Well-known Inequalities

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Algebraic Inequalities

- **Jensen**

  Let $f : [a, b] \to \mathbb{R}$ be a convex function (i.e., $(1-\lambda)f(x) + \lambda f(y) \geq f((1-\lambda)x + \lambda y)$ for all x, $y \in [a, b]$, $\lambda \in [0, 1]$). Then for any $x_1, \ldots, x_n \in [a, b]$ and any nonnegative reals $\lambda_1, \ldots, \lambda_n$ which sum to 1,

  $$\lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \cdots + \lambda_n x_n).$$

  Other criteria for convexity:

  1. Graphically, f is convex iff the set of points $\{(x, y) \in [a, b] \times \mathbb{R} \mid y \geq f(x)\}$ is convex (the line segment joining any two points of the set lies entirely within the set).
  2. If f is continuous, then it suffices to check the statement for $\lambda = \frac{1}{2}$ (or any other fixed $0 < \lambda < 1$). (Conversely, if f is convex, then f is continuous except possibly at a and b.)
  3. If f is continuous on $[a, b]$ and twice differentiable on (a, b), then it suffices to check that $f''(x) \geq 0$ for $x \in (a, b)$.

  Exercise: Prove Jensen's inequality from the definition of convexity using smoothing.

- **Weighted Power Mean**

  If $x_1, \ldots, x_n$ are nonnegative reals and $\lambda_1, \ldots, \lambda_n$ are nonnegative reals with sum 1 then

  $$(\lambda_1 x_1^r + \cdots + \lambda_n x_n^r)^{1/r}$$

  is a nondecreasing function of r. When $r = 0$, this expression represents the weighted geometric mean of the x_i. This function is strictly increasing unless the x_i are all equal (except when some x_i is 0; then it is identically 0 for $r \leq 0$).

  Exercise: Prove the Weighted Power Mean inequality using Jensen.

- **Hölder**

  Let $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$, $\ldots$, $(z_1, \ldots, z_n)$ be several sequences of nonnegative real numbers and $\lambda_a, \ldots, \lambda_z$ nonnegative reals which sum to 1. Then

  $$(a_1 + \cdots + a_n)^{\lambda_a}(b_1 + \cdots + b_n)^{\lambda_b} \cdots (z_1 + \cdots + z_n)^{\lambda_z} \geq a_1^{\lambda_a} b_1^{\lambda_b} \cdots z_1^{\lambda_z} + \cdots + a_n^{\lambda_a} b_n^{\lambda_b} \cdots z_n^{\lambda_z}.$$

  Exercise: Prove Hölder's inequality by reducing to the case of two sequences, then normalizing each sequence to have sum 1.

  Homework: By dividing both sides of this inequality by n, we may rephrase it in the following form: Given a matrix

  $$\begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \\ \vdots & & \vdots \\ z_1 & \cdots & z_n \end{pmatrix}$$

  the arithmetic mean of the (weighted) geometric means of the columns is at most the (weighted) geometric mean of the arithmetic means of the rows. Using Hölder, prove **Minkowski's inequality**, the following generalization: for any reals $r > s$, the rth power mean of the sth power means of the columns is at most the sth power mean of the rth power means of the rows. Another special case of Minkowski's inequality is the **triangle inequality**,

  $$\sqrt[p]{x_1^p + \cdots + x_n^p} + \sqrt[p]{y_1^p + \cdots + y_n^p} \geq \sqrt[p]{(x_1 + y_1)^p + \cdots + (x_n + y_n)^p}$$

  for $x_1, \ldots, x_n, y_1, \ldots, y_n$, nonnegative reals, $p \geq 1$.

Ordering Inequalities

- **Rearrangement** If $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ are two nondecreasing sequences of real numbers, then for any permutation π of $\{1, 2, \ldots, n\}$, we have

  $$a_1b_1 + a_2b_2 + \cdots + a_nb_n \ge a_1b_{\pi(1)} + a_2b_{\pi(2)} + \cdots + a_nb_{\pi(n)} \ge a_1b_n + a_2b_{n-1} + \cdots + a_nb_1.$$

  Exercise: This is obvious, but prove it formally.

- **Chebyshev** If $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ are two nondecreasing sequences of real numbers, then

  $$\frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} \ge \frac{a_1 + a_2 + \cdots + a_n}{n} \cdot \frac{b_1 + b_2 + \cdots + b_n}{n} \ge \frac{a_1b_n + a_2b_{n-1} + \cdots + a_nb_1}{n}.$$

  Exercise: Prove Chebyshev's inequality using rearrangement.

- **Schur** Let x, y, z be nonnegative reals, and let $r > 0$; then

  $$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \ge 0.$$

  Equality holds iff $x = y = z$ or two of x, y and z are equal and the other is 0.

  Exercise: Prove Schur's inequality using rearrangement.

Symmetric Polynomial Inequalities

In this section let $x_1, \ldots, x_n$ be nonnegative real numbers. We define the elementary symmetric polynomials $s_0, s_1, \ldots, s_n$ by $(t + x_1)\cdots(t + x_n) = s_nt^n + \cdots + s_1t + s_0$. The symmetric averages d_i are then defined by $d_i = s_i/\binom{n}{i}$.

