MATHEMATICAL ANALYSIS 2

Worksheet 7.

Taylor formula. Power series. Taylor-Maclaurin series

Theory outline and sample problems

We have seen that the derivative of the function can be used in order to approximate the function, in a vicinity of a given point, by a linear function:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0). \tag{1}$$

The approximate identity sign '≈' here can be understood various ways, most of them involving an information about the *approximation error*, or the *residual term*

$$R(x, x_0) = f(x) - f(x_0) - f'(x_0)(x - x_0)$$

Theorem 1. *Let function f be differentiable on an interval $[a, b]$ and $x_0 \in (a, b)$. Then*

(a)

$$\frac{R(x, x_0)}{|x - x_0|} \to 0, \quad x \to x_0;$$

(b) there exists a point θ, intermediate between points x and x_0, such that

$$R(x, x_0) = (f'(\theta) - f'(x_0))(x - x_0), \quad x \in [a, b].$$

Statement (a) in the above theorem tells us that, *infinitesimally*, i.e. when $x - x_0$ is (infinitely) small, the residue of the approximation is negligible w.r.t. the linear part. Statement (b) is of the principal importance, because it gives a bound for the approximation error for the given pair of points x, x_0:

$$|R(x, x_0)| \leqslant |x - x_0| \sup_{\theta \in [x_0, x]} |f'(\theta) - f'(x_0)|$$

Statement (b) is actually *the Lagrange theorem*, properly re-written; in its original form the Lagrange theorem (AKA the *Mean Value theorem*) states that

$$f(x) - f(x_0) = f'(\theta)(x - x_0).$$

The Taylor formula can be understood an extension of the above approximation formula, where instead of linear functions polynomials are used as approximations.

Theorem 2. *Let function f have n derivatives on an interval $[a, b]$ and $x_0 \in (a, b)$. Then*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + R_n(x, x_0),$$

where $n! = 1 \cdot 2 \cdot \cdots \cdot n$, and

(a)

$$\frac{R_n(x, x_0)}{|x - x_0|^n} \to 0, \quad x \to x_0;$$

(b) there exists a point θ, intermediate between points x and x_0, such that

$$R(x, x_0) = \frac{1}{n!}(f^{(n)}(\theta) - f^{(n)}(x_0))(x - x_0)^n.$$

If the function f have n derivatives on an interval $[a, b]$, then there exists a point ϑ, intermediate between points x and x_0, such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + R_n(x, x_0), \quad (2)$$

$$R_n(x, x_0) = \frac{1}{(n+1)!}f^{(n+1)}(\vartheta)(x - x_0)^{n+1}. \quad (3)$$

Identities (2), (3) are called the *Taylor formula of the order n with the residue in the Lagrange form.* These identities give a practical tool for approximating functions, with increasing accuracy, by polynomials. The accuracy of approximation can be estimated using the formula

$$|R_n(x, x_0)| \leqslant \frac{1}{(n+1)!}|x - x_0|^{n+1} \sup_{y \in [a,b]} |f^{(n+1)}(y)|.$$

Sample problem **1:** *:* Write the Taylor formula of the orders $n = 2, 3$ at the point $x_0 = 0$ for the function $f(x) = \sin x^2$. Estimate respective approximation errors at the interval $[-1, 1]$.

Solution: We have

$$f'(x) = 2x\cos x^2, \quad f''(x) = 2\cos x^2 - 4x^2 \sin x^2, \quad f'''(x) = -12x\sin x^2 - 8x^3\cos x^2,$$

$$f^{(4)}(x) = (14x^4 - 12)\sin x^2 - 24(x^2 + 1)\cos x^2,$$

and

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 2, \quad f'''(0) = 0.$$

In addition,

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24.$$

Then the 2-nd and the 3rd order Taylor formulae have the form

$$\sin(x^2) = 0 + 0(x - 0) + \frac{1}{2}2(x - 0)^2 + R_2(x, 0) = x^2 + R_2(x, 0),$$
$$\sin(x^2) = 0 + 0(x - 0) + \frac{1}{2}2(x - 0)^2 + R_3(x, 0).$$

Since

$$|f'''(x)| = |12x\sin x^2 + 8x^3\cos x^2| \leqslant 20, \quad |f^{(4)}(x)| \leqslant |14x^4 - 12| + 24(x^2 + 1) \leqslant 50, \quad x \in [-1, 1],$$

we have

$$|R_2(x, 0)| \leqslant \frac{20}{6}||x - 0|^3 = \frac{10}{3}|x|^3,$$
$$|R_3(x, 0)| \leqslant \frac{50}{24}|x - 0|^4 = \frac{25}{12}|x|^4.$$

The above example shows clearly that, while n is increasing, the approximation accuracy for the Taylor formula typically improves. The *Taylor series* appears when, in this approximation, $n \to \infty$; in this setting, an *approximation formula* transforms to a true *identity*. To deal with such an identity rigorously, we need to introduce several new notions.

