MATHEMATICAL ANALYSIS 2
Problems List 2.
Second and higher order derivatives. Convexity. Sylvester's criterion. Local and global extrema. Conditional extrema.

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1. Calculate all partial derivatives of the 2nd order of the functions

(a) $f(x,y) = \sin(x^3+y^3)$, (b) $f(x,y) = ye^{xy}$, (c) $f(x,y) = x^2 + \dfrac{y^3}{x}$,
(d) $f(x,y) = y\ln\dfrac{x}{y}$, (e) $f(x,y,z) = \dfrac{y}{\sqrt{1+x^2+z^2}}$ (f) $f(x,y,z,w) = \ln(x+y^2+z^3+w^4+1)$.

2. Write the Hessian of the function and specify the domains where the Hessian is positively/negatively defined.

(a) $f(x,y) = \sin(x^2+y^2)$, (b) $f(x,y) = xe^{xy}$, (c) $f(x,y) = x^3 + \dfrac{y^2}{x}$,
(d) $f(x,y) = x\ln\dfrac{y}{x}$, (e) $f(x,y) = \dfrac{y}{\sqrt{1+x^2}}$ (f) $f(x,y) = \ln(x+y^2+1)$.

3. Study the following matrices for being positively/negatively definite or semi-definite.

(a) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$, (b) $\begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} -1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}$,

(d) $\begin{pmatrix} 2 & 4 & 3 \\ 4 & 5 & 4 \\ 3 & 4 & 12 \end{pmatrix}$, (e) $\begin{pmatrix} 3 & 1 & 5 \\ 1 & 1 & 2 \\ 5 & 2 & 7 \end{pmatrix}$, (f) $\begin{pmatrix} 2 & 2 & 3 & 1 \\ 2 & 5 & 1 & 2 \\ 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}$.

4. Find and classify all the critical points of $f(x,y) = 4 + x^3 + y^3 - 3xy$.

5. Find and classify all the critical points of $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 5$.

6. Find and classify all the critical points of $f(x,y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$.

7. Given the objective function $f(x,y) = x^2+y^2$, subject to the constraint $g(x,y) = x^2+y^2-4x-2y-15$ find extremal points.

8. Determine the point on the plane $4x-2y+z=1$ that is closest to the point $(-2,-1,5)$.

9. On the surface $x^2+y^2+z^2-xy-xz+x+y-z=1$ find the points of maximal and minimal values for x. Check that the surface is bounded.

10. Find the global extrema of $f(x,y) = x^2+4y^2$ on the domain bounded by the curves $x^2+(y+1)^2=4$, $y=-1$, and $y=x+1$.

11. Find the global extrema of $f(x,y) = x^2+y^2-6x+6y$ on the disk of radius 2, centred at the origin.

12. Find the maximal and the minimal values of the functions on the given domains:

(a) $f(x,y) = 2x^3+4x^2+y^2-2xy$, $D = \{(x,y) : x^2 \leqslant y \leqslant 4\}$;

(b) $f(x,y) = \sqrt{y-x^2} + \sqrt{x-y^2}$, $D = \{(x,y) : y \geqslant x^2, x \geqslant y^2\}$;

(c) $f(x,y) = \sqrt{1-x^2} + \sqrt{4-x^2-y^2}$, $D = \{(x,y) : x^2 \leqslant 1, x^2+y^2 \leqslant 4\}$;

(d) $f(x,y) = x^2-y^2$, D is the triangle with the vertices $(0,1),(0,2),(1,2)$;

(e) $f(x,y) = x^4 + y^4$, $D = \{(x,y) : x^2 + y^2 \leqslant 9\}$

13. Krzysztof and Szymon consume two products. If x and y are the quantities of the products (respectively), then Krzysztof's utility function is $U(x,y) = \ln x + 2\ln y$, and Szymon's is $U(x,y) = xy^2$. The prices of the products per unit are: $P_x = 5$ zł and $P_y = 2$ zł. Krzysztof and Szymon have identical incomes, of 90 zł each. Find the optimal consumed quantity from each of the two products, for each of Krzysztof and Szymon.