- **Newton** $d_i^2 \ge d_{i-1}d_{i+1}$.

- **Maclaurin** $d_1 \ge d_2^{1/2} \ge d_3^{1/3} \ge \cdots \ge d_n^{1/n}$. Note that this strengthens AM-GM ($d_1 \ge d_n^{1/n}$).

  These two inequalities can be proved using the following fact.

- If $x_1, \ldots, x_m$ are nonnegative reals and $m > n$ then there exist nonnegative reals $x'_1, \ldots, x'_{m-1}$ with the same first through mth symmetric averages as the x_i. Thus an inequality in the symmetric averages $d_1, \ldots, d_n$ need only be checked in the case of n variables $x_1, \ldots, x_n$.

  Homework: Prove this fact by taking the roots of the derivative of the polynomial $(t-x_1)\cdots(t-x_m)$ for $x'_1, \ldots, x'_{m-1}$; then check Newton's inequality for the case of $i+1$ variables and finally deduce Maclaurin's inequality.

  As another example, Schur's inequality for $r = 1$ can be written in the form $3d_1^3 + d_3 \ge 4d_1d_2$, which in four variables w, x, y, z becomes $9(w^3 + \cdots) - 5(w^2x + \cdots) + 6(wxy + \cdots) \ge 0$.

Majorization

Let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be two nonincreasing sequences of real numbers. If $x_1 + \cdots + x_k \ge y_1 + \cdots + y_k$ for all $k = 1, \ldots, n$, with equality for $k = n$, then we say that the sequence $(x_1, x_2, \ldots, x_n)$ majorizes $(y_1, y_2, \ldots, y_n)$ and denote this by $(x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n)$. If $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are any two sequences of real numbers, not necessarily nonincreasing, then we apply the same definition after sorting the two sequences into nonincreasing order.

- **An alternate criterion**

  The sequence $(x_1, x_2, \ldots, x_n)$ majorizes the sequence $(y_1, y_2, \ldots, y_n)$ iff for every $t \in \mathbb{R}$,

  $$|t - x_1| + |t - x_2| + \cdots + |t - x_n| \ge |t - y_1| + |t - y_2| + \cdots + |t - y_n|.$$

  Intuitively, the y_i are "more bunched together" than the x_i.

  Exercise: Verify the equivalence of these definitions.

- **Majorization** (also known as **Hardy-Littlewood(-Polya)** or **Karamata**)

  Let $f : [a, b] \to \mathbb{R}$ be a convex function and suppose the sequence $(x_1, x_2, \ldots, x_n)$ majorizes the sequence $(y_1, y_2, \ldots, y_n)$ with $x_i,\ y_i \in [a, b]$. Then

  $$\sum_{i=1}^{n} f(x_i) \geq \sum_{i=1}^{n} f(y_i).$$

  Homework: Prove this using the technique of Abel summation together with the fact that the slope of the segment joining two points on the graph of a convex function is an increasing function of each point's x-coordinate.

  Note that since $f(x) = |x|$ is a convex function, this theorem extends the previous criterion.

- **Popoviciu**

  Let $f : [a, b] \to \mathbb{R}$ be a convex function, and let $x,\ y,\ z \in [a, b]$. Then

  $$f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) \geq 2\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right).$$

  Exercise: Prove this by checking it for the function $f(x) = |x|$, then using the previous two results.

- **Muirhead** (or **"Bunching"**)

  If the sequence $(a_1, a_2, \ldots, a_n)$ majorizes the sequence $(b_1, b_2, \ldots, b_n)$ then for any positive real numbers $x_1,\ x_2,\ \ldots,\ x_n$ we have

  $$\sum_{\pi \in S_n} x_{\pi(1)}^{a_1} x_{\pi(2)}^{a_2} \cdots x_{\pi(n)}^{a_n} \geq \sum_{\pi \in S_n} x_{\pi(1)}^{b_1} x_{\pi(2)}^{b_2} \cdots x_{\pi(n)}^{b_n}.$$

  Exercise: Deduce this from the Majorization inequality.

  Muirhead's theorem is a handy way to remember the direction of simple polynomial inequalities; while it is a "named theorem", it is not customarily cited in the formal write-up of a problem, since in any particular instance it is a simple matter to determine the appropriate application of AM-GM.

Other

- **Bernoulli**

  For $r \geq 1$ and $x \geq -1$, $(1 + x)^r \geq 1 + xr$.

  Exercise: Deduce this from convexity of the function $(1 + x)^r$.

- **Titu**

  Let $x_1,\ x_2,\ \ldots,\ x_n$ be nonnegative reals which sum to 1. Then

  $$\prod_{i=1}^{n} \frac{1 - x_i}{x_i} \geq (n-1)^n.$$

  Exercise: Prove this by creative application of AM-GM.

  This inequality sometimes takes other forms, such as the following: Let $x_1,\ x_2,\ \ldots,\ x_n$ $(n \geq 2)$ be positive reals satisfying

  $$\frac{1}{t + x_1} + \frac{1}{t + x_2} + \cdots + \frac{1}{t + x_n} = \frac{1}{t}$$

  where t is a positive real number; then

  $$\sqrt[n]{x_1 x_2 \cdots x_n} \geq (n-1)t.$$