MATHEMATICAL ANALYSIS 2

Worksheet 5.

Change of variables formula for an integral of two variables. Polar coordinates

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Theory outline and sample problems

The change of variables formula for an integral of two variables is formulated in the following setting. Consider a pair of variables (x, y) taking values in a domain D, and assume that the values x, y can be obtained as functions of two other variables u, v taking values in a domain Δ; that is, there exist a pair of functions $F(u, v) = (F_1(u, v), F_2(u, v))$ such that

$$\begin{cases} x = F_1(u, v), \\ y = F_2(u, v), \end{cases} \qquad (u, v) \in \Delta. \tag{1}$$

The function $F(u, v)$ is actually a function of two variables, with its values being vectors in $\mathbb{R}^2$. For such functions, the analogue of the notion of the derivative is given by the *Jacobian matrix*.

Definition 1. Let $F(u, v) = (F_1(u, v), F_2(u, v))$ and the scalar-valued functions $F_1(u, v), F_2(u, v)$ are differentiable. The *Jacobian matrix* of the function $F(u, v)$ is given by

$$DF(u, v) = \begin{pmatrix} \frac{\partial}{\partial u} F_1(u, v) & \frac{\partial}{\partial v} F_1(u, v) \\ \frac{\partial}{\partial u} F_2(u, v) & \frac{\partial}{\partial v} F_2(u, v) \end{pmatrix}.$$

In other words, the Jacobian matrix is the matrix composed from the gradients of F_1, F_2, considered as row vectors:

$$DF(u, v) = \begin{pmatrix} \nabla F_1(u, v) \\ \nabla F_2(u, v) \end{pmatrix}.$$

The determinant of the Jacobian matrix is called *the Jacobian determinant*, or simply *the Jacobian* of F, and is denoted by J_F:

$$J_F(u, v) = \det \begin{pmatrix} \frac{\partial}{\partial u} F_1(u, v) & \frac{\partial}{\partial v} F_1(u, v) \\ \frac{\partial}{\partial u} F_2(u, v) & \frac{\partial}{\partial v} F_2(u, v) \end{pmatrix}.$$

Theorem 1. *Let D, Δ be regular domains and $F : \Delta \to D$ be a differentiable mapping such that*

- *for each point $(x, y) \in D$ there exists $(u, v) \in D$ such that $F(u, v) = (x, y)$ (i.e. F is a* surjection*, or a '*mapping on*');*
- *for each internal point $(u, v) \in \Delta$, for any other point $(u', v') \in \Delta$ $F(u, v) \neq F(u', v')$ (i.e. on the interior of Δ, F is an* injection*, or a '*mapping in*').*

Then for any continuous function $f(x, y)$ the following change of variables formula *holds true:*

$$\iint_D f(x, y)\, dxdy = \iint_\Delta f(F_1(u, v), F_2(u, v))\, |J_F(u, v)|\, dudv. \tag{2}$$

To give a better understanding of the change of variables formula, let me give it once again, now emphasizing different parts in color:

$$\iint_D f(x, y)\, dxdy = \iint_\Delta f(F_1(u, v), F_2(u, v))\, |J_F(u, v)|\, dudv. \tag{3}$$

Now its visible that, to perform a change of variables $(x, y) = F(u, v)$, we have to do three things:

(a) change the old variables (x, y) in the function $f(x, y)$ by their expressions through the new variables (u, v);

(b) change the old *area element* $dxdy$ by the new one following the rule $dxdy \rightsquigarrow |J_F(u, v)|\, dudv$;

(c) change the domain of integration from D (for (x, y)) to Δ (for (u, v)).

Sample problem **1:** Let D be the domain bounded by $y = -x + 4$, $y = x - 1$, and $y = \dfrac{x-4}{3}$. Perform the change of variables $x = \dfrac{1}{2}(u + v), y = \dfrac{1}{2}(u - v)$ in the integral

$$\iint_D x\, dxdy$$

and then calculate the integral.

