MATHEMATICAL ANALYSIS 2

Worksheet 3.

Definite integrals of two variables. Double and iterated integrals.

Preliminaries

Integration of functions of two (and more) variables strongly relies of the integration of functions of one variable. Hence, before proceeding with further, please recall the following facts and methods from Analysis I course about the integration:

- definitions of definite and indefinite integrals, the Newton-Leibnitz formula;
- tables of indefinite integrals;
- integration-by-parts for definite and indefinite integrals;
- change of variables for definite and indefinite integrals.

Theory outline and sample problems

Definite integral of two (and more) variables is defined, similarly to single variable case, as a limit of integral sums. Namely, let $R = [a, b] \times [c, d]$ be a rectangle, take its *partition* $\{R_i\}$ into a finite union of smaller rectangles which can intersect only by their sides and such that their union coincides with R. Take also a set of sample points $(x_i, y_i) \in R_i$; then for a given function $f(x, y)$ the corresponding *integral sum* is equal to

$$\sum_i f(x_i, y_i) \triangle x^i \triangle y^i,$$

where $\triangle x^i, \triangle y^i$ are the x- and y-sizes of the rectangle R_i. Clearly, $\triangle x^i \triangle y^i$ is the area of the rectangle R_i, and $f(x_i, y_i)\triangle x^i \triangle y^i$ is the signed volume of the parallelepiped with the base R_i and the height $f(x_i, y_i)$ (this number can be negative if $f(x_i, y_i)$, hence we are speaking about the signed volume).

Definition 1. The integral of a function $f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$ is defined by the formula

$$\iint_{[a,b]\times[c,d]} f(x, y)\, dxdy = \lim_{|\{R_i\}|\to 0} \sum_i f(x_i, y_i) \triangle x^i \triangle y^i,$$

where $|\{R_i\}|$ denotes the maximal size of a rectangle in the partition $\{R_i\}$. The above limit should exist for any sequence of partitions with their maximal sizes converging to 0 and for any respective sets of sample points, and should not depend neither on the partitions nor on the sets of sample points. In this case, the function $f(x, y)$ is called *integrable* on R.

The integral $\iint_{[a,b]\times[c,d]} f(x, y)\, dxdy$ has the meaning of the signed volume of the body contained between the Oxy plane and the graph of the function $f(x, y)$, with the restriction that the argument is located in R. The sign of the volume is + if $f(x, y) \geqslant 0$, i.e. the graph is located over the Oxy plane, and is − otherwise, i.e. when the graph is located under the Oxy plane. If the sign of the function is changed, there actually will be some cancelations of positive and negative parts of the volume.

Below, some **basic facts and properties** of the definite integral of two variables are listed.

1. *(Integrability of a continuous function)* If $f(x, y)$ is continuous on $[a, b] \times [c, d]$, then it is integrable on $[a, b] \times [c, d]$.

2. *(Linearity of the integral)* If functions $f(x, y)$, $g(x, y)$ are integrable on $[a, b] \times [c, d]$, then for any $\alpha, \beta \in \mathbb{R}$ function $\alpha f(x, y) + \beta g(x, y)$ is also integrable on $[a, b] \times [c, d]$, and

$$\iint_{[a,b]\times[c,d]} \Big(\alpha f(x, y) + \beta g(x, y)\Big)\, dxdy = \alpha \iint_{[a,b]\times[c,d]} f(x, y)\, dxdy + \beta \iint_{[a,b]\times[c,d]} g(x, y)\, dxdy.$$

3. *(Additivity of the integral)* Let a rectangle R be divided in two rectangles $R = R_1 \cup R_2$, which do not have common interior points. If $f(x, y)$ is integrable over R_1 and R_2, then it is integrable over R and

$$\iint_R f(x, y)\, dxdy = \iint_{R_1} f(x, y)\, dxdy + \iint_{R_2} f(x, y)\, dxdy$$

4. *(Equivalence of double and iterated integrals)* If $f(x, y)$ is a continuous function on $[a, b] \times [c, d]$, then

$$\iint_{[a,b]\times[c,d]} f(x, y)\, dxdy = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy. \tag{1}$$

The first integral in (1) is same with the definite integral we have defined; it is also frequently called the *double integral*. The second and the third integrals in (1) are called *iterated integrals*, and are calculated in the two-step procedure: first, we 'freeze' one variable and integrate w.r.t. another one, getting the function which depends on one 'frozen' variable; second, we integrate this new function. The difference between the second and the third integrals in (1) is the integration order: in the second integral, the variable x is first 'frozen', while in the third the 'frozen' variable is y.

