MATHEMATICAL ANALYSIS 2

Worksheet 3.

Local and global extrema. Functions with constraints. Conditional extrema: Lagrange's multipliers method.

Theory outline and sample problems

The notion of a local extremum, studied in the previous worksheet, is closely related to the problem of calculating of the maximum and minimal values of the function; that is, finding the *global extrema*.

Definition 1. A point (x_0, y_0) is a global maximum of a function $f(x, y)$ on the set D if

$$f(x, y) \leqslant f(x_0, y_0), \quad (x, y) \in D.$$

A point (x_0, y_0) is a global minimum, if

$$f(x, y) \geqslant f(x_0, y_0), \quad (x, y) \in D.$$

Global extremum point is a point of either global maximum or global minimum.

The following important fact describes the situation where the global extremal points exist for sure, and thus the question is just to locate them. To formulate it, we have to define the notions of *internal*, *external* and *boundary* points.

Definition 2. A point $(x_0, y_0) \in D$ is internal for D, if there exists (small) $\epsilon > 0$ such that the ball $B_\epsilon(x_0, y_0)$ is contained in D. A point (x_0, y_0) is external for D, if there exists (small) $\epsilon > 0$ such that the ball $B_\epsilon(x_0, y_0)$ does not intersect D. A point which is neither internal nor external is called boundary; that is, (x_0, y_0) is boundary for D if for every $\epsilon > 0$ contains both some points from D and some points outside from D.

Definition 3. A set D is called *closed* if it contains all its boundary points.

Example: Consider $D = \{(x, y) : x^2 + y^2 \leqslant 1\}$, then a point (x_0, y_0) is internal, external, or boundary if $x_0^2 + y_0^2 < 1$, $x_0^2 + y_0^2 > 1$, or $x_0^2 + y_0^2 = 1$, respectively, i.e. the point lies inside, outside or on the unit circle. Since the unit circle is included into D, the set D is closed.

Theorem 1. *(The Weierstrass extreme value theorem) A continuous function $f(x, y)$ defined on a closed bounded set $D \subset \mathbb{R}^2$ attains its maximum and minimum values.*

The conditions for D to be bounded and closed are crucial. Without them, it may happen that the function infinitely approaches its upper/lower bound, without reaching this value.

Example: Consider $D = \{(x, y) : x^2 + y^2 < 1\}$, which is bounded but *not closed*. Then the function $f(x, y) = x^2 + y^2$ does not attain its maximum on D: $f(x, y)$ takes any value < 1, but does not take the value 1. Next, consider the set $D = \{(x, y) : x \geqslant 0, y \geqslant 0, xy \leqslant 1\}$, which is closed but not bounded. Then the function $f(x, y) = x$ does not attain its minimum on D: $f(x, y)$ takes any value > 0, but does not take the value 0.

Now, let us assume that we are in the situation of Theorem 1; that is, the global minimum/maximum exist. How to find them? One natural approach is to observe that any *global* minimum/maximum is a *local* minimum/maximum. The inverse is not true, in general; but still we can try to find the global (say) minimum by finding all the local minima (typically, there will be few of them), and then choosing among them the one where the value of the function is the smallest. The natural tool for finding all the local extrema is the necessary condition, statement I of Theorem 4 in the previous worksheet. However, this simple way of thinking is not completely satisfactory because of the hidden limitations, which we illustrate by the following example.

Example: Consider the function $f(x, y) = x + 2y$ on the set $D = \{(x, y) : x^2 + y^2 \leqslant 1\}$. The gradient equals $\nabla f(x, y) = (1, 2)$, thus the equation

$$\nabla f(x, y) = \vec{0}$$

does not have any solutions, and there is no critical points for this function. Applying without enough care the necessary condition for local extrema would lead to the statement that there is no local extrema at all – which is obviously incorrect, because Theorem 1 tells us that there exist global maximum and minimum, which are local maximum and minimum, as well. The careful inspection reveals the source of the mistake: the necessary condition applies only to *internal* points; that is, for $x^2 + y^2 < 1$. What the calculation above actually tells us it that the local (and thus global) extrema of the function can be located on the boundary $x^2 + y^2 = 1$, only.

The previous example shows one natural – but definitely not the only one – reason for a study of a function under additional *constraints*; in the example the objective function is $f(x, y) = x + 2y$, while the constraint is $x^2 + y^2 = 1$.