Solution: We have

$$D_F = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad J_F = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

Then $dxdy \rightsquigarrow \frac{1}{2}dudv$, and since $x = \frac{1}{2}(u + v)$,

$$\iint_D x\, dxdy = \frac{1}{4} \iint_\Delta (u + v)\, dudv.$$

To calculate the latter integral, we have to specify the domain Δ. For that, write the equations of the lines which define the domain D for (x, y), in the new coordinates u, v; these will be the lines defining the new domain Δ for (u, v):

$$\left[\begin{array}{l} y = -x + 4 \\ y = x - 1 \\ y = \frac{x-4}{3} \end{array}\right. \iff \left[\begin{array}{l} \frac{1}{2}(u - v) = -\frac{1}{2}(u + v) + 4 \\ \frac{1}{2}(u - v) = \frac{1}{2}(u + v) - 1 \\ \frac{1}{2}(u - v) = \frac{1}{6}(u + v) - \frac{4}{3} \end{array}\right. \iff \left[\begin{array}{l} u = 4 \\ v = 1 \\ u = 2v - 4 \end{array}\right.$$

Hence Δ is a triangle bounded by the horizontal line $v = 1$, vertical line $u = 4$, and the line $u - 2v + 4 = 0$. It is easy to represent this domain as (say) v-normal: the intersection point of $u = 4$ and $u - 2v + 4 = 0$ corresponds to $v = 4$, thus

$$\Delta = \{(u, v) : 1 \leqslant v \leqslant 4, 2v - 4 \leqslant u \leqslant 4\}.$$

Then

$$\begin{aligned} \iint_D x\, dxdy &= \frac{1}{4} \iint_\Delta (u + v)\, dudv = \frac{1}{4} \int_1^4 dv \int_{2v-4}^4 (u + v)\, du \\ &= \frac{1}{4} \int_1^4 (uv + \frac{v^2}{2})\Big|_{2v-4}^4 dv = \frac{1}{4} \int_1^4 (2v^2 + 2v(4 - v))\, dv = \int_1^4 2v\, dv = v^2\Big|_1^4 = 15. \end{aligned}$$

This example shows a typical situation where, after the change of variables, the domain D is transformed to a new domain which is much more convenient to deal with, because it has a simple representation as a normal domain, or even a rectangle. Namely, in the example above the original domain D is also a triangle, but after the change of variables the new triangle Δ has two sides parallel to the axes, which makes the further integration much simpler. Let us consider one more example of that kind.

Sample problem **2:** Let D be the domain bounded by curves $xy = 1, xy = 2, y = \sqrt{x}, y = 2\sqrt{x}$. Performing the proper change of variables, calculate the integral

$$\iint_D (x^3 + y^3)\, dxdy.$$

Solution: Domain D is a 'curvilinear rectangle', bounded by two curves of the form $xy = c$, and two curves of the form $x^{-1/2}y = c$. This gives a hint that the good choice of the new variables is $u = xy, v = x^{-1/2}y$; indeed, under such a choice the domain Δ will be just a (true) rectangle $\Delta = [1, 2] \times [1, 2]$. To perform the change of variables, we first have to express (x, y) as a function of (u, v). We have

$$\begin{cases} xy = u \\ x^{-1/2}y = v \end{cases} \Longleftrightarrow \begin{cases} x = u^{2/3}v^{-2/3} \\ y = u^{1/3}v^{2/3} \end{cases} ;$$

that is, the required change of variables is $(x, y) = F(u, v) = (F_1(u, v), F_2(u, v))$ with

$$F_1(u, v) = u^{2/3}v^{-2/3}, \quad F_2(u, v) = u^{1/3}v^{2/3}.$$

Then

$$J_F(u, v) = \begin{vmatrix} \frac{2}{3}u^{-1/3}v^{-2/3} & -\frac{2}{3}u^{2/3}v^{-5/3} \\ \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}u^{1/3}v^{-1/3} \end{vmatrix} = \frac{4}{9}v^{-1} + \frac{2}{9}v^{-1} = \frac{2}{3}v^{-1},$$

and

$$\begin{aligned} \iint_D (x + y)\, dxdy &= \iint_{[1,2]\times[1,2]} (u^2v^{-2} + uv^2)\frac{2}{3}v^{-1}\, dudv \\ &= \frac{2}{3}\int_1^2 u^2\, du \int_1^2 v^{-3}\, dv + \frac{2}{3}\int_1^2 u\, du \int_1^2 v\, dv \\ &= \frac{2}{3}\frac{1}{3}(2^3 - 1)\frac{1}{2}(1 - 2^{-2}) + \frac{2}{3}\frac{1}{2}(2^2 - 1)\frac{1}{2}(2^2 - 1) = \frac{25}{12} \end{aligned}$$

