# CALCULUS II

*Romuald Lenczewski* [1]
*Department of Mathematics*
*Wroclaw University of Sience and Technology*

## Lecture 5

## Power series

Up till now we have dealt with number series, i.e. each series was a sum of infinitely many numbers. It is easy to think of a more general case and study series of the form $\sum_{n=n_0}^{\infty} f_n(x)$, in other words, series of functions. For each fixed $x$ such a series is defined as the usual number series. However, it is not hard to imagine that it is practically impossible to handle such a series for all possible values of $x$ separately. One needs some ways to study convergence of such series at least on some subintervals of $\mathbf{R}$. In general, it is not an easy task and quite sophisticated techniques are needed to study convergence of such series. We will study the simple case of power series when the problem of convergence simplifies considerably.

Definition 1. A *power series* is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

where $a_n \in \mathbf{R}$ for all $n \geq 0$ and $x_0 \in \mathbf{R}$. For those values of $x$ for which we get a convergent series, we can define a function

$$f: \quad x \to \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

It turns out that the domain of such a function is always a (possibly infinite or trivial, i.e. consisting of one point) subinterval of $\mathbf{R}$. The theorem given below tells us more precisely what happens.

Theorem 1. *Let $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ be a given power series. Then there are three possibilities:*

(1) *there exists $0 < R < \infty$ such that the given power series converges for $|x-x_0| < R$*
(2) *the series converges only for $x = x_0$*
(3) *the series converges for all $x \in \mathbf{R}$.*

*The number $R$ is called the radius of convergence and is given by the formula*

$$R = \frac{1}{\limsup_{n\to\infty} (a_n)^{1/n}}.$$

*Moreover, if* $\limsup_{n\to\infty}(a_n)^{1/n} = 0$*, then the series converges for all* $x \in \mathbf{R}$ *and we say that the radius of convergence is infinite and we write* $R = \infty$*. In turn, if* $\limsup_{n\to\infty}(a_n)^{1/n} = \infty$*, then the series converges only for* $x = x_0$ *and we write* $R = 0$*.*

*Remarks.* [1]. We have here some mysterious limit, denoted

$$\limsup_{n\to\infty} b_n$$

for some sequence $(b_n)$. It is a kind of 'upper limit' which is very useful in calculus. What do we mean by an 'upper limit'? For instance, if $(b_n) = (-1)^n$, then we know that $\lim_{n\to\infty} b_n$ does not exist since $(b_n)$ has two constant subsequences, $(b_{2k})$ and $(b_{2k-1})$, where $b_{2k} = 1$ and $b_{2k-1} = -1$ for any $k$. Of course, these subsequences have limits: $\lim_{k\to\infty}(b_{2k}) = 1$ and $\lim_{k\to\infty}(b_{2k-1}) = -1$. Let us define $\limsup_{n\to\infty}(b_n) = \sup S$, where $S$ is the set of limit points of $(b_n)$, that is the set of limits of subsequences of $(b_n)$. We should probably still explain here what $\sup S$ really is. It is 'almost' the maximum of $S$. If the maximum exists, it is the maximum. If it doesn't exist, $\sup S$ is the smallest number among those which are bigger or equal to all numbers in the set $S$. For instance,

$$\sup([0,1)) = 1, \quad \sup([0,1]) = 1, \quad \sup(\{1-1/2, 1-1/3, 1-1/4, \ldots\}) = 1$$

Finally, we obtain in our example

$$\limsup_{n\to\infty}(-1)^n = 1.$$

In all our exercises, we will have a finite number of limit points and in that case computation of the 'upper limit' is not hard.

[2]. In the most general case we need to use $\limsup_{n\to\infty}(a_n)^{1/n}$ to calculate $R$. However, the good news is that in many applications, it is enough to use

$$R = \frac{1}{\lim_{n\to\infty}(a_n)^{1/n}} \quad \text{or} \quad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

when the limits on the right-hand-side exist.

[4]. Unfortunately, the theorem doesn't say what happens for $x = x_0 - R$ and $x = x_0 + R$ and convergence for such $x$ has to be treated separately.

[5]. The set of all $x$ for which the power series converges is usually called the *region of convergence* (RC). Apart from the worst case, when the region of convergence is a one-point set $\{x_0\}$, and the best case when it is given by all real numbers, there are the following four possibilities for RC:

$$(x_0 - R, x_0 + R), \quad [x_0 - R, x_0 + R), \quad (x_0 - R, x_0 + R], \quad [x_0 - R, x_0 + R]$$

Example 1. The series

$$\sum_{n=0}^{\infty}(n+1)x^n$$

has the radius of convergence $R = 1$. For $x = 1$ the series diverges to $\infty$ and for $x = -1$ it diverges (in both cases the necessary convergence condition is not satisfied). Therefore, $RC = (-1, 1)$.

EXAMPLE 2. The series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^{1/n}}$$

has the radius of convergence $R = 1$. For $x = 3$ it diverges to $\infty$ and for $x = 1$ it diverges (in both cases the necessary convergence condition is not satisfied). Thus, $RC = (1, 3)$.

EXAMPLE 3. The series

$$\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

has the radius of convergence $R = \infty$, hence $RC = (-\infty, \infty)$.

EXAMPLE 4. Consider the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n \ln^2 n}$$

We have

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{n+1}{n} \lim_{n\to\infty} \frac{\ln^2(n+1)}{\ln^2 n} = 1$$

(de L'Hospital's rule can be used to show that). Hence $R = 1$. Note now that the series converges for $x = 1$ and converges for $x = -1$ (by absolute convergence). Hence, $RC = [-1, 1]$.