Definition 4. A point (x_0, y_0) is a local maximum of a function $f(x, y)$ under the constraint $g(x, y) = 0$, if $g(x_0, y_0) = 0$ and there exists a (small) ball $B_\epsilon(x_0, y_0)$ centered at this point, such that

$$f(x, y) \leqslant f(x_0, y_0) \quad \text{for any } (x, y) \in B_\epsilon(x_0, y_0) \text{ such that } g(x, y) = 0.$$

A point (x_0, y_0) is a local minimum under the constraint $g(x, y) = 0$, if

$$f(x, y) \geqslant f(x_0, y_0) \quad \text{for any } (x, y) \in B_\epsilon(x_0, y_0) \text{ such that } g(x, y) = 0.$$

The following theorem gives a convenient tool for calculation of local extrema under constraints.

Theorem 2. *(Necessary condition for a local extremum under constraint) Let (x_0, y_0) be a local maximum/minimum of a function $f(x, y)$ under the constraint $g(x, y) = 0$. Let also $f(x, y), g(x, y)$ nbe differentiable in some (small) ball $B_\epsilon(x_0, y_0)$ and g satisfy the non-degeneracy condition*

$$\nabla g(x_0, y_0) \neq \vec{0} \iff (g'_x(x_0, y_0))^2 + (g'_y(x_0, y_0))^2 > 0. \tag{1}$$

Then there exists number $\lambda_0 \in \mathbb{R}$ such that, for the function

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y),$$

the following identities hold:

$$\partial_x F(x_0, y_0, \lambda_0) = 0, \quad \partial_y F(x_0, y_0, \lambda_0) = 0. \tag{2}$$

Remark. The number λ is called the *Lagrange multiplier*, and the function $F(x, y, \lambda)$ is called the lagrange function. Note that

$$\partial_\lambda F(x, y, \lambda) = g(x, y),$$

hence because (x_0, y_0) satisfies the constraint $g = 0$ we always have $\partial_\lambda F(x_0, y_0, \lambda_0) = 0$. Thus identities (2) can be equivalently written as

$$\nabla F(x_0, y_0, \lambda_0) = \vec{0} \Longleftrightarrow \begin{cases} \partial_x F(x_0, y_0, \lambda_0) = 0 \\ \partial_y F(x_0, y_0, \lambda_0) = 0 \\ \partial_\lambda F(x_0, y_0, \lambda_0) = 0 \end{cases} \tag{3}$$

Remark. The similar statement is available for larger number of variables and, possibly, for more number of constraints. Say, for a function $f(x, y, z)$ of three variables under one constraint $g(x, y, z) = 0$ the Lagrange function is defined as $F(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z)$ and equation (3) takes the form

$$\nabla F(x_0, y_0, z_0 \lambda_0) = \vec{0} \Longleftrightarrow \begin{cases} \partial_x F(x_0, y_0, z_0, \lambda_0) = 0 \\ \partial_y F(x_0, y_0, z_0, \lambda_0) = 0 \\ \partial_z F(x_0, y_0, z_0, \lambda_0) = 0 \\ \partial_\lambda F(x_0, y_0, z_0, \lambda_0) = 0; \end{cases}$$

for a function $f(x, y, z)$ of three variables under two constraints $g(x, y, z) = 0, h(x, y, z) = 0$ the Lagrange function is defined as $F(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z) + \mu h(x, y, z)$ with two Lagrange multipliers λ, μ, and equation (3) takes the form

$$\nabla F(x_0, y_0, z_0 \lambda_0, \mu_0) = \vec{0} \Longleftrightarrow \begin{cases} \partial_x F(x_0, y_0, z_0, \lambda_0, \mu_0) = 0 \\ \partial_y F(x_0, y_0, z_0, \lambda_0, \mu_0) = 0 \\ \partial_z F(x_0, y_0, z_0, \lambda_0, \mu_0) = 0 \\ \partial_\lambda F(x_0, y_0, z_0, \lambda_0, \mu_0) = 0 \\ \partial_\mu F(x_0, y_0, z_0, \lambda_0, \mu_0) = 0. \end{cases}$$

In the first case the non-degeneracy condition looks similarly to (1):

$$\nabla g(x_0, y_0, z_0) \neq \vec{0} \Longleftrightarrow (g'_x(x_0, y_0, z_0))^2 + (g'_y(x_0, y_0, z_0))^2 + (g'_z(x_0, y_0, z_0))^2 > 0, \tag{4}$$

in the second case it is required that the matrix of partial derivatives for the constraints has the (maximal) rank 2:

$$\operatorname{rank} \begin{pmatrix} g'_x(x_0, y_0, z_0) & g'_y(x_0, y_0, z_0) & g'_z(x_0, y_0, z_0) \\ h'_x(x_0, y_0, z_0) & h'_y(x_0, y_0, z_0) & h'_z(x_0, y_0, z_0) \end{pmatrix} = 2. \tag{5}$$