One frequently used particular coordinate system is the *polar coordinate system*, where the new variables are ρ, ϕ and the original variables x, y are given by

$$\begin{cases} x = \rho\cos\phi = P_1(\rho, \phi) \\ y = \rho\sin\phi = P_2(\rho, \phi) \end{cases} \tag{4}$$

The geometric meaning of the polar coordinates is close to the trigonometric form of a complex number; namely, $\rho = \sqrt{x^2 + y^2}$ is the modulus of the vector (x, y), and ϕ is defined similarly to the argument of the complex number $x + iy$, i.e. as the angle between the vector (x, y) and the Ox axis, measured counter clock-wise.

The Jacobian matrix and Jacobian determinant of this *polar* mapping $P(\rho, \phi) = (P_1(\rho, \phi), P_2(\rho, \phi))$ are given by

$$D_P(\rho, \phi) = \begin{pmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \end{pmatrix}, \quad J_P(\rho, \phi) = \begin{vmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \end{vmatrix} = \rho(\sin^2\phi + \cos^2\phi) = \rho. \tag{5}$$

Hence, the particular version of the change of variables formula for the polar coordinates is

$$\iint_D f(x,y)\,dxdy = \iint_\Delta f(\rho\cos\phi, \rho\sin\phi)\rho\,d\rho d\phi, \tag{6}$$

and the domain Δ should be such that $P(\rho,\phi)$ maps Δ onto D in a one-to-one way (except, possibly, the points on the boundary).
Typically, polar coordinates are easy to use when, in the description of the domain D, one of the following geometric shapes is involved:

- a ring between two circles centered at the origin;
- an angle between two rays, starting at the origin.

From the point of view of the polar coordinates, these two shapes can be written as follows:

- $r \leqslant \rho \leqslant R$, where r, R are the radii of the circles;
- $\alpha \leqslant \phi \leqslant \beta$, where α, β are the angles between the rays and the Ox axis, measured counter clock-wise.

Any intersection of two such shapes will be, from the point of view of the polar coordinates, just a rectangle, which will make it easy to calculate the integral after the change of the variables.

Sample problem **3:** Let D be quarter of the circle $\{x^2+y^2 \leqslant 2\}$ located in the 2nd quadrant. Calculate the integral

$$\iint_D x^2\,dxdy.$$

Solution: In the polar coordinates, the domain of integration has the form $\Delta = \{(\rho,\phi) : 0 \leqslant \rho \leqslant \sqrt{2}, \frac{\pi}{2} \leqslant \phi \leqslant \pi\}$. Thus

$$\iint_D x^2\,dxdy = \iint_{[0,\sqrt{2}]\times[\pi/2,\pi]} \rho^2\cos^2\phi\,\rho d\rho d\phi = \left[\int_0^{\sqrt{2}} \rho^3\,d\rho\right]\left[\int_{\pi/2}^{\pi} \cos^2\phi d\phi\right].$$

Since

$$\int_0^{\sqrt{2}} \rho^3\,d\rho = \frac{1}{4}\rho^4\Big|_0^{\sqrt{2}} = 1,$$

$$\int_{\pi/2}^{\pi} \cos^2\phi d\phi = \frac{1}{2}\int_{\pi/2}^{\pi}(1+\cos 2\phi)d\phi = \frac{1}{2}(1+\frac{1}{2}\sin 2\phi)\Big|_{\pi/2}^{\pi} = \frac{\pi}{4},$$

we get finally

$$\iint_D x^2\,dxdy = \frac{\pi}{4}.$$

Let us give one more example of such a kind, where one has to pay an extra attention for the choice of the bounds of the angular variable ϕ.

