CALCULUS II

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Lecture 2

Improper Integrals of the Second Kind

In this lecture we will study Riemann integrals over finite intervals $[a, b]$ of functions which are not bounded on those intervals. The definition of such integrals bears some resemblance to the definition of the improper integrals of the first kind since the integrals are defined as *limits of integrals* of continuous functions over finite intervals. The main difference is that in the case of improper integrals of the first kind we have an infinite range of integration, whereas in the case of improper integrals of the second kind we have a finite range of integration, only the integrated function f is not bounded on $[a, b]$.

Definition 1. Suppose that a function $f \in C(a, b]$, where a and b are real numbers and that f is not bounded on $[a, b]$, i.e. $f(x)$ is not bounded when $x \to a^+$. By an *improper integral of the second kind* of the function f over the interval $(a, b]$ we understand the limit

$$\int_a^b f(x)dx := \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^b f(x)dx$$

if it exists. If this limit is finite, we say that the integral *converges*. If this limit is equal to ∞ or $-\infty$, we say that the integral *diverges to* ∞, or $-\infty$, respectively. If this limit does not exist, we say that the integral *diverges*.

Remarks. In an analogous fashion we define an improper integral of the second kind of a function $f \in C[a, b)$ which is not bounded on $[a, b]$, namely

$$\int_a^b f(x)dx := \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)dx$$

if it exists. Note that the continuity assumption ensures existence of the integrals $\int_{a+\epsilon}^b f(x)dx$ and $\int_a^{b+\epsilon} f(x)dx$ for all $\epsilon > 0$ in the two cases considered above, respectively. As in the case of improper integrals of the first kind, we can also integrate functions which are not continuous on $(a, b]$ or $[a, b)$, but we will restrict our attention to the continuous ones.

If an improper integral of the second kind I converges, diverges to ∞ or diverges to $-\infty$, we will write $I < \infty$, $I = \infty$, or $I = -\infty$, respectively.

Using the definition of the improper integral of the second kind, check if the integrals given below converge, diverge to ∞, $-\infty$, or diverge. If possible, evaluate the integrals.

EXAMPLE 1. First consider a function which is unbounded in the left neighborhood of 1, thus

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x}} &= \lim_{\epsilon\to 0^+}\int_0^{1-\epsilon}\frac{dx}{\sqrt{1-x}}\\ &= \lim_{\epsilon\to 0^+}\left(-2\sqrt{1-x}\right)_{x=0}^{x=1-\epsilon}\\ &= \lim_{\epsilon\to 0^+}(-2\sqrt{\epsilon}+2\sqrt{1})\\ &= 2\end{aligned}$$

and the investigated integral converges.

EXAMPLE 2. As a second example, let us integrate over $(\pi/2,\pi]$ a function which is unbounded in the right neighborhood of $\pi/2$. We get

$$\begin{aligned}\int_{\pi/2}^{\pi}\frac{dx}{\cos^2 x} &= \lim_{\epsilon\to 0^+}\int_{\pi/2+\epsilon}^{\pi}\frac{dx}{\cos^2 x}\\ &= \lim_{\epsilon\to 0^+}(\mathrm{tg}\pi-\mathrm{tg}(\pi/2+\epsilon))\\ &= \infty\end{aligned}$$

and thus our integral diverges to ∞.

EXAMPLE 3. Examples of functions with divergent integrals can also be given. For instance,

$$\begin{aligned}\int_0^{2/\pi}\frac{1}{x^2}\cos\frac{1}{x}\,dx &= \lim_{\epsilon\to 0^+}\int_{\epsilon}^{2/\pi}\frac{1}{x^2}\cos\frac{1}{x}\,dx\\ &= \lim_{\epsilon\to 0^+}\left(-\sin\frac{1}{x}\right)_{x=\epsilon}^{x=2/\pi}\\ &= \lim_{\epsilon\to 0^+}(-1+\sin\frac{1}{\epsilon})\end{aligned}$$

but the limit of the expression on the right-hand side doesn't exist, hence the integral diverges.

