y\to y_0} \frac{\delta(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0.$$

Sample problem **5:** The lengths of sides of a rectangular box are 10 cm, 15 cm, 20 cm. How the volume of the box will change when the first two sides are increased by 1 mm, and the last one is decreased by 2 mm?

Solution: The volume is given by the function $V = f(x, y, z) = xyz$, where the x, y, z are the lengths of the sides. Then for $x_0 = 10, y_0 = 15, z_0 = 20$

$$f'_x(x_0, y_0, z_0) = yz|_{x=x_0, y=y_0, z=z_0} = 300, \quad f'_y(x_0, y_0, z_0) = 200, \quad f'_z(x_0, y_0, z_0) = 150.$$

Then using the approximate formula (3) we get

$$f(10 + 0.1, 15 + 0.1, 20 - 0.2) - f(10, 15, 20) \approx 300 \cdot 0.1 + 200 \cdot 0.1 + 150 \cdot (-0.2) = 20\,\text{cm}^3$$

Problems to solve

1. Calculate partial derivatives of the functions
(a) $f(x, y) = \dfrac{\sin x + \ln y}{\text{arcsin}x + \text{arccos}y)}$, (b) $f(x, y) = (x + y)\sqrt{x^2 + y^2}$, (c) $f(x, y, z) = \cos(x \sin(y \cos z))$,
(d) $f(x, y) = e^y \ln(xy)$, (e) $f(x, y) = \ln\left(\sqrt[4]{x^4 + y^4}\right)$ (f) $f(x, y) = \sin(x \arccos(y))$.

2. Calculate the directional derivatives of the functions in given directions
(a) $f(x, y) = \dfrac{x + y^2}{x^2y^3}, \vec{v} = (1, 2)$, (b) $f(x, y) = \sqrt{x^3 + y^3}, \vec{v} = (-1, 1)$.

3. Find the direction such that the function $f(x, y) = e^{xy}(x + y^2)$ at the point $(0, 2)$ has the derivative in this direction equal 0.

4. Find the directional derivative of the function $f(x, y) = y - x^2 + 2\ln(xy)$ at the point $(-1, -1)$ in the direction $\vec{v}(\alpha)$, which is the unit vector that constitutes the angle α with the positive OX-semiaxis. Find the values of α for which the derivative takes its maximal and minimal values.

5. Calculate the gradients of the functions in given points
(a) $f(x, y) = x^4 + x^2y^2 + 2$, $(-1, 2)$, (b) $f(x, y) = (1 + y)^x$, $(1, 1)$,
(c) $f(x, y) = e^x(x + y^2)$, $(2, 1)$, (d) $f(x, y) = 1 + y - x^2 + 2\ln(xy)$, $(1, 1)$.

6. The altitude $H = 100$ mm and the diameter of the base $D = 50$ mm of a cylinder are measured with the error ± 1 mm. With which accuracy one can give the value for the volume of the cylinder?

7. The lengths of the sides of a rectangular box are measured with the error 5mm each, and the values are $3, 4$, and 5cm. With which accuracy one can give the value

(a) of the volume of the box;

(b) of the surface area of the box?

8. Solve the previous problem if the measurement errors for the sides are $3, 4$, and 5mm respectively.

9. Write the general and the directional forms of equation of the tangent plane to the graph of the function in the given point
(a) $f(x, y) = x^2\sqrt{y^2 + 1}$, $(1, 3, z_0)$, (b) $f(x, y) = e^{x+2y}$, $(2, -1, z_0)$,
(c) $f(x, y) = \dfrac{\arcsin x}{\arccos x}$, $(\dfrac{-1}{2}, \dfrac{\sqrt{3}}{2}, z_0)$.

10. Find the points on the graph of the function $f(x, y) = \text{arctg}\dfrac{x}{y}$ where the tangent plane is parallel to the plane $x + y - z = 5$.

11. Find the tangent plane to the graph of the function $f(x, y) = x^2 + y^2$ which is orthogonal to the line $x = t, y = t, z = 2t, t \in \mathbf{R}$.