Sample problem **4:** Let D be defines by inequalities $x^2+y^2 \leqslant 1, y+x \geqslant 0, y-x \leqslant 0$. Calculate the integral

$$\iint_D y^2\,dxdy.$$

Solution: The first inequality $x^2 + y^2 \leqslant 1$ defines the unit circle centered at the origin. Two inequalities $y + x \geqslant 0, y - x \leqslant 0$ define the intersection of two half-planes, the one 'above' $y = -x$, and the one 'below' $y = x$. This is the angle between two rays, staring at the origin, and having their angles with the Ox axis equal $\frac{\pi}{4}$, $\frac{7\pi}{4}$, respectively. One has to be careful here, because writing e.g. $\frac{\pi}{4} \leqslant \phi \leqslant \frac{7\pi}{4}$ will give us the complement to the required angle instead of the angle itself. To get the required angle, one has to rotate the Ox axis in the clock-wise direction, i.e. to decrease the value of ϕ. That is, the domain of integration has the form $\Delta = \{(\rho, \phi) : 0 \leqslant \rho \leqslant 1, -\frac{\pi}{4} \leqslant \phi \leqslant \frac{\pi}{4}\}$. The rest of calculation is similar:

$$\begin{aligned}\iint_D y^2\,dxdy &= \iint_{[0,1]\times[-\pi/4,\pi/4]} \rho^2 \sin^2\phi\rho\,d\rho d\phi \\ &= \left(\frac{1}{4}\rho^4\right)\Big|_0^1 \left(\frac{1}{2}(1 - \frac{1}{2}\sin 2\phi)\right)\Big|_{-\pi/4}^{\pi/4} = \frac{1}{4}\left(\frac{\pi}{4} - \frac{1}{2}\right) = \frac{\pi - 2}{16}.\end{aligned}$$

The final example shows that the polar coordinates can be also useful when the domain D, parametrized in polar coordinates, is not a rectangle, but the function under the integral depends on ρ, ϕ in a simple way.

Sample problem **5:** Let D be defined by inequality $x^2 + y^2 \leqslant x$. Calculate the integral

$$\iint_D \sqrt{x^2 + y^2}\,dxdy.$$

Solution: The domain D can be written as

$$x^2 + y^2 - x \leqslant 0 \Longleftrightarrow x^2 + y^2 - 2\frac{1}{2}x + \frac{1}{4} \leqslant \frac{1}{4} \Longleftrightarrow \left(x - \frac{1}{2}\right)^2 + y^2 \leqslant \frac{1}{4},$$

hence it is a circle centered at the point $(\frac{1}{2}, 0)$ with the radius $\frac{1}{2}$. Let us discuss two different possibilities, both involving the polar coordinates.

I. Let us first change the variables $u = x - 1, v = y$, and then change u, v to the polar coordinates: $u = \rho\cos\phi, v = \rho\sin\phi$. The first change gives the domain $D_1 = \{(u, v) : u^2 + v^2 \leqslant \frac{1}{4}\}$, which is the circle centered at the origin, thus the second leads to $\Delta = \{(\rho, \phi) : \rho \leqslant \frac{1}{2}.\}$ Since $x = u + 1 = 1 + \rho\cos\phi$, $v = \rho\sin\phi$, we have

$$\sqrt{x^2 + y^2} = \sqrt{1 + 2\rho\cos\phi + \rho^2\cos^2\phi + \rho^2\sin^2\phi} = \sqrt{1 + 2\rho\cos\phi + \rho^2},$$

and the original integral transforms to

$$\int_0^{2\pi} d\phi \int_0^{\frac{1}{2}} \sqrt{1 + 2\rho\cos\phi + \rho^2}\,\rho d\rho$$

(mind the multiplier ρ, which is the Jacobian of the polar transform!). This integral is difficult to calculate; the main reason is that the function under the original integral and the change of variables were not adjusted to each other.

II. Let us change the original variables x, y to the polar coordinates directly: $x = \rho\cos\phi, y = \rho\sin\phi$. The function under the integral will be $\sqrt{x^2 + y^2} = \rho$. The domain Δ can be treated as

ϕ-normal in the following way: since D is located in the right half-plane w.r.t. Oy axis, ϕ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. For a given ϕ, the interval where ρ varies can be found as follows:

$$x^2+y^2-x\leqslant 0 \Longleftarrow \rho^2-\rho\cos\phi\leqslant 0 \Longleftarrow \rho\leqslant\cos\phi.$$

