LISTS OF PROBLEMS FOR THE COURSE MATHEMATICAL ANALYSIS 1 (MAT 1653)

The problems are intended for solving at classes and for homework. Two additional lists REPETITION 1 and REPETITION 2 are preparation for colloquiums. The exams from the previous years can be found on the Mathematics Department website: *http://wmat.pwr.edu.pl*

LIST 0

(material for self-repetition).

Actions in a set of real numbers

In the problems $0.2 - 0.5$ $n \in \mathbf{N}$, *and* a, b, x, y *are real numbers for which the expressions and transformations occurring in the problems have sense.*

0.1. Recall the order in which the actions are performed in expressions without parentheses and in expressions with parentheses. Calculate the value of the expression: $4 + 6 : 2 \cdot 3 - 8 \cdot 2$. Insert the brackets so that the value of the expression received is equal.

(a) -1, (b) -11, (c) -10.

0.2. Provide and memorize the patterns of "reduced multiplication":

(a) $(a+b)^2 = \cdots$, (b) $(a+b)^3 = \cdots$, (c) $(a+b)(a-b) = \cdots$, (d) $(a+b)(a^2-ab+b^2) = \cdots$.

Is it possible to replace "b" with "$-b$" in the above expressions? What will we receive?

0.3. Simplify rational expressions:

(a) $\dfrac{3a^2-6ab+3b^2}{6a^2-6b^2}$, (b) $\dfrac{9+6x+x^2}{x^2-9}$, (c) $\dfrac{a^3+8}{a^2-4}$, (d) $\dfrac{1-x^3}{3x^2+3x+3}$,

(e) $\dfrac{x^3-x^2-x+1}{x^4-2x^2+1}$, (f) $\dfrac{2x^2+4xy+2y^2}{9x^2-9y^2}$, (g) $\dfrac{x^3+x^2+2x+2}{x^4+4x^2+4}$.

0.4. Write expressions in a simpler form

(a) $\dfrac{2^n+3\cdot 2^{n+2}}{4^{2n}}$, (b) $\dfrac{(\sqrt{2})^{3n+2}-(\sqrt{8})^n}{2^n}$, (c) $\dfrac{21\cdot 27^n}{9^{n+2}+3^{2n+1}}$, (d) $\left(\dfrac{1}{\sqrt{a^3}}\cdot b\cdot\sqrt{\dfrac{b^2}{a}}\cdot\sqrt[3]{a^2}\right)^3$.

0.5. Perform actions. Write the result in the simplest form.

(a) $\dfrac{b}{ay+ax}-\dfrac{a}{by+bx}$, (b) $\dfrac{1}{a-b}-\dfrac{3ab}{a^3-b^3}-\dfrac{b-a}{a^2+ab+b^2}$,

(c) $\dfrac{8x}{x-9x^3}+\dfrac{3x}{x+3x^2}-\dfrac{2-6x}{(1-3x)^2}$, (d) $\dfrac{x\sqrt{4-x^2}-(2-x^2)\cdot\dfrac{x}{\sqrt{4-x^2}}}{4-x^2}$.

0.6. In the expressions given, remove the non-rationality from the denominator

(a) $\dfrac{1}{4+\sqrt{1+x}}$, (b) $\dfrac{n-2}{\sqrt{n}+\sqrt{2}}$, (c) $\dfrac{n+1}{\sqrt{5n+4}-\sqrt{4n+3}}$,

(d) $\dfrac{a-b}{\sqrt[3]{a}-\sqrt[3]{b}}$, (e) $\dfrac{x}{\sqrt[3]{x+1}+\sqrt[3]{x-1}}$, (f) $\dfrac{n-1}{\sqrt[3]{n^2}+\sqrt[3]{n}+1}$.

LIST 1.

(for 4 classes)

Elemets of mathematical logic. Functions: general definitions, elementary functions.

1.1. Logical sentences, quantifiers.

For sentences, which are logical sentences, give their logical values.

(a) $\frac{6}{4} \geqslant \frac{3}{2}$, (b) $x^2 - 7 < 0$, (c) $\bigwedge_{x \in \mathbf{R}} x^2 - 7 < 0$,

(d) $\bigvee_{x \in \mathbf{R}} x^2 - 7 < 0$, (e) $\bigvee_{x \in \mathbf{R} - \{0, -2\}} \frac{1}{x+2} = \frac{1}{x} + \frac{1}{2}$, (f) $\bigwedge_{x \in \mathbf{R}} \bigvee_{y \in \mathbf{R}} x^2 - y^2 = 0$.

1.2. Negation. Equability. De Morgan's rules for conjunction and alternatives.

Write using the logic connectivities "and", "or" the solutions to equality (inequality). Mark in a plane a set of points that are co-ordinated by the given condition.

(a) $(x+3)(y-2) = 0$, (b) $(x+3)(y-2) \neq 0$, (c) $(x+3)(y-2) > 0$, (d) $x^2 - 4y^2 < 0$,

(e) $\frac{a+b}{a-b} = 0$, (f) $\frac{a+2}{a-b+1} > 0$, (g) $\frac{2a+b}{a+b} \leqslant 0$, (h) $\frac{a^2+b-1}{a^2-b^2} \geqslant 0$.

1.3. Implication. Proposition.

(**A**) The following proposition is true:

If the natural number is divisible by 12, it is divisible by 3.

Indicate the assumption and statement of the proposition.
Based on the above proposition, give:
(a) condition sufficient for divisibility by 3. Why is this not a necessary condition?
(b) the necessary condition of divisibility by 12. Why is this not a sufficient condition?
(c) The natural number is not divisible by 12. Does the statement allow you to deduce the divisibility of this number by 3?
(d) The natural number is divisible by 3. Does the statement allow you to you to deduce the divisibility of this number by 12?
(e) The natural number is not divisible by 3. Does the statement allow to you deduce the (non-) divisibility of this number by 12?
Formulate a necessary and sufficient condition for divisibility by 3.

(**B**) Let $x, y \in \mathbf{R}$. The following implication is true:

$$(x > 0 \quad \text{and} \quad y > 0) \Longrightarrow (xy > 0).$$

Indicate the assumption and statement of the proposition.

(a) It is known that $\alpha > 1$ and $\beta > -1$. Does the proposition allow you to conclude about the sign of the product $(\alpha - 1) \cdot (\beta + 1)$? And of the product $\alpha \cdot \beta$? Give examples.

(b) It is known that $ab > 0$. Does the proposition allow you to conclude about the sign of the number a? Give examples.

(c) It is known that $uv \leqslant 0$. What does the proposition allow you to conclude about the numbers u and v?

1.4. De Morgan's rules for quantifiers.

Write the equivalent form of the sentence:

(a) $\neg\left(\bigwedge_{x\in\mathbf{R}} 2^x = 2^{-x}\right)$, (b) $\neg\left(\bigvee_{x<0} x^2 = x^4\right)$,

(c) $\neg\left(\bigvee_{M\in\mathbf{R}}\bigwedge_{n\in\mathbf{N}} \frac{n^2+1}{n} < M\right)$, (d) $\neg\left(\bigwedge_{\epsilon>0}\bigvee_{n_0\in\mathbf{N}}\bigwedge_{n\in\mathbf{N}} (n > n_0) \Longrightarrow \left(\frac{n}{n+5} < \epsilon\right)\right)$.