Theorem 2 tells us that any local extremum under a constraint is actually a critical point, without

any constraints, for the Lagrange function with a proper choice of the Lagrange mutliplier λ. Then the global extrema with a given constraint are contained in the list of the critical points for the Largange function, i.e. the solutions to (3). So, a practical algorithm for finding the global extrema is to

- list all the 'suspicious points' (x, y), which correspond to critical points for the Lagrange function;
- calculate the values of the objective function in each point from the list, and choose the maximal and the minimal one.

Let us illustrate this algorithm by completing the solution of our original example.

Sample problem **1:** Find the extrema of the function $f(x,y) = x + 2y$ under the constraint $x^2 + y^2 = 1$.
Solution: Write the equation for the critical points of the Lagrange function:

$$\begin{cases} 1 + 2\lambda x = 0 \\ 2 + 2\lambda y = 0 \\ x^2 + y^2 = 1 \end{cases}$$

This is a system of 3 equations with three unknowns, and theoretically we can find x, y, λ. However, the particular value of λ is not important for us, hence we find only x, y. Namely, it follows from the 1st and 2nd equations that $y = 2x$, then from the third equation

$$(1+4)x^2 = 1 \Longrightarrow x = \pm\frac{1}{\sqrt{5}}.$$

Thus we have the following 'suspicious points': $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}), (-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$, and the corresponding values of the function $f(x,y)$ are $\sqrt{5}, -\sqrt{5}$. Thus, the maximal point is $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, the minimal point is $(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$. This answers the original question: the function $f(x,y) = x + 2y$ takes on the disk $\{x^2 + y^2 \leqslant 1\}$ its maximal value $\sqrt{5}$ at the (boundary) point $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, and its minimal value $-\sqrt{5}$ at the (boundary) point $(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$.

Let us consider another, more sophisticated, example.
Sample problem **2:** A rectangular box that is open at the top must have a volume of 32 cm^3 What must its dimensions be so that its total area will be minimal?
Solution: Let x, y, z be the lengths of the sides of the box, then the constraint is $g(x,y,z) = xyz - 32 = 0$, and the objective function is $f(x,y) = xy + 2yz + 2xz$, where x, y correspond to horizontal sides and z to the vertical one. The equation for the critical points is

$$\begin{cases} y + 2z + \lambda yz = 0 \\ x + 2z + \lambda xz = 0 \\ 2x + 2y + \lambda xy = 0 \\ xyz = 32. \end{cases}$$

Multiplying the 1st equation by x, 2nd by y, and 3rd by z, we get

$$-\lambda xyz = xy + 2xz = xy + 2yz = 2xz + 2yz,$$

which yields

$$2xz = 2yz, \quad xy = 2xz.$$

Since $xyz = 23$, none of these numbers equal 0 and thus we can divide the above identities by $2z$ and x to get

$$x = y, \quad y = 2z.$$

Combined with $xyz = 32$ this gives the unique 'suspicious point' $(x,y,z) = (4,4,2)$. This will be the point of global minimum; the maximum is not attained because the function $f(x,y,z)$ is unbounded on the surface $xyz = 32$. To see this, one can take $y = x$, $z = 32x^{-2}$, and observe that

$$f(x, x, 32x^{-2}) \to \infty, \quad x \to \infty.$$

In the previous problem we had to provide an additional analysis because the surface $xyz = 32$, specified by the constraint, was unbounded and thus the Weierstrass extreme value theorem is not directly applicable. Let us consider one more example, where we will see one how this question can be treated can be treated for some classes of constraints.

Sample problem **3:** On the surface $2x^2 + 3y^2 + 3z^2 - 2xy + 4xz = 15$ find the points of maximal and minimal values for z.
Solution: The objective function is $f(x, y, z) = z$, and the equation for the critical points is

$$\begin{cases} \lambda(4x - 2y + 4z) = 0 \\ \lambda(6y - 2x) = 0 \\ 1 + \lambda(6z + 4x) = 0 \\ 2x^2 + 3y^2 + 3z^2 - 2xy + 4xz = 15. \end{cases}$$