Definition 1. (I) An infinite (number) series is a sum of the form $\sum_{n=0}^{\infty} a_n$, where $a_0, a_1, \dots$ are real numbers. This infinite sum is defined as a limit, as $N \to \infty$, of the partial sums $S_N = \sum_{n=0}^{N} a_n$.

(II) A functional series is a sum of the form $\sum_{n=0}^{\infty} f_n(x)$ where $f_0(x), f_1(x), \dots$ are functions defined on some interval $[a, b]$. The infinite sum is obtained as a collection of sums of number series in each point $x \in [a, b]$.

(III) A power series is a functional series with $f_n(x) = a_n(x - x_0)^n$, where $a_0, a_1, \dots$ are real numbers and x_0 is a given number.

The notion of *convergence* of a functional series (that is, the sum of an infinite number of functions) requires a certain accuracy. It is highly desirable for the standard operations of differentiation and integration to be adjusted with this notion. It appears that the *point-wise* convergence introduced above is not well adjusted with these basic analysis tools. This motivates the following

Definition 2. A functional series $\sum_{n=0}^{\infty} f_n(x)$ converges *uniformly* to a function $f(x)$ on a segment $[a, b]$ if

$$\sup_{x \in [a,b]} \left| f(x) - \sum_{n=0}^{\infty} f_n(x) \right| \to 0, \quad N \to \infty.$$

Theorem 3. *(I) Let functional series $\sum_{n=0}^{\infty} f_n(x)$ converge uniformly to a function $f(x)$ on a segment $[a, b]$. Then for every $[c, d] \subset [a, b]$,*

$$\int_c^d f(x)\, dx = \sum_{n=0}^{\infty} \int_c^d f_n(x)\, dx$$

(II) Let functional series $\sum_{n=0}^{\infty} f_n(x)$ converge to a function $f(x)$, and the series $\sum_{n=0}^{\infty} f_n'(x)$ converge uniformly on a segment $[a, b]$. Then $f(x)$ is differentiable and

$$f'(x) = \sum_{n=0}^{\infty} f_n'(x).$$

For a power series, it is quite easy to describe the interval of convergence.

Theorem 4. *(The Cauchy-Hadamard theorem) For any power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ there exists unique number $\Lambda \in [0, \infty]$ such that the sequence $|a_n \lambda^n|$ is bounded whenever $|\lambda| < \Lambda$ and $|a_n \lambda^n|$ is unbounded whenever $|\lambda| > \Lambda$. The power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges uniformly on any segment $[a, b] \subset (x_0 - \Lambda, x_0 + \Lambda)$ and diverges at any point x outside of $[x_0 - \Lambda, x_0 + \Lambda]$.*

The interval $(x_0-\Lambda, x_0+\Lambda)$ is called the interval of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, and Λ is caller the radius of convergence. Frequently, the radius of convergence can be calculated as a limit, if of either of the following limits exists:

$$\Lambda = \lim_{n \to \infty} \frac{1}{|a_n|^{1/n}}, \quad \Lambda = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|. \tag{4}$$

With these preliminaries made, we can proceed to the main topic of this section, which is the Taylor-Maclaurin series.

Definition 3. The *Taylor series* of a function $f(x)$ at a point x_0 is the power series

$$f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n.$$

This series has a certain convergence interval $I = (x_0 - \Lambda, x_0 + \Lambda)$. If for $x \in I$ the residues in the Taylor formula (2) satisfy

$$R_n(x, x_0) \to 0, \quad n \to \infty,$$

then the function $f(x)$ has the *Taylor series representation*

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n, \quad x \in (x_0 - \Lambda, x_0 + \Lambda).$$

The Taylor series with $x_0 = 0$ is called the *Maclaurin series.*

Sample problem **2:** *:* Write the Taylor-Maclaurin series representation for the function $f(x) = \frac{1}{1+x}$.