1.5. Define the natural domains of functions. Determine which of the functions is even, which is odd, and which has none of these properties.

(a) $f(x) = \frac{|x|+3}{x^2-9}$, (b) $f(x) = \frac{x}{6x^2-x-1}$, (c) $f(x) = \sqrt{3x-x^3}$, (d) $f(x) = \sqrt{8-\frac{1}{x^3}}$.

1.6. Transforming the graph of the corresponding linear function, draw a graph of the given function. Find the range of the function using the graph.

(a) $f(x) = |4-2x|$, (b) $f(x) = 4 - 2|x|$,

(c) $f(x) = \sqrt{x^2+4x+4}$, (d) $f(x) = \begin{cases} x+2 & \text{for} \quad |x| \leqslant 1 \\ 1 & \text{for} \quad |x| > 1 \end{cases}$.

1.7. Transforming the graph of the function $y = ax^2$ draw a graph of the function $y = f(x)$. Find the range of the function using the graph.

(a) $f(x) = x^2 - 4x + 5$, (b) $f(x) = x^2 - 2|x| + 1$,

(c) $f(x) = -4 - 4x - 2x^2$, (d) $f(x) = \operatorname{sgn}(x^2 - 3x)$.

1.8. Transforming the graph of the function $y = \frac{a}{x}$ or $y = \frac{a}{x^2}$ draw a graph of the function $y = f(x)$. Find the range of the function using the graph.

(a) $f(x) = \frac{x}{x-1}$, (b) $f(x) = \frac{x-1}{x+1}$, (c) $f(x) = \frac{1}{(x-2)^2}$, (d) $f(x) = \frac{x^2+4x+3}{x^2+4x+4}$.

1.9. Write formulae for the complex functions $f\circ g, g\circ f, f\circ f, g\circ g$ given the functions f and g. Draw the graphs for the functions $y = f(g(x))$ and $y = g(f(x))$.

(a) $f(x) = x^2, \quad g(x) = x - 2$, (b) $f(x) = \sqrt{x}, \quad g(x) = 4x^2$,

(c) $f(x) = |x|, \quad g(x) = \frac{1}{x+1}$, (d) $f(x) = x^2 - 2, \quad g(x) = \operatorname{sgn} x$.

1.10. Propose the presentation of functions in a form $g\circ h$. Is there only the one pair of functions g, h such that $f = g\circ h$?

(a) $f(x) = \sqrt{x^2+16}$, (b) $f(x) = \frac{1}{x^4+3}$, (c) $f(x) = 4x^2 + 12x$.

1.11. Calculate

$$\log_2 2\sqrt{2}, \quad \log 0{,}01, \quad \log_3 2 - \log_3 18, \quad 3\log 5 + 0{,}5\log 64, \quad \log_3 \operatorname{tg}\frac{\pi}{6},$$

$$\ln e^3, \quad 2^{\log_2 3}, \quad \left(\frac{1}{3}\right)^{\log_3 5}, \quad 3^{\log_{\sqrt{3}} \frac{\sqrt{6}}{2}}, \quad e^{2\ln 10}, \quad e^{1-\ln 10}, \quad \log_2 3 \cdot \log_3 8.$$

1.12. Mark in a plane a set of points whose coordinates (x, y) satisfy the given condition
(a) $\log_2 y = \log_2 x + \log_2 3$, (b) $\log_{0,5} y = 2\log_{0,5}(x+1)$, (c) $\log|y| = \log|x| + \log 0{,}5$.

1.13. Sketch the graphs of functions

(a) $f(x) = 2^{|x|}$, (b) $f(x) = \left|\left(\frac{1}{2}\right)^x - 1\right|$, (c) $f(x) = 1 + \frac{1}{e^x}$, (d) $f(x) = -e^{-|x|}$,

(e) $f(x) = \log_2(x-1)$, (f) $f(x) = \left|\log_{0,5} x\right|$, (g) $f(x) = \ln|x|$, (h) $f(x) = \ln x^2$.

1.14. Solve the following equations and inequalities

(a) $\left(\frac{1}{2}\right)^{(x-2)^2-5x} = \left(\frac{1}{4}\right)^5$, (b) $4^x + 24 = 5 \cdot 2^{x+1}$, (c) $|2^x - 5| < 2$,

(d) $|3\log x - 1| = 2$, (e) $\log_2(x+1) - \log_2 x < 1$, (f) $\ln^2 x + \ln x \geqslant 2$.

1.15. Give the formula that defines the inverse function to f. Sketch in one coordinate system the graphs of functions $y = f(x)$ and $y = f^{-1}(x)$.

(a) $f(x) = \log_2(x+1)$, (b) $f(x) = 1 - 2^x$, (c) $f(x) = 2 - \sqrt{x}$,

(d) $f(x) = x^2 - 2x + 2$ for $x \geqslant 1$, (e) $f(x) = x^2 - 2x + 2$ for $x \leqslant 1$,

(f) $f(x) = 2^x - 2^{-x}$, (g) $f(x) = 3^{1/x}$.

1.16. Using the periodicity of functions and the trigonometric circle, calculate the values of expressions

(a) $\cos\frac{\pi}{3} + \sin\frac{4}{3}\pi$, (b) $\sin\frac{13}{6}\pi + \sin\frac{11}{3}\pi$, (c) $\cos\frac{14}{3}\pi + \cos\frac{19}{6}\pi$,

(d) $\sin\left(-\frac{9}{4}\pi\right) + \cos\left(-\frac{13}{4}\pi\right)$, (e) $\sin\frac{17}{2}\pi + \cos\frac{17}{2}\pi$, (f) $\operatorname{tg}\frac{20}{3}\pi + \operatorname{ctg}\frac{19}{3}\pi$.

1.17. Prove identities. Specify their domains.

(a) $\cos^2 x = \frac{1}{1+\operatorname{tg}^2 x}$, (b) $\sin^2 x = \frac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x}$, (c) $\cos x = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}$, (d) $\sin x = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}$,

(e) $1 + \operatorname{tg} x + \operatorname{tg}^2 x + \operatorname{tg}^3 x = \frac{\sin x + \cos x}{\cos^3 x}$, (f) $\sin^4 x + \cos^4 x = 1 - 0{,}5\sin^2 2x$.

1.18. We call *sine curve* a curve given by the equation $y = a\sin(bx + c) + d$ with parameters $a \neq 0, b \neq 0, c, d$. Show that each of the following curves is a sine curve and draw it.

(a) $y = \sin x \cos x$, (b) $y = (\sin x + \cos x)^2$, (c) $y = \cos^2 x$.

1.19. Draw the graph of a function $y = f(x)$. Read from the graph the period and the range of the function.

(a) $f(x) = \cos\left(x + \frac{\pi}{3}\right)$, (b) $f(x) = \sin x + |\sin x|$, (c) $f(x) = \operatorname{tg}\frac{x}{2}$, (d) $f(x) = |\operatorname{ctg}(\pi x)|$.

1.20. Solve the equations and inequalities.

(a) $\cos 2x = 0$, (b) $\sin\left(3x + \frac{\pi}{3}\right) = -1$, (c) $\operatorname{tg}\frac{x}{2} = 1$,

(d) $\sin\left(x + \frac{\pi}{4}\right) \leqslant 0$, (e) $\cos\frac{x}{3} > 0$, (f) $\operatorname{ctg}^2 x < 1$.