EXAMPLE 4. Let us consider an important class of integrals of the form

$$\int_0^b \frac{dx}{x^p}$$

where $b>0$ and $p\in\mathbf{R}$. One can show that the above integrals converge for $p<1$ and diverge to ∞ for $p\geq 1$. The proof is left to the reader. The result should be compared with Example 3 of Chapter 1, where the results were quite different. The reason is clear: in Example 3 of Chapter 1, the integrated function has to decrease fast enough at ∞ for the integral to converge, whereas here the function cannot grow too fast in the neighborhood of 0 if the integral is to converge. The difference is profound, but people often confuse the two cases.

EXAMPLE 5. Using this fact and easy substitutions we obtain convergence or divergence to ∞ of seemingly more complicated integrals. For instance,

$$\int_{-1/2}^{1} \frac{dx}{(2x+1)^{3/2}} = \infty, \quad \int_{-2}^{1} \frac{dx}{(x+2)^{2/3}} < \infty$$

(the details are left to the reader).

We often encounter integrals over intervals $[a,b]$ of functions f which are not bounded in the neighborhood of $c \in (a,b)$, i.e. somewhere inside the interval (a,b) and not near the end-points of $[a,b]$. If there is one such point c, we define

$$\int_a^b f(x)dx := \int_a^c f(x)dx + \int_c^b f(x)dx$$

and thus the integral I on the left-hand side is expressed in terms of two integrals I_1 and I_2, of which at least one is an improper integral of the second kind. Such an integral is said to *converge* if both I_1 and I_2 converge. It is said to *diverge to* ∞ if I_1 diverges to ∞ and I_2 converges or diverges to ∞ or vice versa. Similar conditions can be formulated for divergence to $-\infty$. In the remaining cases, the integral is said to diverge.

By spliting up the range of integration in the appropriate way, one can define Riemann integrals of f over intervals $[a,b]$, which contain more "singular points" of f in $[a,b]$. In particular, if f is unbounded in the right neighborhood of a and in the left neighborhood of b, we set

$$\int_a^b f(x)dx := \int_a^c f(x)dx + \int_c^b f(x)dx$$

for any $c \in (a,b)$ (here, we assume that there are no "singular points" in (a,b)).

EXAMPLE 6. The two integrals given below provide good examples:

$$\int_{-1}^{1} \frac{dx}{x^{2/3}}, \quad \int_0^{\pi} \frac{dx}{\sin^2 x}$$

(in the first integral, the integrated function has a "singularity" near 0, whereas the second integral has "singularities" near 0 and π). The first integral can be easily shown to converge by splitting up the range of integration into $[-1,0)$ and $(0,1]$ and using Example 4. In turn,

$$\begin{aligned} \int_0^{\pi} \frac{dx}{\sin^2 x} &= \int_0^{\pi/2} \frac{dx}{\sin^2 x} + \int_{\pi/2}^{\pi} \frac{dx}{\sin^2 x} \\ &= \lim_{\epsilon \to 0^+} (-\mathrm{ctg}\pi/2 + \mathrm{ctg}\epsilon) + \lim_{\epsilon \to 0^+} (-\mathrm{ctg}(\pi - \epsilon) + \mathrm{ctg}\pi/2) \\ &= \infty + \infty = \infty \end{aligned}$$

and the integral diverges to ∞.

Again, as in the case of improper integrals of the first kind, we will study two simple convergence tests: the comparison test and the limit ration test. They are very similar to their counterparts studied in Chapter 1. It is important to notice that they refer to integrals which converge, diverge to ∞ or diverge to $-\infty$, i.e. they do not treat the (rather rare) cases of divergent integrals. This is because they deal with functions which do not change signs in the range of integration. It follows from the definition of Riemann integral that if a function is positive on the whole interval or negative on the whole interval, then the improper integral of that function converges or diverges to ∞ or $-\infty$, but it cannot diverge, i.e. the limit which defines the integral has to exist. The only "chance" for an integral to diverge is when the integrated function is not everywhere of the same sign.

THEOREM 1. (COMPARISON TEST). *Suppose that* $0 \le f(x) \le g(x)$ *for all* $x \in (a,b]$, *where* a, b *are (finite) real numbers and* f *and* g *are continuous on* $(a,b]$. *Then*

(a) *if* $\int_a^b g(x)dx$ *converges then* $\int_a^b f(x)dx$ *converges*

(b) *if* $\int_a^b f(x)dx$ *diverges to* ∞, *then* $\int_a^b g(x)dx$ *diverges to* ∞.