In order to avoid writing $\Big[\dots\Big]$, the following notational convention is used:

$$\int_a^b \left[\int_c^d f(x, y)\, dy\right] dx = \int_a^b dx \int_c^d f(x, y)\, dy, \quad \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy = \int_c^d dy \int_a^b f(x, y)\, dx.$$

Identity (1) actually tells us that the calculation of an (double) integral of two variables can be reduced to calculation of two integrals of one variable. This is the efficient tool, which makes it possible to apply the standard methods of integral calculus to integrals of two (and more) variables.

Sample problem **1:** Calculate $\iint_{[-1,1]\times[0,1]} (x + y)^2\, dxdy$

Solution: Write the double integral as an iterated one (say, with integration first w.r.t. y and then w.r.t. x):

$$\iint_{[-1,1]\times[0,1]} (x + y)^2\, dxdy = \int_{-1}^1 dx \int_0^1 (x + y)^2\, dy.$$

Now we have to calculate two integrals; to illustrate all the possibilities available let us consider two different ways to calculate.

1. Write $(x+y)^2 = x^2+2xy+y^2$, then, since x is treated as a constant in the first integral,

$$\int_0^1 (x+y)^2\,dy = \int_0^1 (x^2+2xy+y^2)\,dy = x^2\int_0^1 1\,dy + 2x\int_0^1 y\,dy + \int_0^1 y^2\,dy.$$

Using the *Newton-Leibnitz formula* $\int_a^b f(x)\,dx = F(b)-F(a)$, where $F(x) = \int_a^b f(x)\,dx$ is an indefinite integral for $f(x)$ and the *table integral* $\int x^\alpha\,dx = \frac{1}{\alpha+1}x^{\alpha+1}+C, \alpha\neq -1$, we get

$$\int_0^1 (x+y)^2\,dy = x^2\left(y\Big|_0^1\right) + 2x\left(\frac{y^2}{2}\Big|_0^1\right) + \frac{y^3}{3}\Big|_0^1,$$

where we use the standard notation $F(x)\Big|_a^b$ for the increment $F(b)-F(a)$. Calculating the increments, we get

$$\int_0^1 (x+y)^2\,dy = x^2(1-0) + 2x\left(\frac{1}{2}-0\right) + \frac{1}{3} = x^2+x+\frac{1}{3}.$$

Then we calculate the second integral:

$$\int_{-1}^1 (x^2+x+\frac{1}{3})\,dx = \frac{x^3}{3}\Big|_{-1}^1 + \frac{x^2}{2}\Big|_{-1}^1 + \frac{1}{3}x\Big|_{-1}^1 = \frac{2}{3}+0+\frac{2}{3}.$$

This gives the answer

$$\int_0^1 (x+y)^2\,dy = \frac{4}{3}.$$

2. There is another possibility to calculate the first integral directly, i.e. without decomposing the polynomial $(x+y)^2$. Since x is a constant when we integrate in y, we can make a change of variables $z = x+y$, and get

$$\int_0^1 (x+y)^2\,dy = \left| \begin{array}{c} y = z-x \\ dy = (z-x)'_z dz = dz \end{array} \right| = \int_x^{x+1} z^2\,dz,$$

mind that the limits of the integration are changed to the value of the new variable z at the points $y=0$ and $y=1$, respectively. Then

$$\int_0^1 (x+y)^2\,dy = \int_x^{x+1} z^2\,dz = \frac{z^3}{3}\Big|_x^{x+1} = \frac{1}{3}\Big((x+1)^3 - x^3\Big).$$