1.21. Calculate the values of the expressions

(a) $w = \arcsin\frac{x}{2} - \arccos\frac{x}{2} + \operatorname{arctg}\frac{1}{x}$, if $\operatorname{arcctg} x = \frac{\pi}{6}$;

(b) $w = \arcsin(-x) + \arccos 2x + \operatorname{arctg} 2x$, if $\arccos x = \frac{2\pi}{3}$;

(c) $\operatorname{tg}\left(\arccos\frac{1}{3}\right)$; (d) $\sin\left(\arcsin\frac{3}{5} + \arcsin\frac{8}{17}\right)$.

1.22. Solve the equations

(a) $\operatorname{tg} 2x = 5$, (b) $\sin x = \frac{1}{3}$, (c) $\sin x = -\frac{1}{4}$, (d) $\cos\left(x + \frac{\pi}{5}\right) = \frac{\sqrt{3}}{3}$, (e) $\cos x = -\frac{3}{4}$.

Similar problems (with solutions) can be found in the book
M.Gewert, Z.Skoczylas, Wstęp do analizy i algebry. Teoria, przykłady, zadania, Oficyna Wydawnicza GiS, Wrocław 2014.

LIST 2
(for 1 class)

Sequences

2.1. Show that the given sequences are monotone and bounded.

(a) $a_n = \dfrac{n}{2n+1}$, (b) $b_n = \dfrac{2^n}{3^n+2}$, (c) $c_n = \dfrac{(n!)^2}{(2n)!}$, (d) $d_n = \sin\dfrac{\pi}{2n+1}$,

(e) $e_n = \dfrac{(n+2)^2}{2^{n+2}}$, (f) $f_n = \sqrt{n+8} - \sqrt{n+3}$, (g) $g_n = \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n}$.

2.2. Using the definition of a limit of a sequence, show that

(a) $\lim\limits_{n\to\infty} \dfrac{n}{n+2} = 1$, (b) $\lim\limits_{n\to\infty} \dfrac{n^2+1}{2n} = +\infty$, (c) $\lim\limits_{n\to\infty} \dfrac{n+4}{n+2} \neq 2$.

2.3. By giving the appropriate examples, show that the following expressions are not well defined

$$\frac{0}{0},\qquad \frac{\infty}{\infty},\qquad 0\cdot\infty,\qquad \infty-\infty,\qquad 1^{\infty},\qquad \infty^0,\qquad 0^0.$$

2.4. Find the limits of the sequences

(a) $a_n = \dfrac{2n-3}{3n+4}$, (b) $b_n = \dfrac{n^2+3n-8}{2n+5}$, (c) $c_n = \dfrac{n^2+n-3}{n^3+2n+1}$,

(d) $d_n = \dfrac{(2n^3+3)^8}{(2n^4+7)^6}$, (e) $e_n = \dfrac{n+\sqrt{n^3+7}}{\sqrt[3]{n^2+5}+4n}$, (f) $f_n = \dfrac{8^{n+2}+2^n}{2^{3n+1}+3^n+4}$,

(g) $g_n = \dfrac{1+2+3+\cdots+n}{n^2}$, (h) $h_n = \sqrt{n+8} - \sqrt{n+3}$,

(i) $i_n = \sqrt{n^2+4n+1} - \sqrt{n^2+3}$, (j) $j_n = \sqrt{2n+1} - \sqrt{n+23}$,

(k) $k_n = \sqrt{9^n+4\cdot 3^n+1} - \sqrt{9^n+3}$, (l) $l_n = n^{30} - 2\cdot n^{21} - 3\cdot n^9 + 3$,

(m) $m_n = 7^n - 2\cdot 5^{2n} + 3\cdot 9^{n+5} + 4$, (n) $m_n = \left(\dfrac{n+4}{n+1}\right)^{n+3}$, (o) $o_n = \left(\dfrac{n^2+3}{n^2+1}\right)^{n^2}$,

(p) $p_n = \left(\dfrac{2n+1}{2n+5}\right)^{1-3n}$, (r) $r_n = \left(\dfrac{4n+1}{2n-1}\right)^{n+6}$, (s) $s_n = \left(\dfrac{3^n+2^n}{5^n+3^n}\right)^{n}$.

2.5. For a given sequence (a_n) find a sequence (b_n) of the form $b_n = n^p$ or $b_n = \alpha^n$ such that (a_n) i (b_n) are of the same order. (*We say that the sequences* (a_n), (b_n) *are of the same order if* $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c$, *for a certain positive number* c.)

(a) $a_n = \dfrac{1}{n^2+4n+3}$, (b) $a_n = \dfrac{n^2}{n^3+7}$, (c) $a_n = \sqrt{n+9} - \sqrt{n+1}$,

(d) $a_n = \dfrac{1}{3\cdot 2^n + 2\cdot 3^n}$, (e) $a_n = \dfrac{3^n}{4^n+5^n}$, (f) $a_n = \dfrac{4^{n+2}}{5\cdot 2^{n+1} + 2\cdot 3^n}$.

Similar problems (with solutions) can be found in
M.Gewert, Z.Skoczylas, Analiza matematyczna 1. Przykłady i zadania, Oficyna Wydawnicza GiS, Wrocław 2017, rozdział 1.

LIST 3

(for 2 classes)

Limits of a function. Asymptotes. Continuous functions

3.1. Draw graphs of functions that meet all the given conditions

(a) $f(0)=1$, $\lim_{x\to-\infty} f(x)=+\infty$, $\lim_{x\to 0} f(x)=-1$, $\lim_{x\to 2} f(x)=-\infty$, $\lim_{x\to+\infty} f(x)=\pi$;

(b) $g(0)=4$, $\lim_{x\to-\infty} g(x)=0$, $\lim_{x\to-3^-} g(x)=+\infty$, $\lim_{x\to-3^+} g(x)=0$, $\lim_{x\to+\infty} g(x)$ nie istnieje;

(c) $\lim_{x\to-\infty} h(x)=-\infty$, $\lim_{x\to-1} h(x)$ nie istnieje, $\lim_{x\to 0} h(x)\neq h(0)$, $\lim_{x\to 1} h(x)=+\infty$, $h(3)<0$.

3.2. Find the limits

(a) $\lim\limits_{x\to+\infty}\dfrac{x^2-3x+2}{x^2-1}$, (b) $\lim\limits_{x\to 1}\dfrac{x^2-3x+2}{x^2-1}$, (c) $\lim\limits_{x\to-\infty}\dfrac{8-x^3}{x^2-4}$, (d) $\lim\limits_{x\to 2}\dfrac{8-x^3}{x^2-4}$,

(e) $\lim\limits_{x\to 1}\dfrac{x+3}{|x^2-1|}$, (f) $\lim\limits_{x\to 1^-}\dfrac{e^x}{x-1}$, (g) $\lim\limits_{x\to+\infty}\dfrac{\sqrt[3]{x}+x^2}{x+\sqrt{x}}$, (h) $\lim\limits_{x\to 0^+}\dfrac{\sqrt[3]{x}+x^2}{x+\sqrt{x}}$,

(i) $\lim\limits_{x\to 0}\dfrac{\sin \pi x}{x}$, (j) $\lim\limits_{x\to 0}\dfrac{x^2}{\sin^2 3x}$, (k) $\lim\limits_{x\to+\infty}\dfrac{\operatorname{tg}\frac{1}{x}}{\operatorname{tg}\frac{2}{x}}$, (l) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos x}{\sin 6x}$.