Now the original integral transforms to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \int_0^{\cos\phi} \rho\cdot\rho d\rho,$$

which is easy to calculate:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \int_0^{\cos\phi} \rho^2 d\rho = \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\phi\, d\phi = \left| \begin{array}{c} v=\sin\phi, dv=\cos\phi\, d\phi \\ \cos^2\phi = 1-\sin^2\phi = 1-v^2 \end{array} \right|$$

$$= \frac{1}{3}\int_{-1}^{1}(1-v^2)\,dv = \frac{1}{3}\left(v-\frac{v^3}{3}\right)\Big|_{-1}^{1} = \frac{1}{3}\left(2-\frac{2}{3}\right) = \frac{4}{9}.$$

Problems to solve

Part A

1. Calculate the Jacobians of the transformations:

(a) $x=4u-3v^2 \qquad y=u^2-6v$;

(b) $x=u^2v^3 \qquad y=4-2\sqrt{u}$;

(c) $x=\dfrac{v}{u} \qquad y=u^2-4v^2$.

2. Determine the domain Δ which is transformed by the given mapping to the given domain D.

(a) D is the ellipse $x^2+\dfrac{y^2}{36}\leqslant 1$, the transformation $x=\dfrac{u}{2}, y=3v$.

(b) D is the parallelogram with the vertices $(1,0),(4,3),(1,6)$ and $(-2,3)$, the transformation $x=\dfrac{1}{2}(u+v), y=\dfrac{1}{2}(u-v)$.

(c) D is the parallelogram with vertices $(2,0),(5,3),(6,7)$ and $(3,4)$, the transformation $x=\dfrac{1}{3}(v-u), y=\dfrac{1}{3}(4v-u)$.

(d) D is the domain bounded by $xy=1, xy=3, y=2$ and $y=6$, the transformation $x=\dfrac{v}{6u}, y=2u$.

3. Propose a transformation that will represent the triangle D with vertices $(1,0),(6,0)$ and $(3,8)$ as an image of a right triangle with the right angle occurring at the origin of the u,v system.

4. Propose a transformation that will represent the parallelogram D with vertices $(1,2)$, $(3,5)$, $(-1,0)$, $(1,3)$ as an image of a rectangle.

5. Perform the change of variables to the polar coordinates and evaluate the integrals. Draw the domain of integration in the Cartesian and polar coordinates

(a) $\iint_D xy\,dxdy,\ D: x^2+y^2 \leqslant 1, \dfrac{x}{\sqrt{3}} \leqslant y \leqslant x\sqrt{3};$

(b) $\iint_D y^2 e^{x^2+y^2}\,dxdy,\ D: x^2+y^2 \leqslant 1, x \geqslant 0, y \geqslant 0;$

(c) $\displaystyle\iint\limits_D (y^2+3x)\,dxdy$, D is the region in the 3rd quadrant between $x^2+y^2=1$ and $x^2+y^2=9$;

(d) $\displaystyle\iint\limits_D (4xy-7)\,dxdy,\ D: x^2+y^2 \leqslant 1, -x\sqrt{3} \leqslant y \leqslant x;$

(e) $\displaystyle\iint\limits_D (x^3+y^3)\,dxdy,\ D: x^2+y^2 \leqslant 1, x\sqrt{3} \leqslant y \leqslant -x.$

Part B

6. Performing an appropriate change of variables, evaluate the integrals

(a) $\displaystyle\iint\limits_D 6x-3y\,dxdy$ where R is the parallelogram with vertices $(1,0),(4,3),(5,7)$ and $(2,4)$;

(b) $\displaystyle\iint\limits_D xy^3\,dxdy$ where D is the domain bounded by $xy=1, xy=2, y=3$ and $y=4$;

(c) $\displaystyle\iint\limits_D (x+2y)\,dxdy$ where D is the triangle with vertices $(0,3),(4,1)$ and $(2,6)$;

(d) $\displaystyle\iint\limits_D x^2\,dxdy$, where D is the ellipse $x^2+\dfrac{y^2}{9} \leqslant 1$;

(e) $\displaystyle\iint\limits_D \sqrt{4-x^2-y^2}\,dxdy$, where D is the circle with radius 1 centered at $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$.

7. Find the area of the ellipse $(x-3)^2+4(y+1)^2 \leqslant 10$.