Solution: Writing $f(x) = (1 + x)^{-1}$, we can calculate the derivatives:

$$f'(x) = -(1 + x)^{-2}, \quad f''(x) = (-1)(-2)(1 + x)^{-3} = 2(1 + x)^{-3}, \ldots,$$

$$f^{(n)}(x) = (-1)(-2)\ldots(-n)(1 + x)^{-n-1} = (-1)^n n!(1 + x)^{-n-1}, \ldots.$$

Then the Taylor series at $x_0 = 0$ has the form

$$\sum_{n=0}^{\infty} (-1)^n x^n.$$

and its follows from (4) that the radius of convergence $\Lambda = 1$. Using the formula for the sum of an infinite geometric progression, we get that, for any $x \in (-1, 1)$,

$$\sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1 - (-x)} = \frac{1}{1 + x},$$

i.e. $f(x) = \frac{1}{1+x}$ has the Taylor-Maclaurin representation

$$\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

Sample problem **3:** *:* Write the Taylor-Maclaurin series representation for the function $f(x) = \sin x$.

Solution: Calculate the derivatives:

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{(4)}(x) = \sin x = f(x),$$

and then all the higher order derivatives can be calculated cyclically:

$$f^{(4k+j)}(x) = \quad f^{(j)}(x), \quad j = 0, 1, 2, 3, \quad k \geqslant 1.$$

Since $\sin(0) = 0, \cos(0) = 1$, the Taylor-Maclaurin series has the form

$$0 + 1x + \frac{1}{2}0x^2 + \frac{1}{6}(-1)x^3 + \dots.$$

The even terms in the sum are zero, while an odd term with the number n (i.e., with the overall number $2n-1$) equals $(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$. That is, after eliminating the zero terms and renumbering the series has the form

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}.$$

The sequence

$$\left|(-1)^n\frac{x^{2n-1}}{(2n-1)!}\right| = \frac{|x|}{1}\cdot\frac{|x|}{2}\cdots\frac{|x|}{2n-1}, \quad n \geqslant 1$$

is bounded for any x, hence the radius of convergence $\Lambda = \infty$.
Finally, since

$$|R_n(x)| = \frac{1}{(n+1)!}|f^{(n+1)}(\theta)| \leqslant \frac{1}{(n+1)!} \to 0,$$

we have the Taylor-Maclaurin series representation

$$\sin x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}.$$

Knowing the Taylor-Maclaurin series representation for some function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad x \in (-\Lambda, \Lambda),$$

we can provide the representation for other functions, which are obtained from this one by natural transformations

1. Scaling of the argument: if $g(x) = f(cx)$, then

$$g(x) = \sum_{n=0}^{\infty} a_n c^n x^n, \quad x \in (-\frac{\Lambda}{c}, \frac{\Lambda}{c});$$

2. Shift of the argument: if $g(x) = f(x-b)$, then

$$g(x) = \sum_{n=0}^{\infty} a_n (x-b)^n, \quad x \in (b - \frac{\Lambda}{c}, b + \frac{\Lambda}{c});$$

3. Multiplying by a monomial: if $g(x) = x^k f(x)$, then

$$g(x) = \sum_{n=0}^{\infty} a_n x^{n+k} = \sum_{n=k}^{\infty} a_{n-k} x^n, \quad x \in (-\frac{\Lambda}{c}, \frac{\Lambda}{c});$$

4. Differentiation: for $g(x) = f'(x)$,

$$f(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, \quad x \in (-\Lambda, \Lambda).$$

5. Integration: for $g(x) = \int_0^x f(v)\, dv$,

$$g(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n, \quad x \in (-\Lambda, \Lambda).$$

Sample problem **4:** *:* Write the Taylor-Maclaurin series representation for the functions $(1-x)^{-1}, \ln(1-x)$.

Solution: We have by Sample problem 2

$$(1-x)^{-1} = (1+(-x))^{-1} = \sum_{n=0}^{\infty} (-1)^n (-x)^n = \sum_{n=0}^{\infty} x^n, \quad x \in (-1, 1).$$

Since

$$\ln(1-x) = -\int_0^x (1-v)^{-1}\, dv,$$

after integration we get

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Such transformations often makes it possible to calculate values of particular infinite sums.
Sample problem **5:** *:* Find $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Solution: We know that

$$\sum_{n=0}^{\infty} x^n = (1-x)^{-1},$$

and hence

$$\sum_{n=1}^{\infty} n x^{n-1} = \left((1-x)^{-1}\right)' = (1-x)^{-2}.$$

Then

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} \sum_{n=1}^{\infty} n x^{n-1} \Big|_{x=1/2} = \frac{1}{2} \left(1 - \frac{1}{2}\right)^{-2} = 2.$$

Knowing the Taylor-Maclaurin series representation of a function actually gives us a knowledge of all its derivatives at the point 0:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \Longleftrightarrow a_n = \frac{1}{n!} f^{(n)}(0) \Longleftrightarrow f^{(n)}(0) = a_n n!. \tag{5}$$

Sample problem **6:** *:* Find $f^{(1001)}(0), f^{(2020)}(0)$ for $f(x) = x^5 \sin x$.