3.3. Prove that the limits given below do not exist

(a) $\lim\limits_{x\to-0{,}5}\dfrac{2x-1}{4x^2-1}$, (b) $\lim\limits_{x\to 0} 2^{1/x}$, (c) $\lim\limits_{x\to 0}\dfrac{1}{2^x-3^x}$, (d) $\lim\limits_{x\to\pi}\operatorname{sgn}(\sin \mathrm{x})$.

3.4. Using the corresponding propositions (about three functions, about the product of a limited function and a function convergent to zero, about two functions) find the limits

(a) $\lim\limits_{x\to 0^+}\sqrt{x}\cos\dfrac{1}{x^2}$, (b) $\lim\limits_{x\to-\infty}\dfrac{\sin x^2}{x}$, (c) $\lim\limits_{x\to+\infty}\dfrac{2x+\sin x^2}{3x+\cos\sqrt{x}}$, (d) $\lim\limits_{x\to 0^+}\dfrac{2+\sin\frac{1}{x}}{x^3}$.

3.5. Draw graphs of functions that meet all the given conditions

(a) the line $x=1$ is a two-sided vertical asymptote for the function f, $y=2$ is a horizontal asymptote at $-\infty$, $y=-x+2$ is a slant asymptote at $+\infty$;

(b) the line $x=-2$ is a left-sided vertical asymptote for the function g and is not a right-sided vertical asymptote, the function g does not have and asymptote at $-\infty$, $\lim_{x\to+\infty} g(x)=3$;

(c) the line $x=0$ is a two-sided vertical asymptote for the function h, $\lim_{x\to 0} h(x)$ does not exist, $\lim_{x\to-\infty}[h(x)+2x]=0$, $\lim_{x\to+\infty}[h(x)+x-1]=0$.

3.6. Find all asymptotes of the function f. Draw a sketch of the graph.

(a) $f(x) = \dfrac{8x^3+1}{4x^2-1}$, (b) $f(x) = \dfrac{x^2-6}{x-1}$, (c) $f(x) = \dfrac{3}{2^x-8}$,

(d) $f(x) = \dfrac{e^x}{e^x-2}$, (e) $f(x) = \sqrt{x^2-2x}$, (f) $f(x) = \dfrac{\cos x}{2\pi - x}$.

3.7. Is it possible to choose parameters $a, b \in \mathbf{R}$ such that the given function is continuous on $\mathbf{R}$? Draw the picture.

(a) $f(x) = \begin{cases} |x+2| & \text{for } x < 0 \\ a - x & \text{for } x \geqslant 0 \end{cases}$, (b) $f(x) = \begin{cases} \operatorname{arctg} x & \text{for } |x| \leqslant 1 \\ ax^2 + bx & \text{for } |x| > 1 \end{cases}$,

c) $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{for } x \neq 2 \\ a & \text{for } x = 2 \end{cases}$, d) $f(x) = \begin{cases} \dfrac{x+1}{x^2-1} & \text{for } |x| \neq 1 \\ a & \text{for } x = -1 \\ b & \text{for } x = 1 \end{cases}$.

3.8. Prove, using the Darboux theorem, that the equation has a solution in the given interval. In examples (a), (b), (c) show the uniqueness of the solution. Give a graphical interpretation of the equation.

(a) $\sin x = 2 - 2x$, $\left(0, \dfrac{\pi}{2}\right)$; (b) $e^x = \dfrac{1}{x^2}$, $\left(\dfrac{1}{2}, 1\right)$;

(c) $x^2 = -\ln x$, $(0, +\infty)$; (d) $10\sin(\pi x) = x + 1$, $\left(-\dfrac{1}{2}, 1\right)$.

3.9. Show that the equation has exactly one solution and find it (without using a calculator) with an error not bigger than 0,25.

(a) $x^3 + 6x = 2$, (b) $x^3 + x^2 + 2x + 1 = 0$, (c) $x^3 = 4 + 2^{-x}$.

Similar problems (with solutions) can be found in
M.Gewert, Z.Skoczylas, Analiza matematyczna 1. Przykłady i zadania, Oficyna Wydawnicza GiS, Wrocław 2017, rozdział 2 i 3.

REPETITION 1

P1.1. Draw the graphs of functions.

(a) $f(x) = \left|\frac{|x|}{2} - 4\right|$, (b) $f(x) = \frac{x-1}{x-2}$, (c) $f(x) = x^2 - 4|x| + 7$,

(d) $f(x) = 1 - \sqrt{|x| - 2}$, (e) $f(x) = 2^{-x} - 2$, (f) $f(x) = ||\log_2(x-2)| - 1|$,

(g) $f(x) = 1 + \mathrm{tg}\frac{x}{2}$, (h) $f(x) = \cos\left(|x| + \frac{\pi}{3}\right)$, (i) $f(x) = 2\sin 2x - |\sin 2x|$,

(j) $f(x) = \frac{|\mathrm{ctg}x|}{\mathrm{ctg}x}$, (k) $f(x) = \pi - \mathrm{arctg}x$, (l) $f(x) = \frac{\pi}{2} + \arcsin x$.

P1.2. Find the domains of functions.

(a) $f(x) = \frac{x+3}{\sqrt{x^2+4x}}$, (b) $f(x) = \sqrt{\frac{x+2}{x-4}}$, (c) $f(x) = 1 - \ln\sin x$,

(d) $f(x) = \frac{x-5}{\log_2(x^2-3)}$, (e) $f(x) = \frac{1}{2^{-x} - 2}$, (f) $f(x) = \ln^2\left(6 - \frac{1}{x}\right)$,

(g) $f(x) = 3\mathrm{ctg}\frac{x}{4}$, (h) $f(x) = \frac{e^x}{\pi^2 - 16\mathrm{arctg}^2 x}$, (i) $f(x) = \arcsin\ln x$.

P1.3. Solve the equations and inequalitites.

(a) $x(x-1) < 2(x+2)$, (b) $x^4 - 5x^2 \geqslant -4$, (c) $\frac{1}{x^3} \leqslant 8$,

(d) $\left|e^{-x} - 3\right| = 1$, (e) $2^x - \frac{3}{2^x} > 2$, (f) $\frac{1}{\ln x} < 3$,

(g) $\sin\left(2x + \frac{\pi}{4}\right) \geqslant 0$, (h) $\cos^2\frac{x}{5} = 1$, (i) $\mathrm{tg}3x = 2$.

P1.4. Prove identities and specify their domains.

(a) $\cos x \cdot (\mathrm{tg}x + \mathrm{ctg}x) = \frac{1}{\sin x}$, (b) $\mathrm{tg}^2 x - \mathrm{ctg}^2 x = \frac{1}{\cos^2 x} - \frac{1}{\sin^2 x}$, (c) $\frac{\sin x}{1 - \cos x} = \mathrm{ctg}\frac{x}{2}$.