By the 3rd equality, $\lambda \neq 0$, hence the 2nd and the 1st equalities give subsequently

$$x = 3y, \quad 12y - 2y + 4z = 0 \Longrightarrow z = -\frac{5}{2}z.$$

Substituting to the 4th equality, we get

$$y^2(2 \cdot 9 + 3 + 3 \cdot \frac{25}{4} - 2 \cdot 3 - 4 \cdot 3 \cdot \frac{5}{2}) = 15 \Longleftrightarrow y^2 \frac{15}{4} = 15,$$

which gives two solutions $y = \pm 2$ and thus two 'suspicious points'

$$(6, 2, -5), \quad (-6, -2, 5).$$

Thus the maximal and the minimal values for z are equal 5 and -5, respectively - but to make such a conclusion we have to be sure that the maximum and the minimum are attained. To check this, note that the constraint function is *quadratic*, i.e. its a second order polynomial of the variables x, y, z. To check if the corresponding surface is bounded, one has to take the principal part (i.e. the one with the terms of the degree 2 - in our case it is the entire function) and write it into the form

$$(x, y, z)A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

with a symmetric matrix A. The surface $g(x, y, z) = c$ is bounded for any c if, and only if, the matrix A is either positive or negative definite. In the particular example:

$$A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix}, \quad \det A_1 = 2 > 0, \quad \det A_2 = 5 > 0, \quad \det A_3 = 3 > 0,$$

and by the Sylvester criterion the matrix A is positive definite. Thus the surface $2x^2 + 3y^2 + 3z^2 - 2xy + 4xz = 15$ is bounded and the Weierstrass extreme value theorem applies. By this theorem, there should be at least one minimal point, and at least one maximal - and since we have in our list of 'suspicious points' only two points, we easily identify the maximum and the minimum.

Now, let us return to the problem of finding the global maximum and minimum of a function f on a certain closed domain D. As he have already seen, the 'suspicious points' for that are

(i) critical points for f, if they are interior for D;

(ii) critical points for f under the constraint $g(x,y) = 0$, if the boundary of D is a curve Γ described by this constraint.

It may also happen, that, instead of one curve Γ, the boundary is given by a union of several curves $\Gamma_1, \dots, \Gamma_m$, each of them described by its own constraint $g_i(x,y) = 0, i = 1, \dots, m$. Then the points where these parts of the boundary intersect, the 'angles', cannot be treated by means of the Lagrange multipliers method, and we have to include into the list of 'suspicious points' the third group

(iii) 'angles', i.e. the points of intersections of the different curves which constitute the boundary.

Sample problem **4:** Find the maximal and the minimal values of the function $f(x,y) = \sqrt{y - x^2} + \sqrt{x - y^2}$, on the domain $D = \{(x,y) : y \geqslant x^2, x \geqslant y^2\}$.
Solution: The boundary of the domain consists of two curves

$$\Gamma_1 = \{(x,y), y = x^2, x \in [0,1]\}, \quad \Gamma_2 = \{(x,y), y^2 = x, x \in [0,1]\},$$

which intersect in the points $(0,0), (1,1)$. Thus the first two 'suspicious points' (the 'angles') are $(0,0), (1,1)$.
Next, let us calculate the critical points under the constraints Γ_1, Γ_2. For Γ_1, $g(x,y) = y - x^2$, and the function $f(x,y)$ can be written in the simpler form $f(x,y) = \sqrt{x - y^2}$. The equation for the critical points of the Largange function has the form

$$\begin{cases} \frac{1}{2\sqrt{x-y^2}} - 2\lambda x = 0 \\ \frac{-2y}{2\sqrt{x-y^2}} + \lambda = 0 \\ y - x^2 = 0 \end{cases}$$

From the 1st and the 2nd equations, $y = \frac{1}{4x}$, and then by the 3rd equation

$$\frac{1}{4x} = x^2 \iff x = 2^{-\frac{2}{3}}.$$

Since $2^{-\frac{2}{3}} \in (0,1)$, this gives us another 'suspicious point' $(2^{-\frac{2}{3}}, 2^{-\frac{4}{3}})$ on the curve Γ_1. The calculation for the constraint Γ_2 is quite analogous, and we just give the answer: respective 'suspicious point' is $(2^{-\frac{4}{3}}, 2^{-\frac{2}{3}})$. Finally, we calculate the critical points for $f(x,y)$ without constraints. For that, write the equation $\nabla f = \vec{0}$:

$$\begin{cases} \frac{-2x}{2\sqrt{y-x^2}} + \frac{1}{2\sqrt{x-y^2}} = 0 \\ \frac{1}{2\sqrt{y-x^2}} - \frac{2y}{2\sqrt{x-y^2}} = 0 \end{cases}$$