The intuitive sense of this test is identical to the case of improper integrals of the first kind (see Theorem 1, Chapter 1). A similar test can be phrased for integrals over the interval $[a,b)$. Clearly, the above test concerns mainly improper integrals of the second kind although we didn't say that the functions f and g are unbounded on $[a,b]$ (if they were bounded, the theorem still holds although is trivial).

EXAMPLE 7. Using the comparison test, check if the integrals given below converge, diverge to ∞ or $-\infty$:

$$\int_0^1 \frac{x+1}{\sin^2 x}dx, \quad \int_0^{\pi/2} \frac{\sin x dx}{(2x-\pi)^{1/3}}$$

Let us solve the first example. For $x \in (0,1]$ we have

$$g(x) = \frac{x+1}{\sin^2 x} > \frac{1}{\sin^2 x} = f(x) \ge 0$$

and

$$\int_0^1 \frac{dx}{\sin^2 x} = \infty$$

hence, by the comparison test (case (b)), our integral also diverges to ∞.

THEOREM 2. (LIMIT RATIO TEST) *Suppose that* f *and* g *are both positive (or, both negative) continuous functions on* $(a,b]$ *and that*

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = K$$

where $0 < K < \infty$*. Then*

(a) $\int_a^b f(x)dx$ *converges if and only if* $\int_a^b g(x)dx$ converges

(b) $\int_a^b f(x)dx$ *diverges to* ∞ *if and only if* $\int_a^b g(x)dx$ *diverges to* ∞*.*

(c) $\int_a^b f(x)dx$ *diverges to* $-\infty$ *if and only if* $\int_a^b g(x)dx$ *diverges to* $-\infty$*.*

A similar test holds for improper integrals of the secodn kind over $[a, b)$*.*

Again, as in the case of improper integrals of the first kind, it is very important that $K \neq 0$ and $K \neq \infty$ simply because if $K = 0$ or $K = \infty$, the test does not hold in general.

EXAMPLE 8. Check for convergence the integrals

$$\int_0^{\pi/2} \frac{dx}{(\sin x)^{1/3}}, \quad \int_0^1 \frac{dx}{\ln(1+x)}$$

Let us treat the second example and let

$$f(x) = \frac{1}{\ln(1+x)}, \quad g(x) = \frac{1}{x}$$

Then

$$\frac{f(x)}{g(x)} = \frac{x}{\ln(1+x)} \to 1$$

as $x \to 0^+$. Both functions f and g are positive on the interval $(0, 1]$. Therefore, by the limit ratio test (case (b)) our integral diverges to ∞ since the integral $\int_0^1 dx/x = \infty$ ($p = 1$, see Example 4).

Up till now we have mainly dealt with functions which were positive or negative on the whole interval (Example 3 was the only exception). If, as in Example 3, we can calculate the integral explicitly, then there is no immediate need to look for easy ways to check whether it converges or not. However, very often we encounter integrals which cannot be easily calculated. Then, we would like to have at least some convergence conditions. In order to give the most typical sufficient convergence condition, let us introduce the following definition.

DEFINITION 2. We say that an improper integral of the first or second kind $\int_a^b f(x)dx$, where $-\infty \leq a < b \leq \infty$, *converges absolutely* if the integral $\int_a^b |f(x)|dx$ converges.

THEOREM 3. (SUFFICIENT CONVERGENCE CONDITION) If an improper integral $\int_a^b f(x)dx$ of Definition 2 converges absolutely, then it converges.

Notice that this theorem only says that absolute convergence is stronger than convergence. In general, if an integral converges, it doesn't have to converge absolutely.

But if it doesn't converge, then it certainly doesn't converge absolutely.

EAXMPLE 9. The integral

$$\int_0^{\infty} \frac{|\sin x|}{x^2+1} dx$$

converges by the comparison test since

$$\frac{|\sin x|}{x^2+1} \leq \frac{1}{x^2+1}$$

and $\int_0^{\infty} dx/(x^2+1) = \pi/2 < \infty$. Therefore, the integral

$$\int_0^{\infty} \frac{\sin x}{x^2+1} dx$$

converges absolutely.

EXAMPLE 10. We can easily see that the integral $\int_0^{\infty} |\sin x| dx$ does not converge (it diverges to ∞) since $\int_0^{\infty} \sin x dx$ is not convergent.