P1.5. Write formulae for the complex functions $f \circ g$, $g \circ f$ and draw their graphs.

(a) $f(x) = x^2 - 4x$, $g(x) = |x|$, (b) $f(x) = e^{-x}$, $g(x) = 2x + 1$,

(c) $f(x) = \log_{0,5} x$, $g(x) = |x| + 2$, (d) $f(x) = \cos 2x$, $g(x) = 0{,}5x$,

(e) $f(x) = \sin\left(x + \frac{\pi}{4}\right)$, $g(x) = 2x$, (f) $f(x) = \sqrt{x}$, $g(x) = x^2$.

P1.6. Give the formula that defines the inverse function to f. Draw in one coordinate system the graphs of functions $y = f(x)$ and $y = f^{-1}(x)$.

(a) $f(x) = 4 - 2x$, (b) $f(x) = \sqrt{x} + 1$, (c) $f(x) = 1 + 2^x$, (d) $f(x) = 2\ln(x+1)$,

(e) $f(x) = x^2 + 2x$ for $x \geqslant -1$, (f) $f(x) = x^2 + 2x$ for $x \leqslant -1$.

P1.7. Show that the sequence (a_n) is monotone (starting from certain place) and bounded

(a) $a_n = \dfrac{n+1}{3n+4}$, (b) $a_n = \dfrac{2^n + 4^n}{5^n}$, (c) $a_n = \dfrac{12^n}{(n+1)!}$,

(d) $a_n = \cos^2 \dfrac{\pi}{4n+7}$, (e) $a_n = \sqrt{n+4} - \sqrt{n}$, (f) $a_n = \dfrac{1}{2} \cdot \dfrac{1}{2^2} \cdot \dfrac{1}{2^3} \cdot \ldots \cdot \dfrac{1}{2^n}$.

P1.8. Fin the limits of the sequences:

(a) $a_n = \dfrac{\sqrt{5n+4}}{4n+\sqrt{5}}$, (b) $a_n = \dfrac{3^{n+1} + 6 \cdot 2^n}{5 \cdot 4^{n-1} - 3^n}$, (c) $a_n = \dfrac{1}{\sqrt{4^n + 3 \cdot 2^n} - \sqrt{4^n + 4}}$,

(d) $a_n = 7^{3n+4} - 9^{2n+7}$, (e) $a_n = \dfrac{1+n^2}{1+2+3+\cdots+n}$, (f) $a_n = \dfrac{n\sqrt{n+3} - \sqrt{n^3+9}}{\sqrt{n}}$,

(g) $a_n = \left(\dfrac{n^3+2}{n^2+2n}\right)^{3n+1}$, (h) $a_n = \left(\dfrac{2n}{2n+1}\right)^n$, (i) $a_n = \left(\dfrac{3n+5}{3n+2}\right)^{2-5n}$,

(j) $a_n = \sqrt{\pi^n} - \sqrt{e^n}$, (k) $a_n = \dfrac{\operatorname{arctg}(2n+1)}{1 + 2\operatorname{arcctg} n^2}$, (l) $a_n = \ln(4n+5) - \ln(2n+3)$.

P1.9. Draw graphs of functions that meet all the given conditions

(a) $\lim\limits_{x\to 0^+} f(x) = 1$, $\lim\limits_{x\to 3} f(x) = +\infty$, $\lim\limits_{x\to+\infty} f(x) = +\infty$, f is an odd function;

(b) the line $y = 1$ is a horizontal asymptote at $-\infty$, the line $x = 0$ is a two-sided vertical asyptote, $\lim\limits_{x\to 2} g(x)$ does not exist, g is an even function;

(c) $\lim\limits_{x\to-\infty} [h(x) - x + 2] = 0$, $\lim\limits_{x\to 1^-} h(x) = -2$, $\lim\limits_{x\to 1^+} h(x) = -\infty$, $\lim\limits_{x\to+\infty} h(x) = 1$,
h is not continuous at the point $x_0 = 0$.

P1.10. Find the limits of functions:

(a) $\lim\limits_{x\to 3}\dfrac{x^2-2x-3}{x^2-9}$, (b) $\lim\limits_{x\to -3^+}\dfrac{x^2-2x-3}{x^2-9}$, (c) $\lim\limits_{x\to +\infty}\dfrac{x^2-2x-3}{x^2-9}$, (d) $\lim\limits_{x\to 9}\dfrac{\sqrt{x-5}-2}{x-9}$,

(e) $\lim\limits_{x\to +\infty}\dfrac{3^x+2^x}{4+2\cdot 3^x}$, (f) $\lim\limits_{x\to -\infty}\dfrac{3^x+2^x}{4+2\cdot 3^x}$, (g) $\lim\limits_{x\to -\infty}\dfrac{1}{4^x-3^x}$, (h) $\lim\limits_{x\to -\infty}(\sqrt{x^2+x}+x)$,

(i) $\lim\limits_{x\to +\infty}\dfrac{\sin^2 x}{\sqrt{x+\pi}}$, (j) $\lim\limits_{x\to \pi}\dfrac{\sin x}{x-\pi}$, (k) $\lim\limits_{x\to 0}\dfrac{3\sin 3x-5\sin 5x}{x}$, (l) $\lim\limits_{x\to 0}\dfrac{\mathrm{tg}4x}{\sqrt{1+3x}-1}$.

P1.11. Investigate whether the following limits exist:

(a) $\lim\limits_{x\to 0}\dfrac{\sin(\pi x)}{|x|}$, (b) $\lim\limits_{x\to 2} e^{\frac{x+2}{x-2}}$, (c) $\lim\limits_{x\to 0}\mathrm{arcctg}\dfrac{1}{x}$, (d) $\lim\limits_{x\to e}\dfrac{1}{1-\ln x}$.

P1.12. Find the asymptotes of the functions:

(a) $f(x)=\dfrac{x}{x^2-1}$, (b) $f(x)=\dfrac{x^2+1}{x^2-1}$, (c) $f(x)=\dfrac{x^3-1}{x^2-1}$, (d) $f(x)=\sqrt{x^2-4x}$,

(e) $f(x)=\dfrac{\sqrt{x^2-4}}{x}$, (f) $f(x)=\dfrac{\ln x}{2+\ln x}$, (g) $f(x)=\dfrac{3^x}{3^x-2}$, (h) $f(x)=x+\dfrac{\sin\sqrt{x}}{x}$.

P1.13. Is it possible to choose parameters $a,b\in\mathbf{R}$ such that the given function is continuous on $\mathbf{R}$? Calculate the corresponding limits and draw the graphs.

(a) $f(x)=\begin{cases} x+2 & \text{for } x<1\\ b & \text{for } x=1\\ x^2+ax+1 & \text{for } x>1\end{cases}$, (b) $f(x)=\begin{cases} ax+b & \text{for } |x|<1\\ \mathrm{arctg}x & \text{for } |x|\geqslant 1\end{cases}$,

P1.14. Prove, using the Darboux theorem, that the equation has a solution in the given interval. Give a graphical interpretation of the equation.

(a) $4^x=\dfrac{2}{x}$, $(0{,}5,1)$; (b) $\ln x = 1-2x$, $(0{,}5,1)$;

(c) $3^x=-x^3$, $(-1,-0{,}5)$; (d) $2^x=4-\sqrt{x}$, $(1,2)$.