From the 1st and the 2nd equations, $y = \frac{1}{4x}$, which substituted to the 1st equation gives

$$\frac{2x}{\sqrt{\frac{1}{4x} - x^2}} = \frac{1}{\sqrt{x - \frac{1}{16x^2}}} \iff \frac{4x^2}{\frac{1}{4x} - x^2} = \frac{1}{x - \frac{1}{16x^2}} \iff \frac{16x^3}{1 - 4x^3} = \frac{16x^2}{16x^3 - 1}$$

This equation has a solution $x = 0$, which however is not appropriate for us since then $y = \frac{1}{4x}$ is not well defined. When $x \neq 0$, the above equation is equivalent to

$$x(16x^3 - 1) = 1 - 4x^3 \iff 16x^4 + 4x^3 - x - 1 = 0$$

The polynomial $16x^4 + 4x^3 - x - 1$ has two roots $\pm\frac{1}{2}$, which can be found using the polynomial roots theorem. Dividing this polynomial by $4x^2-1 = 4(x-\frac{1}{2})(x+\frac{1}{2})$, we get polynomial $4x^2+x+1$, which does not have real roots. That is, the above equation has two solutions $x = \pm\frac{1}{2}$, which gives respective values for $y = \frac{1}{4x} = \pm\frac{1}{2}$. The point $(-\frac{1}{2}, -\frac{1}{2})$ lies outside the domain D, thus the only critical point for f inside D is $(\frac{1}{2}, \frac{1}{2})$. Summarizing, we get the following table of 'suspicious' points and the values of the function in these points:

(x, y)	$(0,0)$	$(1,1)$	$(2^{-\frac{2}{3}}, 2^{-\frac{4}{3}})$	$(2^{-\frac{4}{3}}, 2^{-\frac{2}{3}})$	$(\frac{1}{2}, \frac{1}{2})$
$f(x, y)$	0	0	$\sqrt{2^{-\frac{2}{3}}\frac{3}{4}}$	$\sqrt{2^{-\frac{2}{3}}\frac{3}{4}}$	1

Comparing the values of the function, we get that $f(x, y)$ takes its maximal value 1 at the point $(\frac{1}{2}, \frac{1}{2})$ and its minimal value 0 at the points $(0, 0)$ and $(1, 1)$.

Problems to solve

1. Find extremal points for the function $f(x, y) = x^2 + y^2$ under the constraint $g(x, y) = x^2 + y^2 - 4x - 2y - 15$.

2. Krzysztof and Szymon consume two products. If x and y are the quantities of the products (respectively), then Krzysztof's utility function is $U(x, y) = \ln x + 2\ln y$, and Szymon's is $U(x, y) = xy^2$. The prices of the products per unit are: $P_x = 5$ and $P_y = 2$. Krzysztof and Szymon have identical incomes, of 90 złeach. Find the optimal consumed quantity from each of the two products, for each of Krzysztof and Szymon.

3. Determine the point on the plane $4x - 2y + z = 1$ that is closest to the point $(-2, -1, 5)$.

4. On the surface $2x^2 + 2y^2 + 2z^2 - xy + 2yz + 2xz = 30$ find the points of maximal and minimal values for x. Check that the surface is bounded.

5. On the surface $x^2 + y^2 + z^2 - xy - xz + x + y - z = 1$ find the points of maximal and minimal values for x. Check that the surface is bounded.

6. Find the global extrema of $f(x, y) = x^2+4y^2$ on the domain bounded by the curves $x^2+(y+1)^2 = 4, y = -1$, and $y = x + 1$.

7. Find the global extrema of $f(x, y) = x^2 + y^2 - 6x + 6y$ on the disk of radius 2, centred at the origin.

8. Find the maximal and the minimal values of the functions on the given domains:

(a) $f(x, y) = 2x^3 + 4x^2 + y^2 - 2xy$, $D = \{(x, y) : x^2 \leqslant y \leqslant 4\}$;

(b) $f(x, y) = \sqrt{1 - x^2} + \sqrt{4 - x^2 - y^2}$, $D = \{(x, y) : x^2 \leqslant 1, x^2 + y^2 \leqslant 4\}$;

(c) $f(x, y) = x^2 - y^2$, D is the triangle with the vertices $(0, 1), (0, 2), (1, 2)$;

(d) $f(x, y) = x^4 + y^4$, $D = \{(x, y) : x^2 + y^2 \leqslant 9\}$