EXAMPLE 5. In a similar way, we show that the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{n \ln n}$$

has radius of convergence $R = 1$. If $x = -1$, then the series diverges to $\infty$. If $x = 1$, then the series converges by Leibniz's theorem. Thus $RC = (-1, 1]$.

It is interesting that functions which are regular enough can be expanded in power series. Such expansions are sometimes valid on the whole domains of the functions, but it is usually not the case. The easiest example is given by the geometric series, which is a power series expansion of the function $f(x) = 1/(1-x)$ around $x_0 = 0$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

which is valid for $x \in (-1, 1)$.

Under suitable assumptions such series are obtained from Taylor's theorem. Recall that Taylor's theorem allows us to represent locally a suitably regular function as a polynomial plus a remainder. Namely, if $f$ and all its derivatives up to and including order $n$ are continuous on the interval $(x_0 - r, x_0 + r) \subset D_f$, where $r > 0$, then there exists $c \in (x, x_0)$ (if $x < x_0$) or $c \in (x_0, x)$ (if $x_0 < x$) such that

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n)}(c)}{n!}(x - x_0)^n$$

If we let $n \to \infty$ and the remainder converges to zero, then we can hope that we can represent a function as a power series and the remainder will disappear.

THEOREM 2. (TAYLOR SERIES EXPANSION) *Suppose that $f : D_f \to \mathbf{R}$ and all its derivatives are continuous on the interval $\mathcal{O} = (x_0 - r, x_0 + r) \subset D_f$ and that $\lim_{n\to\infty} R_n(x) = 0$ for $x \in \mathcal{O}$. Then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

*for all $x \in \mathcal{O}$. The series is called Taylor's series of $f$ in the neighborhood of $x_0$.*

EXAMPLE 6. Let us show that for all $x \in \mathbf{R}$ we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Note that on each $\mathcal{O} = (-r, r)$, $r > 0$, we have

$$R_n(x) = \frac{e^c}{n!}x^n \to 0$$

on $\mathcal{O}$. Therefore, since $f^{(n)}(0) = e^0 = 1$, we get the desired expansion on each $\mathcal{O}$. Since $r$ was in this case arbitrary, the expansion is valid for all $x \in \mathbf{R}$. Note that the radius of convergence of the series is $R = \infty$.

EXAMPLE 7. In a similar way one can show the following power series expansions:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}, \quad x \in \mathbf{R}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}, \quad x \in \mathbf{R}$$

$$\ln(1 + x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}x^{n+1}, \quad x \in (-1, 1]$$

It is important that expansions in power series are unique since it is sometimes convenient to derive a power series expansion of a function without using the Taylor series expansion. First, let us state the uniqueness result.

THEOREM 3. *If*

$$f(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

*for all* $x \in (x_0 - r, x_0 + r) \subset D_f$, *then* $b_n = f^{(n)}(x_0)/n!$.

EXAMPLE 8. For the function $f(x) = 1/(1 + x^3)$ we will calculate $f^{(12)}(0)$. We have for $x_0 = 0$ the following expansion

$$\frac{1}{1 + x^3} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

for $-1 < x < 1$. Hence

$$(-1)^4 = b_{12} = \frac{f^{(12)}(0)}{12!}$$

and thus $f^{(12)}(0) = 12!$.

EXAMPLE 9. Let us expand $f(x) = \ln(1+x^2)$ in the power series in the neighborhood of $x_0 = 0$. Note that it is not easy to apply Theorem 2 directly. But, using the expansion for $\ln(1 + y)$ around $y_0 = 0$ and setting $y = x^2$ gives

$$\ln(1 + x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + 1)} x^{2n+2}$$

which gives Taylor's series of $f$ by Theorem 3. It is easy to check that the radius of convergence is $R = 1$.

It turns out that power series are nice enough to be differentiated and integrated term-by-term inside (i.e. in the open neighborhoods) their regions of convergence. That is why we have the theorems given below.

THEOREM 4. *Suppose that the series* $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ *has a nonzero radius of convergence* $R$. *Then the series can be differentiated term-by-term, namely*

$$\frac{d}{dx} \left( \sum_{n=0}^{\infty} a_n (x - x_0)^n \right) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

*for all* $x \in (x_0 - R, x_0 + R)$.

THEOREM 5. *Suppose that the series* $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ *has a nonzero radius of convergence* $R$. *Then the series can be integrated term-by-term, namely*

$$\int_{x_0}^{x} \sum_{n=0}^{\infty} a_n (t - x_0)^n dt = \sum_{n=0}^{\infty} \frac{a_n}{n + 1} (x - x_0)^{n+1}$$

*for all* $x \in (x_0 - R, x_0 + R)$.

The above theorems say that one can interchange limits, namely the derivative with taking the limit of partial sums, or the limit of Riemann sums with the limit of partial sums. In general, it is a very subtle point and one has to be careful when doing that. The theorems given above enable us to do it for power series. So also in that respect power series are “nice” series. Some applications are given below.

EXAMPLE 10. Let us derive the Taylor series of $f(x) = \mathrm{arctg} x$ around $x_0 = 0$. Doing this directly from Theorem 2 is very cumbersome, but by using Theorem 4 we can get the result easily. Namely, note that $f'(x) = 1/(1+x^2)$ and derive the power series expansion for $f'$. We have

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

for $t \in (-1, 1)$. Let us integrate this series term-by-term. We get

$$\mathrm{arctg} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n} dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

EXAMPLE 11. Evaluate

$$\sum_{n=0}^{\infty} \frac{2n+1}{4^n}, \quad \sum_{n=1}^{\infty} \frac{n+2}{n5^n}$$

using Theorems 3-4 (exercise).