Similar problems (with solutions) can be found in
M.Gewert, Z.Skoczylas, Analiza matematyczna 1. Przykłady i zadania, Oficyna Wydawnicza GiS, Wrocław 2014,
M.Gewert, Z.Skoczylas, Wstęp do analizy i algebry. Teoria, przykłady, zadania, Oficyna Wydawnicza GiS, Wrocław 2017.

LIST 4
(for 3 classes)

Differential calculus of functions of single variable

4.1. Using the definition, investigate whether the one-sided derivatives and the derivative of the given function exits at the indicated point. Draw the graph of the function.

(a) $y(x) = \left|x^2 - 4\right|, \quad x_0 = 2;$ (b) $f(x) = \left|\sin^3 x\right|, \quad x_0 = 0;$

(c) $g(x) = x^2 \mathrm{sgn} x, \quad x_0 = 0;$ (d) $h(x) = \begin{cases} 1 - x^2 & \text{for } x \leqslant 1 \\ (x-2)^2 & \text{for } x > 1 \end{cases}, \quad x_0 = 1.$

4.2. Using the formula on the derivative of the function $f(x) = x^\alpha$ and the rules of dirrerentiation, calculate the derivatives of the functions:

(a) $y = \sqrt{2}x^4 + 4\sqrt{x} - x\sqrt{x} + 3x^2\sqrt[3]{x},$ (b) $y = 5 \cdot \frac{1}{x^4} + \frac{1}{\sqrt[3]{x}} - \frac{2}{x \cdot \sqrt{x}} + \frac{1}{3x \cdot \sqrt[3]{x}},$

(c) $y = \frac{x}{\sqrt[4]{x^3}} - \frac{5x^2}{\sqrt{x}} + \frac{\sqrt[3]{x^2}}{7x^2} + \frac{x^{-2}}{x};$ (d) $y = \frac{\sqrt{3}}{\sqrt{9x}} - \frac{8}{(2x)^2} + \sqrt[3]{\frac{x}{16}} + \sqrt[4]{2^3}.$

4.3. Using the formula on the derivative of the product or the fraction, calculate the derivatives of the functions:

(a) $y = e^x \cdot \cos x,$ (b) $y = x^2 \cdot \ln x,$ (c) $y = x \cdot 2^x \cdot \sin x,$ (d) $y = x^2 \cdot \mathrm{tg} x \cdot \mathrm{arctg} x,$

(e) $y = \frac{x^2}{x^2 + x + 2},$ (f) $y = \frac{2^x - 3^x}{x},$ (g) $y = \frac{x \ln x}{2x - 3},$ (h) $y = \frac{x \sin x + \cos x}{\sin x - x \cos x}.$

4.4. Calculate the derivatives of the functions:

(a) $y = \ln(2x),$ (b) $y = \frac{1}{(2x-3)^2},$ (c) $y = 3x \sin\left(5x - \frac{\pi}{4}\right),$ (d) $y = \mathrm{arctg}\frac{1}{x},$

(e) $y = \sqrt{\frac{x+1}{x+2}},$ (f) $y = \sin^2 x,$ (g) $y = \cos^3\left(2x - \frac{\pi}{6}\right),$ (h) $y = x^3 \cos^2 \pi x.$

4.5. Write the equation of the tangent line for the graph of the function $y = f(x)$ at the point $(x_0, f(x_0))$. Draw the picture.

(a) $f(x) = \sin 2x, \quad x_0 = 0;$ (b) $f(x) = \mathrm{ctg} x, \quad x_0 = 1{,}5\pi;$ (c) $f(x) = \ln(x - 3), \quad f(x_0) = 0.$

4.6. Write the equation of the tangent line to the graph of the function $y = f(x)$, which has the given property.

(a) $f(x) = x \cdot \ln x$, the tangent line is parallel to the line $5x + 5y - 1 = 0$;
(b) $f(x) = \sqrt{x^2 + 1}$, the tangent line is orthogonal to the line $2x - y = 0$;
(c) $f(x) = \frac{x}{x^2 + 1}$, styczna jest horizontal;
(d) $f(x) = 3 - x^2$, the tangent line has angle $\frac{\pi}{3}$ with the positive half-axis OX.

4.7. Using derivative of the function, calculate the approximate value of the expression:

(a) $\frac{1}{\sqrt{4{,}02}}$, (b) $\frac{\ln 0{,}99}{1{,}99}$, (c) $1{,}03 \cdot \sqrt[3]{8{,}03}$, (d) $\mathrm{tg}^2\, 44\,^\circ$.

4.8. W wyniku pomiaru długości krawędzi czworościanu foremnego otrzymano $1{,}00 \pm 0{,}01$ m. Z jakim błędem bezwzględnym i względnym zostaną obliczone: wysokość, pole powierzchni i objętość tego czworościanu?

4.9. Using the L'Hospital rule calculate the limits:

(a) $\lim\limits_{x\to 1} \frac{\ln\left(\sin\left(\frac{\pi}{2}x\right)\right)}{\ln x}$, (b) $\lim\limits_{x\to -\infty} \frac{\ln(1+2^x)}{3^x}$, (c) $\lim\limits_{x\to 0^-} \left(\frac{1}{x} - \mathrm{ctg} x\right)$,

(d) $\lim\limits_{x\to +\infty} \left(\sqrt{x} - \ln x\right)$, (e) $\lim\limits_{x\to 0^+} \sqrt{x}\ln x$, (f) $\lim\limits_{x\to \pi^-} (\pi - x)\cdot \mathrm{tg}\frac{x}{2}$.

4.10. Find all the asymptotes of the function:

(a) $f(x) = \frac{x - \mathrm{arctg} x}{x^2}$, (b) $f(x) = \frac{\ln(x+1)}{\sqrt{x}}$, (c) $f(x) = \frac{x}{\mathrm{arctg} x}$, (d) $f(x) = \ln\left(x^2 - 4\right)$.

4.11. Find the intervals of monotonicity and local extrema of the function. Draw its graph.

(a) $y(x) = \frac{x^4}{4} - \frac{x^3}{3} - x^2$, (b) $y(x) = \frac{x^2}{x+1}$, (c) $f(x) = x^3 \cdot e^x$, (d) $g(x) = \sqrt{x}\cdot \ln x$.

4.12. Find the maximum and the minimum values of the function on the given interval:

(a) $f(x) = x - 2\sqrt{x}$, $[0, 5]$, (b) $f(x) = \mathrm{arctg} x - \frac{x}{2}$, $[0, 2]$, (c) $f(x) = 2\sin x + \sin 2x$, $\left[0, \frac{3}{2}\pi\right]$.

4.13.

(a) Determine two positive numbers, the sum of which is equal to 20, and the product of the square of the first and third power of the second power has the largest possible value.

(b) Determine which parallelepiped with the given surface area has the largest possible volume.

(c) The transport company accepts for transportation packages in the form of a parallelepiped, for which the sum of the height and the perimeter of the base is not more than 108 cm. Find the dimensions of the package with a square base and the largest volume that can be sent through this company.

(d) Define the line crossing the point $P = (1, 3)$ such that triangle formed by this line and the positive coordinate semi-axes has the minimal area.

