MATHEMATICAL ANALYSIS 2

Problems List 4.

Improper integrals. Number and power series. Taylor-Maclaurin series

---

1. Check if the following improper integrals are convergent:

(a) $\int_1^\infty \frac{x^2+1}{\sqrt{x^6+\sin x}}\,dx$, (b) $\int_1^\infty \frac{\ln x}{\sqrt{x^3+1}}\,dx$,

(c) $\int_2^\infty \frac{x^{2022}}{(1.1)^x}\,dx$, (d) $\int_5^\infty \frac{e^x}{2^{2x}}\,dx$,

(e) $\int_1^\infty \frac{x^2+\sin x}{x^4+\cos x}\,dx$, (f) $\int_1^\infty \sin\frac{1}{x}\,dx$.

2. Check if the following number series are convergent:

(a) $\sum_{n=1}^\infty \frac{\ln n}{n^2}$, (b) $\sum_{n=1}^\infty \sqrt{\frac{n+\sin n}{n^3+\cos n}}$,

(c) $\sum_{n=1}^\infty \frac{(n+1)^2}{(n^2+2)^2}$, (d) $\sum_{n=1}^\infty (-1)^n\frac{\ln n}{n}$,

(e) $\sum_{n=1}^\infty \frac{\ln n}{n}$, (f) $\sum_{n=1}^\infty \frac{n}{3^n}$.

3. Find the radius of convergence and the interval of convergence for the power series:

(a) $\sum_{n=0}^\infty \frac{n}{n^{2022}+1}(x-1)^n$, (b) $\sum_{n=0}^\infty \frac{2^n}{\sqrt{3^n+n}}(x+1)^n$,

(c) $\sum_{n=0}^\infty \frac{(n^2+1)^n}{(2n+1)^n(3n+1)^n}(x-3)^n$, (d) $\sum_{n=0}^\infty (-1)^n\frac{(1+\frac{1}{n})^{n^2}}{2^n}(x+2)^n$,

(e) $\sum_{n=1}^\infty \frac{(2n)!}{(n!)^2}(x-2)^n$, (f) $\sum_{n=1}^\infty \frac{(4n)!}{(n!)(2n!)^2(3n)!)}(x-1)^n$.

4. Determine the Taylor-Maclaurin series for the given function

(a) $f(x) = \cos(4x)$;

(b) $f(x) = x^6 e^{2x^3}$;

(c) $f(x) = x\cos(2x^3)$;

(d) $f(x) = \frac{x^{100}}{1+x^3}$.

5. For each function from the previous problem find $f^{(2022)}(0)$.

6. Determine the Taylor series for the given function $f(x)$ and x_0. Provide **two** solutions: using the formula for the coefficients and the change of variables.

(a) $f(x) = e^{-6x}, x_0 = -4$;

(b) $f(x) = \ln(3+4x), x_0 = 1$;

(c) $f(x) = \frac{7}{x^4}, x_0 = -3$;

7. For each of the series in the previous problem determine the interval of convergence.

8. Using the Taylor-Maclaurin series and differentiation/integration calculate the infinite sums

(a) $\sum_{n=1}^{\infty} \frac{1}{n3^n}$, (b) $\sum_{n=2}^{\infty} \frac{2^n - n}{3^n}$,

(c) $\sum_{n=0}^{\infty} \frac{n(n+1)}{5^n}$, (d) $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^n}$.

9. * Using the Taylor-Maclaurin series and differentiation/integration calculate $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ (*Hint:* consider the limit of $\sum_{n=1}^{\infty} \frac{x^n}{n(n+2)}$ as $x \nearrow 1$).