Now we integrate in x, doing the similar change of variables:

$$\int_{-1}^1 \frac{1}{3}\Big((x+1)^3-x^3\Big)\,dz = \frac{1}{3}\int_{-1}^1 (x+1)^3\,dx - \frac{1}{3}\int_{-1}^1 x^3\,dx = \left| \begin{array}{c} z = x+1 \\ x = z-1 \\ dx = (z-1)'_z dz = dz \end{array} \right|$$

$$= \frac{1}{3}\int_0^2 z^3\,dz - \frac{1}{3}\int_{-1}^1 x^3\,dx = \frac{1}{3}\left(\frac{z^4}{4}\Big|_0^2\right) - \frac{1}{3}\left(\frac{x^4}{4}\Big|_{-1}^1\right) = \frac{1}{3}\left(\frac{2^4}{4}-\frac{0}{4}\right) - \frac{1}{3}\left(\frac{1}{4}-\frac{1}{4}\right) = \frac{1}{3}4 = \frac{4}{3}.$$

The notion of the integral have natural extension for more general domains of integration than just rectangles. The general definition is the following. Let $D\subset\mathbb{R}^2$ be a bounded set, and R be some rectangle which contains D. For $f(x,y)$ defined on D, define its extension $f^*(x,y)$ to R by

$$f^*(x,y) = \begin{cases} f(x,y), & (x,y)\in D; \\ 0, & (x,y)\in R\setminus D. \end{cases}$$

Definition 2. The function $f(x, y)$ is integrable on D if the function $f^*(x, y)$ is integrable on R; in this case

$$\iint_D f(x, y)\, dxdy = \iint_R f^*(x, y)\, dxdy.$$

There is a large class of special domains, where the double integral can be calculated as an iterated one.

Definition 3. A domain D is called *normal w.r.t. x variable* (or simply *x-normal*) if it can be written in the form

$$D = \{(x, y) : a \leqslant x \leqslant b, g(x) \leqslant y \leqslant h(x)\} \tag{2}$$

with some numbers $a < b$ and functions $g(x) \leqslant h(x)$. Similarly, a *normal w.r.t. y variable* (or simply *y-normal*) domain has the form

$$D = \{(x, y) : c \leqslant y \leqslant d, g(y) \leqslant x \leqslant h(y)\}. \tag{3}$$

Theorem 1. *If $f(x, y)$ is a continuous function on an x-normal domain D of the form* (2)*, then $f(x, y)$ is integrable on D and*

$$\iint_D f(x, y)\, dxdy = \int_a^b dx \int_{g(x)}^{h(x)} f(x, y)\, dy.$$

If D is y-normal of the form (3)*, then*

$$\iint_D f(x, y)\, dxdy = \int_c^d dy \int_{g(y)}^{h(y)} f(x, y)\, dx.$$

In simple words, normal domain is a 'curvilinear rectangle', with two straight opposite sides (each of these sides can reduce to a point), and two curved sides which correspond to the graphs of the functions $g(\cdot), h(\cdot)$. To calculate the double integral over such a 'curvilinear rectangle', one has to perform iterative integration, first taking the integral w.r.t. dependent variable (y for x-normal domains, x for y-normal), and then w.r.t. the basic variable (e.g. x for x-normal domains).

Sample problem **2:** Calculate the integral of the function $f(x, y) = \frac{x}{y}$ over the domain $D = \{(x, y) : 1 \leqslant x \leqslant 2, x \leqslant y \leqslant x^2\}$.
Solution:

$$\iint_D \frac{x}{y}\, dxdy = \int_1^2 dx \int_x^{x^2} \frac{x}{y}\, dy = \int_1^2 x\Big(\ln y\Big|_{y=x}^{y=x^2}\Big)\, dx = \int_1^2 x \ln x\, dx.$$