4.14. Find the range of the function:

(a) $f(x) = \frac{\sqrt{1+x^2}}{x}$, (b) $g(x) = (x+1)\cdot e^{-2x}$, (c) $h(x) = x\cdot \ln^2 x$, (d) $h(x) = \sin x - \sin^2 x$.

Similar problems (with solutions) can be found in
M.Gewert, Z.Skoczylas, Analiza matematyczna 1. Przykłady i zadania, Oficyna Wydawnicza GiS, Wrocław 2017, rozdział 4, 5, 6.

LISTA 5.

(na 4 ćwiczenia)

The indefinite integral. The definite integral

5.1. Using the definition and formulae of derivatives of basic functions find the primitive F of the function f:

(a) $f(x) = 2x - 1$, (b) $f(x) = \dfrac{3}{1+x^2}$, (c) $f(x) = \sin\left(x + \dfrac{\pi}{3}\right)$, (d) $f(x) = e^{-4x}$.

5.2. Find the integrals:

(a) $\displaystyle\int \frac{x^4 - x^3 + x - 1}{x - 1}\,dx$, (b) $\displaystyle\int \left(\frac{x-2}{x}\right)^2 dx$, (c) $\displaystyle\int \left(\sqrt{x}+1\right)\left(x - \sqrt{x} + 1\right) dx$,

(d) $\displaystyle\int \frac{\sqrt[3]{x^2} - \sqrt[4]{x}}{\sqrt{x}}\,dx$, (e) $\displaystyle\int \frac{3 \cdot 2^x - 2 \cdot 3^x}{2^x}\,dx$, (f) $\displaystyle\int \frac{\cos 2x}{\cos^2 x \cdot \sin^2 x}\,dx$,

(g) $\displaystyle\int \operatorname{ctg}^2 x\,dx$, (h) $\displaystyle\int \sin x \cdot \cos x\,dx$, (i) $\displaystyle\int \left(4\sin\left(x + \frac{\pi}{4}\right) - 6\cos 3x + 1\right) dx$.

5.3. Find the integrals using corresponding substitutions

(a) $\displaystyle\int x\sqrt{1+2x^2}\,dx$, (b) $\displaystyle\int \frac{4x}{\sqrt[3]{x^2-4}}\,dx$, (c) $\displaystyle\int x^2\left(x^3 - 2\right)^5 dx$, (d) $\displaystyle\int \sin x \cdot \cos^2 x\,dx$,

(e) $\displaystyle\int \frac{\ln^2 x}{x}\,dx$, (f) $\displaystyle\int xe^{-x^2}\,dx$, (g) $\displaystyle\int \sin^2 x \cdot \cos^3 x\,dx$, (h) $\displaystyle\int \frac{1}{4x^2+4x+5}\,dx$.

5.4. Find the integrals using that $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$.

(a) $\displaystyle\int \frac{1}{3x+2}\,dx$, (b) $\displaystyle\int \frac{x}{1+2x^2}\,dx$, (c) $\displaystyle\int \frac{1}{x\ln x}\,dx$, (d) $\displaystyle\int \frac{e^x}{e^x+1}\,dx$.

5.5. Find the integrals, using the integration-by-parts formula

(a) $\displaystyle\int xe^{-3x}\,dx$, (b) $\displaystyle\int x\cos\frac{x}{2}\,dx$, (c) $\displaystyle\int x^2 \sin\left(x + \frac{\pi}{3}\right) dx$, (d) $\displaystyle\int \ln(x+1)\,dx$,

(e) $\displaystyle\int \sqrt{x}\ln x\,dx$, (f) $\displaystyle\int \frac{\ln x}{x^2}\,dx$, (g) $\displaystyle\int \operatorname{arcctg} x\,dx$, (h) $\displaystyle\int e^{-x}\sin 2x\,dx$.

5.6. Write an integral sum for the given definite integral. Use the uniform partition of the interval of integration, and the values of the integrand function in the right ends of the subintervals. Using the definition, calculate the definite integrals

(a) $\displaystyle\int_0^1 x^2\,dx$, (b) $\displaystyle\int_1^2 x\,dx$, (c) $\displaystyle\int_0^\pi \sin x\,dx$, (d) $\displaystyle\int_0^1 \frac{1}{1+x}\,dx$.

5.7. Calculate the definite integral. Give a geometric interpretation by a corresponding picture.

(a) $\int\limits_0^1 (1+x)\,dx$, (b) $\int\limits_0^{\pi} \sin 2x\,dx$, (c) $\int\limits_{-1}^{1} e^{-x}\,dx$, (d) $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} \mathrm{ctg}x\,dx$, (e) $\int\limits_{e^{-2}}^{e^2} \ln x\,dx$.

5.8. Find a mean value of the function f on the interval $[a,b]$. Draw the picture.

(a) $f(x)=\sin^2 x,\ [a,b]=[0,\pi]$; (b) $f(x)=|x-2|\,,\ [a,b]=[0,3]$.

5.9. Find the area of the domain bounded by the given curves. Draw the picture.

(a) $y=x^2-2x+3,\ y=x+3$; (b) $y=\dfrac{4}{x^2+2},\ y=1$;

(c) $y=x^2,\ y=\dfrac{x^2}{2},\ y=3x$; (d) $y=-\ln(x+2)\,,\ x=0,\ y=0$.

5.10. Write the formula for the length of the curve of the graph of a differentiable function and calculate the length of the given curves. Draw them.

(a) $y=-x\sqrt{x},\ x\in\left[0,\dfrac{4}{9}\right]$; (b) $y=\sqrt{4-x^2},\ x\in[-1,1]$;

(c) $y=\ln\sin x,\ x\in\left[\dfrac{\pi}{3},\dfrac{\pi}{2}\right]$; (d) $y=\ln x,\ x\in[1,e]$.

5.11. Write the formula for the volume of a body of rotation obtained by a rotation around the axis OX of the domain bounded by the graph of a continuous non-negative function $y=f(x)$, axis OX, and the lines $x=a, x=b$. Using this formula, find the volume:

(a) the ball or the radius R,

(b) the conoid with the radii of the bases r, R and the height H,

(c) the body obtained by a rotation around the axis OX of the domain

$$T=\left\{(x,y)\in R^2 : 0\leqslant x\leqslant\frac{\pi}{4},\ 0\leqslant y\leqslant \mathrm{tg}x\right\},$$

(d) the body obtained by a rotation around the axis OX of the domain

$$T=\left\{(x,y)\in R^2 : -\frac{\pi}{2}\leqslant x\leqslant\frac{\pi}{2},\ 0\leqslant y\leqslant \cos^2 x\right\}.$$

5.12. Find the integrals of rational functions

(a) $\displaystyle\int \frac{8x^2}{x^2-1}\,dx$, (b) $\displaystyle\int \frac{3x^2}{x^3+x^2-4x-4}\,dx$, (c) $\displaystyle\int \frac{x^3-x^2+3}{x^4+3x^2}\,dx$,

(d) $\displaystyle\int \frac{2}{x^2+6x+18}\,dx$, (e) $\displaystyle\int \frac{5-4x}{x^2-4x+20}\,dx$, (f) $\displaystyle\int \frac{x^2+2x+1}{x^3+2x^2+2x}\,dx$.

