MATHEMATICAL ANALYSIS 2

Problems List 3.

Definite integrals of two variables. Double and iterated integrals. Change of variables formula for an integral of two variables. Polar coordinates. Applications of double integrals in geometry and physics.

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1. Calculate the definite integrals over the given rectangles.

(a) $\iint_R (x^2 + y^3 - xy)\, dxdy,\, R = [0,1] \times [0,1]$, (b) $\iint_R \frac{x}{y^2}\, dxdy,\, R = [1,2] \times [2,4]$,

(c) $\iint_R (1 + x + y)^3\, dxdy,\, R = [0,2] \times [0,1]$, (d) $\iint_R x \sin(xy)\, dxdy,\, R = [0,1] \times [\pi, 2\pi]$,

(e) $\iint_R \frac{x+y}{e^x}\, dxdy,\, R = [0,1] \times [0,1]$, (f) $\iint_R e^{2x-y}\, dxdy,\, R = [0,1] \times [-1,0]$.

2. Calculate the iterated integrals and draw the domains of integration

(a) $\int_0^1 dx \int_x^{x^2} \frac{y}{x^2}\, dy$, (b) $\int_1^4 dx \int_x^{2x} x^2\sqrt{y-x}\, dy$, (c) $\int_0^3 dx \int_0^x \sqrt{x^2+16}\, dy$.

3. Calculate the integrals over the normal domains bounded by the given curves

(a) $\iint_D xy^2 dxdy,\, y = x,\, y = 2 - x^2$, (b) $\iint_D x^2 y dxdy,\, y = -2,\, y = \frac{1}{x},\, y = -\sqrt{-x}$,

(c) $\iint_D e^{x/y} dxdy,\, y = \sqrt{x},\, x = 0,\, y = 1$, (d) $\iint_D (xy + 4x^2) dxdy,\, y = x + 3,\, y = x^2 + 3x + 3$.

4. Calculate the Jacobians of the transformations:

(a) $x = 4u - 3v^2 \qquad y = u^2 - 6v$;

(b) $x = u^2 v^3 \qquad y = 4 - 2\sqrt{u}$;

(c) $x = \frac{v}{u} \qquad y = u^2 - 4v^2$.

5. Determine the domain Δ which is transformed by the given mapping to the given domain D.

(a) D is the ellipse $x^2 + \frac{y^2}{36} \leqslant 1$, the transformation $x = \frac{u}{2},\, y = 3v$.

(b) D is the parallelogram with the vertices $(1,0)\,,(4,3)\,,(1,6)$ and $(-2,3)$, the transformation $x = \frac{1}{2}(u+v),\, y = \frac{1}{2}(u-v)$.

(c) D is the parallelogram with vertices $(2,0)\,,(5,3)\,,(6,7)$ and $(3,4)$, the transformation $x = \frac{1}{3}(v-u)\,,\, y = \frac{1}{3}(4v-u)$.

(d) D is the domain bounded by $xy = 1,\, xy = 3,\, y = 2$ and $y = 6$, the transformation $x = \frac{v}{6u},\, y = 2u$.

6. Propose a transformation that will represent the triangle D with vertices $(1,0)\,,(6,0)$ and $(3,8)$ as an image of a right triangle with the right angle occurring at the origin of the u,v system.

7. Propose a transformation that will represent the parallelogram D with vertices $(1,2),(3,5)$, $(-1,0),(1,3)$ as an image of a rectangle.

8. Perform the change of variables to the polar coordinates and evaluate the integrals. Draw the domain of integration in the Cartesian and polar coordinates

(a) $\iint_D xy\,dxdy,\ D: x^2+y^2\leqslant 1,\ \dfrac{x}{\sqrt{3}}\leqslant y\leqslant x\sqrt{3};$

(b) $\iint_D y^2e^{x^2+y^2}\,dxdy,\ D: x^2+y^2\leqslant 1, x\geqslant 0, y\geqslant 0;$

(c) $\iint\limits_D (y^2+3x)\,dxdy$, D is the region in the 3rd quadrant between $x^2+y^2=1$ and $x^2+y^2=9$;

(d) $\iint\limits_D (4xy-7)\,dxdy,\ D: x^2+y^2\leqslant 1, -x\sqrt{3}\leqslant y\leqslant x;$

(e) $\iint\limits_D (x^3+y^3)\,dxdy,\ D: x^2+y^2\leqslant 1, x\sqrt{3}\leqslant y\leqslant -x.$

9. Performing an appropriate change of variables, evaluate the integrals

(a) $\iint\limits_D (6x-3y)\,dxdy$ where R is the parallelogram with vertices $(1,0)\,,(4,3)\,,(5,7)$ and $(2,4)$;

(b) $\iint\limits_D xy^3\,dxdy$ where D is the domain bounded by $xy=1, xy=2, y=3$ and $y=4$;

(c) $\iint\limits_D (x+2y)\,dxdy$ where D is the triangle with vertices $(0,3)\,,(4,1)$ and $(2,6)$;

(d) $\iint\limits_D x^2\,dxdy$, where D is the ellipse $x^2+\dfrac{y^2}{9}\leqslant 1$;

(e) $\iint\limits_D \sqrt{4-x^2-y^2}\,dxdy$, where D is the circle with radius 1 centered at $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$.

10. Find the area of the ellipse $(x-3)^2+4(y+1)^2\leqslant 10$.

11. Find the area of the part of the sphere $x^2+y^2+z^2=R^2$, cutted off by the cylinder $x^2+y^2=r^2$ $(r<R)$.

12. Find the area of the part of the cone $z^2=x^2+y^2$, cutted off by the cylinder $x^2+y^2=2x$.

13. For the body bounded by the cone and cylinder from the previous problem, find its volume.

14. Find the volume of the body bounded by the cone $z^2=\frac{1}{2}(x^2-y^2)$ and the planes $x+y=\pm 1, x-y=\pm 1$. *Hint: After applying the general formula, change the variables (x,y) to a more convenient pair (u,v).*

15. Find the area of the part of the surface[1] $z = axy$, cutted off by the cylinder $x^2 + y^2 = b$.

16. For a given plate D and density function $\gamma(x, y)$, find the mass of D, its center of mass, its static moments and moments of inertia:

(a) D is a rectangle with the sides A, B from which a smaller rectangle with the sides a, b is cutted in the middle. All the sides are parallel to the axes, and the centers of the rectangles coincide and are located in the origin. The density function $\gamma(x, y) = 1$.

(b) D is the quarter of a circle of the radius R, located symmetrically w.r.t. the Ox axis with the center placed at the origin; the density function $\gamma(x, y) = 1$.

(c) D is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leqslant 1$, the density function $\gamma(x, y) = 1 + c(x^2 + y^2)$.

[1] This surface is called a *hyperbolic paraboloid*