To calculate the latter integral, use the integration-by-parts formula:

$$\int_1^2 x \ln x\, dx = \left| \begin{array}{c} f(x) = \ln x, g'(x) = x \\ f'(x) = \frac{1}{x}, g(x) = x^2/2 \end{array} \right| = f(x)g(x)\Big|_1^2 - \int_1^2 f'(x)g(x)\, dx$$
$$= \frac{1}{2}x^2 \ln x\Big|_1^2 - \int_1^2 \frac{x}{2}\, dx = \frac{1}{2}x^2 \ln x\Big|_1^2 - \frac{x^2}{4}\Big|_1^2 = 2\ln 2 - \frac{3}{4}.$$

In some cases, it may be easier to perform the iterated integration for one order of the variables than for the other. To use such a possibility, we should be able to represent a given (say) x-normal domain as y-normal.

Sample problem **3:** Change the order of integration in the iterated integral

$$\int_1^2 dx \int_x^{x^2} \frac{x}{y}\, dy,$$

and then calculate it.

Solution: We have to represent D as a y-normal domain; that is, to indicate the minimal/maximal values for y and in which bounds, for a given y, the variable x takes its values. Both the upper bound $y = x^2$ and the lower bound $y = x$ are increasing on $[1, 2]$, hence the minimal value for y equals $c = 1$ (the lower bound at the left endpoint of the interval $[1, 2]$), and the maximal value equals $d = 4$ (the upper bound at the right endpoint, $2^2 = 4$).
Next, let us write into a system all the inequalities which define the domain D

$$\begin{cases} x \geqslant 1 \\ x \leqslant 2 \\ y \geqslant x \\ y \leqslant x^2 \end{cases},$$

and resolve the last two inequalities w.r.t. x for a fixed $y \in [1, 4]$:

$$\begin{cases} x \geqslant 1 \\ x \leqslant 2 \\ x \leqslant y \\ x \in (-\infty, -\sqrt{y}] \cup [\sqrt{y}, \infty) \end{cases}.$$

Since $x \geqslant 1$, the negative branch of the solutions of the last inequality is not involved, that is, we have the following set of restrictions on x:

$$\begin{cases} x \geqslant 1 \\ x \leqslant 2 \\ x \leqslant y \\ x \in [\sqrt{y}, \infty) \end{cases} \iff \begin{cases} x \geqslant 1 \\ x \leqslant 2 \\ x \leqslant y \\ x \geqslant \sqrt{y} \end{cases}$$

We have here two lower bounds and two upper bounds:

$$\begin{cases} x \geqslant 1 \\ x \geqslant \sqrt{y} \end{cases}, \quad \begin{cases} x \leqslant 2 \\ x \leqslant y \end{cases}$$

These pairs of bounds can be shortly written as

$$x \geqslant \max(1, \sqrt{y}), \quad x \leqslant \min(y, 2).$$

Note that, for $y \in [1, 4]$, $\max(1, \sqrt{y}) = 1$. That is, D is a y-normal domain of the form

$$D = \{(x, y) : y \in [1, 4], \sqrt{y} \leqslant \min(y, 2)\},$$

and the integral equals

$$\int_1^4 dy \int_{\sqrt{y}}^{\min(y,2)} \frac{x}{y}\, dx.$$

To calculate this integral, note that

$$\min(y,2) = \begin{cases} y, & y \in [1,2]; \\ 2, & y \in [2,4], \end{cases}$$

hence

$$\begin{aligned}
\int_1^4 dy \int_{\sqrt{y}}^{\min(y,2)} \frac{x}{y}\, dx &= \int_1^2 dy \int_{\sqrt{y}}^{y} \frac{x}{y}\, dx + \int_2^4 dy \int_{\sqrt{y}}^{y} \frac{x}{y}\, dx \\
&= \int_1^2 \left[\frac{1}{y} \left(\frac{x^2}{2} \right)_{x=\sqrt{y}}^{x=y} \right] dy + \int_2^4 \left[\frac{1}{y} \left(\frac{x^2}{2} \right)_{x=\sqrt{y}}^{x=2} \right] dy \\
&= \frac{1}{2} \int_1^2 (y-1)\, dy + \frac{1}{2} \int_2^4 \left(\frac{4}{y} - 1 \right) dy \\
&= \frac{1}{2} \left(\frac{y^2}{2} - y \right)_{y=1}^{y=2} + \frac{1}{2} \left(4 \ln y - y \right)_{y=2}^{y=4} = \frac{1}{2}\frac{1}{2} + \frac{1}{2}(4 \ln 2 - 2) = 2 \ln 2 - \frac{3}{4}.
\end{aligned}$$