5.13. Find the integrals of trigonomatric functions

(a) $\int \sin^5 x\,dx$, (b) $\int \sin^2 x \cos^3 x\,dx$, (c) $\int \frac{\cos^3 x}{2-\sin x}\,dx$, (d) $\int \frac{1}{4+5\sin^2 x}\,dx$,

(e) $\int \frac{1}{5-3\cos x}\,dx$, (e) $\int_{-\pi}^{\pi} \sin x \sin 3x\,dx$, (f) $\int_{-\pi}^{\pi} \sin 2x \cos 4x\,dx$, (g) $\int_{-\pi}^{\pi} \sin^2 x\,dx$.

Similar problems (with solutions) can be found in
M.Gewert, Z.Skoczylas, Analiza matematyczna 1. Przykłady i zadania, Oficyna Wydawnicza GiS, Wrocław 2017, rozdział 7, 8, 9.

REPETITION 2

P2.1. Find the derivatives of the functions:

(a) $f(x) = \dfrac{\text{arctg}x}{\ln(1+x^2)}$, (b) $f(x) = e^{3\sin x}\cdot \sin 2x$, (c) $f(x) = (x\cos x)^2$, (d) $f(x) = \dfrac{x\cdot\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}$.

P2.2.

(a) Write the equations for the tangent lines to the graph of the function $f(x) = \frac{1}{2}\text{arctg}\,(1-x^2)$ at the zeroes of the function. Under which angles the graph intersects the axis OX?

(b) Write the equation for the tangent line to the graph of the function $f(x) = \ln\sqrt{x} - 0{,}5x^2$ which is parralel to the axis OX.

(c) Write the equation for the tangent line to the graph of the function $f(x) = \ln\left(x^2 + e^{-x}\right)$, which is parallel to the line $l: y = 5 - x$.

(d) Write the equation for the tangent line to the graph of the function $f(x) = \text{tg}(2x) - 3x, \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, which is orthogonal to the line $l: x + 5y = 0$.

(e) For which values of parameters a, b the parabola defined by the equation $y = -x^2 + ax + b$ is tangent to the line $y = x$ at the point $(1, 1)$? Draw the picture.

P2.3. Wykorzystując różniczkę obliczyć, o ile w przybliżeniu zmieni się wartość funkcji

(a) $f(x) = (1+x)\ln x$, gdy jej argument wzrośnie od wartości $x_0 = 1$ do wartości $x_1 = 1{,}1$;

(b) $g(x) = \dfrac{\sqrt{x}}{1+x}$, gdy jej argument zmieni się od wartości $x_0 = 4$ do wartości $x_1 = 3{,}99$.

P2.4. Investigate existence of asymptotes

(a) at the point $x = 0$ of the function $f(x) = \dfrac{1-\cos 3x}{\sin^2 4x}$,

(b) at the point $x = \dfrac{\pi}{2}$ of the function $f(x) = \dfrac{\ln(1+3\cos x)}{\pi - 2x}$,

(c) horizontal at $+\infty$ of the function $f(x) = x\cdot\left(6^{\frac{1}{x}} - 2^{\frac{1}{x}}\right)$,

(d) at the point $x = 0$ of the function $f(x) = \dfrac{1}{x} - \dfrac{1}{\sin 2x}$.

P2.5. Find the minimal and maximal values of the function on the given interval.

(a) $f(x) = \dfrac{x+1}{x^2+2x+2}$, $[-7, 0]$; (b) $y(x) = \dfrac{e^x}{1+x^2}$, $[-2, 2]$;

(c) $g(x) = \sqrt{3}\cos x + \sin x$, $\left[0, \dfrac{\pi}{2}\right]$; (d) $y(x) = \sqrt[3]{(x^2+x)^2}$, $[-2, 3]$.

the intervals of monotonicity and the local extrema of the function f. Draw its graph.

(a) $f(x) = \ln\left(x^3 - 2x^2 + x\right)$, (b) $f(x) = x\cdot\ln^4 x$, (c) $f(x) = \dfrac{e^{2x}}{e^x - 1}$, (d) $f(x) = \dfrac{x}{\ln x}$.

P2.6. Find the range of the function. Draw its graph.

(a) $y(x) = \dfrac{(x-1)^2}{(x-3)^2}$, (b) $f(x) = x \cdot (1 + 2\ln x)$, (c) $f(x) = x \cdot \sqrt{4x - x^2}$, (d) $f(x) = \dfrac{\sqrt{x^2-1}}{x}$.

P2.7.
(a) In the domain bounded by the parabola $y = 16 - x^2$ and the axis ą OX a rectangle inscribed such that one of its sides lies on the axis OX. What are the lengths of the sides of such a rectangle which has a maximal area?

(b) Metodami rachunku różniczkowego uzasadnić, że prostopadłościan o danej sumie długości krawędzi, kwadratowej podstawie i największej objętości jest sześcianem.

(c) How much material will belost if we cut from a metal piece, shaped as a half-circle of the radius R, a rectangle of a maximal area?

P2.8. Find the integrals:

(a) $\int x \cdot \cos(\pi x + 2)\, dx$, (b) $\int \left(\frac{x}{e^x}\right)^2 dx$, (c) $\int \frac{\ln^2 x}{\sqrt{x}}\, dx$,

(d) $\int \frac{\sin 3x}{e^x}\, dx$, (e) $\int \frac{\mathrm{tg}(\ln x)}{x}\, dx$, (f) $\int \frac{\sqrt[3]{x^2+1} + \sqrt{x^2+1}}{x^2+1}\, x dx$,

(g) $\int \frac{1+\ln x}{1+\ln^2 x} \cdot \frac{1}{x} dx$, (h) $\int (1+\cos x) \cdot \sin^3 x\, dx$, (i) $\int 3^{2x} \cdot \sin 3^x\, dx$.

P2.9. Calculate the definite integral. Give a geometric interpretation by a corresponding picture.

(a) $\int_{-1}^{1} e^{2x}\, dx$, (b) $\int_{0}^{\pi} \sin x \cos x\, dx$, (c) $\int_{\frac{1}{e}}^{e^2} \ln x\, dx$, (d) $\int_{0}^{\frac{\pi}{3}} \mathrm{tg} x\, dx$.

P2.10. Find the area of the domain bounded by the given curves. Draw the picture.

(a) $y = x^2 - 2x,\ y = x + 4$; (b) $y = x^2,\ y = 5 - (x+1)^2$;

(c) $y = \sqrt{x},\ y = \sqrt[3]{x}$; (d) $y = \dfrac{4}{x^2+1},\ y = 1$;

(e) $x + y = 4,\ y = \dfrac{3}{x}$; (f) $y = \sin x,\ y = x,\ x = \pi$;

(g) $y = \ln(1+x),\ y = x,\ x = e$; (h) $y = \ln(1+x),\ y = x,\ y = 1$.

Similar problems (with solutions) can be found in
M.Gewert, Z.Skoczylas, Analiza matematyczna 1. Przykłady i zadania, Oficyna Wydawnicza GiS, Wrocław 2017,
M.Gewert, Z.Skoczylas, Analiza matematyczna 1. Kolokwia i egzaminy, Oficyna Wydawnicza GiS, Wrocław 2014.

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YET TO DO: 4.8, P2.3, P2.7(b)