Solution: We have by Sample problem 3

$$x^5 \sin x = x^5 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+4}}{(2n-1)!}.$$

To use (5), we have to return to find the coefficients a_{1001}, a_{2020} in the representation $f(x) = \sum_{n=0}^{\infty} a_n x^n$. We have non-zero coefficients for the even terms starting from 6, only; that is, $a_{1001} = 0$ and thus $f^{(1001)}(0) = 0$. Next, to get the power $2n + 4 = 2020$, we have to take $n = 1008$, hence

$$a_{2020} = (-1)^{1007} \frac{1}{2015!} = -\frac{1}{2015!}, \quad f^{(2020)}(0) = -\frac{2020!}{2015!} = -2020 \cdot 2019 \cdot 2018 \cdot 2017 \cdot 2016.$$

Below, a table of several most important Taylor-Maclaurin series is given; the notation

$$\binom{a}{n} = \frac{a(a-1)\dots(a-n+1)}{n!}$$

is used for the so called *generalized binomial coefficient.*

Name	Function	Series	Interval of convergence
Exponential	e^x	$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$	$\mathbb{R}$
Sine	$\sin x$	$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$	$\mathbb{R}$
Cosine	$\cos x$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$\mathbb{R}$
Generalized Binomial	$(1+x)^a$	$\sum_{n=0}^{\infty} \binom{a}{n} x^n$	$(-1, 1)$
Logarithm	$\ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n}$	$(-1, 1)$

Problems to solve

Part A

1. Determine the Taylor-Maclaurin series for the given function

(a) $f(x) = \cos(4x)$;

(b) $f(x) = x^6 e^{2x^3}$;

(c) $f(x) = x\cos 2x^3$;

(d) $f(x) = \dfrac{x^{100}}{1+x^3}$.

2. For each function from the previous problem find $f^{(2020)}(0)$.

3. Determine the Taylor series for the given function $f(x)$ and x_0. Provide **two** solutions: using the formula for the coefficients and the change of variables.

(a) $f(x) = e^{-6x}, x_0 = -4$;

(b) $f(x) = \ln(3+4x)\,, x_0 = 1;$

(c) $f(x) = \dfrac{7}{x^4}, x_0 = -3;$

4. For each of the series in the previous problem determine the interval of convergence.

5. Using the Taylor-Maclaurin series and differentiation/integration calculate the infinite sums

(a) $\sum_{n=1}^{\infty} \dfrac{1}{n3^n};$

(b) $\sum_{n=2}^{\infty} \dfrac{2^n - n}{3^n}.$

Part B

6. Write the Taylor formula of the orders $n = 2, 3$ at the point $x_0 = 0$ for the given function, and estimate approximation errors on the given interval

(a) $f(x) = e^{3x}$, $x \in [-1, 1]$;

(b) $f(x) = \ln(1 + x^2)$, $x \in [-1/2, 1/2]$.

7. Using the Generalized Binomial function, determine the Taylor-Maclaurin series for the given function

(a) $f(x) = \sqrt{1 - x^2};$

(b) $f(x) = \dfrac{1}{\sqrt[3]{1 + x^3}};$

(c) $f(x) = \dfrac{x^3}{\sqrt{x^2 + 16}};$

(d) $f(x) = \dfrac{x^{100}}{1 + x^3}.$

8. Using the Taylor-Maclaurin series and differentiation/integration calculate the infinite sums

(a) $\sum_{n=0}^{\infty} \dfrac{n(n+1)}{5^n};$

(b) $\sum_{n=1}^{\infty} \dfrac{n}{(n+1)2^n};$

(c)* $\sum_{n=1}^{\infty} \dfrac{1}{n(n+2)}$ (*Hint:* consider the limit of $\sum_{n=1}^{\infty} \dfrac{x^n}{n(n+2)}$ as $x \nearrow 1$).