Note that, changing the integration order, in the above example we get another integral which does not require integration-by-parts to be calculated.

Generally, it is easy to understand if the given domain D is normal. Namely, for D to be x-normal it is necessary and sufficient that, for any vertical line, the intersection of D by this line is either the empty set or a segment (possibly, degenerate one, i.e. one point). The numbers a, b will be then the minimal and the maximal values of x such that the corresponding vertical line with the fixed first coordinate x has non-empty intersection with D. For x between a, b, the values of the functions $g(x), h(x)$ then can be found as the left- and the right-ends of the segment which is obtained as the intersection of D by the vertical line with the first coordinate x. Similar description is true for y-normal domains; in this case instead of vertical lines on the level x one has to consider horizontal lines on the level y. *Control question:* Use the geometric approach to represent D from

the above example as y-normal. For that, draw the domain on the plane and consider its sections by horizontal line on the level y. What is the difference between pictures you got for the values $y \in [1,2]$ and $y \in [2,4]$?

Note that an x-normal domain is not necessarily y-normal, as one can see from the following example.

Example. Let $D = \{(x,y) : x \in [-1,1], 1 - x^2 \leqslant y \leqslant 2 - 2x^2\}$. Then the section of this domain by horizontal line on the level y is non-empty for $y \in [0,2]$. However, for $y \in (0,1)$ this section consists of two segments $[-\sqrt{1-y/2}, -\sqrt{1-y}]$, $[\sqrt{1-y}, \sqrt{1-y/2}]$ and the domain is not y-normal. We can decompose the domain D in three sub-domains, each of them being y-normal, by taking

$$D_1 = \{(x,y) : y \in [0,1], -\sqrt{1-y/2} \leqslant x \leqslant -\sqrt{1-y}\},$$

$$D_2 = \{(x,y) : y \in [0,1], \sqrt{1-y} \leqslant x \leqslant \sqrt{1-y/2}\},$$

$$D_3 = \{(x,y) : y \in [1,2], -\sqrt{1-y/2} \leqslant x \leqslant \sqrt{1-y/2}\}$$

Corresponding change of the integration order formula will have the form

$$\int_{-1}^{1} dx \int_{1-x^2}^{2-2x^2} f(x,y)\,dx$$
$$= \int_0^1 dy \int_{-\sqrt{1-y/2}}^{-\sqrt{1-y}} f(x,y)\,dy + \int_0^1 dy \int_{\sqrt{1-y}}^{\sqrt{1-y/2}} f(x,y)\,dx + \int_1^2 dy \int_{-\sqrt{1-y/2}}^{\sqrt{1-y/2}} f(x,y)\,dx.$$

This example explains the following

Definition 4. Domain D is called *regular* if is can be decomposed into a finite union of normal domains, which do not have common interior points.

For a regular domain, an integral over D is equal to the sum of the integrals over its normal components, which in turn can be calculated as iterated integrals. In many cases, even if the domain is normal it is practical to decompose it into components to make it easier to calculate the integral.

Sample problem **4:** Calculate the integral $\iint_D x^2 y dx dy$, where the bounded domain D is separated by the curves $y=3, y=\frac{1}{x}, y=\sqrt{x}$.

Solution: Drawing three curves $y=3, y=\frac{1}{x}, y=\sqrt{x}$, we see that the only bounded part of the domain determined by these curves is the normal domain bounded from above by $y=3$ and from below by $y=\max(\frac{1}{x},\sqrt{x})$. The intersection points for these bounds are $x=\frac{1}{3}$ and $x=9$; that is, the integration domain is

$$D = \{(x,y) : \frac{1}{3} \leqslant x \leqslant 9, \max(\frac{1}{x},\sqrt{x}) \leqslant y \leqslant 3\}.$$

To avoid using the function max(), we divide it in two domains

$$D_1 = \{(x,y) : \frac{1}{3} \leqslant x \leqslant 1, \frac{1}{x} \leqslant y \leqslant 3\}, \quad D_2 = \{(x,y) : 1 \leqslant x \leqslant 9, \sqrt{x} \leqslant y \leqslant 3\}.$$

Then the required integral equals

$$\begin{aligned}
\iint_D y\,dxdy &= \int_{\frac{1}{3}}^{1} dx \int_{\frac{1}{x}}^{3} y\,dy + \int_1^9 dx \int_{\sqrt{x}}^{3} y\,dy \\
&= \frac{1}{2}\int_{\frac{1}{3}}^{1} (9-x^{-2})\,dx + \frac{1}{2}\int_1^9 (9-x)\,dx \\
&= \frac{1}{2}\Big(9x+\frac{1}{x}\Big)_{x=1/3}^{x=1} + \frac{1}{2}\Big(9x-\frac{x^2}{2}\Big)_{x=1}^{x=9} \\
&= \frac{1}{2}\Big((9+1)-(9\frac{1}{3}+3)\Big) + \frac{1}{2}\Big((81-\frac{81}{2})-(9-\frac{1}{2})\Big) = 18.
\end{aligned}$$

Problems to solve

Part A

1. Calculate the definite integral. *(This problem is meant as a repetition for the integration of functions of one variable)*

(a) $\int\limits_{-1}^{1} e^{-x}\,dx$, (b) $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} \mathrm{ctg} x\,dx$, (c) $\int\limits_{e^{-2}}^{e^2} \ln x\,dx$. (d) $\int\limits_{0}^{\pi} \sin 2x\,dx$,

2. Calculate the definite integral over the given rectangles.

(a) $\iint_R (x^2+y^3-xy)\,dxdy, R=[0,1]\times[0,1]$, (b) $\iint_R \frac{x}{y^2}\,dxdy, R=[1,2]\times[2,4]$,

(c) $\iint_R (1+x+y)^3\,dxdy, R=[0,2]\times[0,1]$, (d) $\iint_R x\sin(xy)\,dxdy, R=[0,1]\times[\pi,2\pi]$,

(e) $\iint_R \frac{x+y}{e^x}\,dxdy, R=[0,1]\times[0,1]$, (f) $\iint_R e^{2x-y}\,dxdy, R=[0,1]\times[-1,0]$.

3. Calculate the iterated integrals and draw the domains of integration

(a) $\int_0^1 dx\int_x^{x^2}\frac{y}{x^2}\,dy$, (b) $\int_1^4 dx\int_x^{2x} x^2\sqrt{y-x}\,dy$, (c) $\int_0^3 dx\int_0^x \sqrt{x^2+16}\,dy$.

Part B

4. Change the order of integration in the iterated integrals and calculate the integrals

(a) $\int_1^4 dx\int_x^{3x} x\sqrt{y-x}\,dy$, (b) $\int_0^3 dx\int_0^x \sqrt{x^2+1}\,dy$, (c) $\int_1^e dx\int_{\ln x}^1 \frac{1}{e^y-1}\,dy$.

5. Calculate the integrals over the normal domains bounded by the given curves

(a) $\iint_D xy^2dxdy, y=x, y=2-x^2$, (b) $\iint_D x^2ydxdy, y=-2, y=\frac{1}{x}, y=-\sqrt{-x}$,

(c) $\iint_D e^{x/y}dxdy, y=\sqrt{x}, x=0, y=1$, (d) $\iint_D (xy+4x^2)dxdy, y=x+3, y=x^2+3x+3$.