CALCULUS II

Romuald Lenczewski [1]
Department of Mathematics
Wroclaw University of Science and Technology

Lecture 6

Functions of several variables

In this lecture we begin our study of functions of several variables. These are natural generalizations of functions of one variable $f(x)$. For instance, in the case of two variables, say x, y, we have $f(x, y)$, in the case of three variables, say x, y, z, we get $f(x, y, z)$. More generally, we can consider functions $f(x_1, \ldots, x_n)$ of n variables $x_1, \ldots, x_n$. It should be obvious that if we can develop a theory for functions of two variables, we can then extend it to more variables. Therefore, we will concentrate on functions of two variables. Most definitions and theorems can be easily generalized to the case of more variables.

As compared with $\mathbf{R}$, where we use open and closed intervals, we introduce open and closed disks. Namely, by an *open disk* of radius $r > 0$, centered at (x_0, y_0), we understand the set

$$\mathcal{O}((x_0, y_0), r) = \{(x, y) : (x - x_0)^2 + (y - y_0)^2 < r^2\},$$

and by a *closed disk* of radius $r > 0$, centered at (x_0, y_0), we understand the set

$$\bar{\mathcal{O}}((x_0, y_0), r) = \{(x, y) : (x - x_0)^2 + (y - y_0)^2 \leq r^2\}$$

(the only difference is that the first one does not contain the "boundary"). Open disks are used to define open sets.

Definition 1. A subset A of $\mathbf{R}^2$ is *open* if for each $(x_0, y_0) \in A$ there exists an *open disk* of radius $r > 0$, centered at (x_0, y_0), namely $\mathcal{O}((x_0, y_0), r)$ such that $\mathcal{O}((x_0, y_0), r) \subset A$. The *interior* of A, denoted $\text{Int}A$, is the set of all points $(x_0, y_0) \in A$ such that there exists $\mathcal{O}((x_0, y_0), r) \subset A$. The *boundary* of A, denoted $\text{Bd}A$ is the set of all points $(x_0, y_0) \in \mathbf{R}^2$ such that every open disk $\mathcal{O}((x_0, y_0), r)$ for any $r > 0$ contains points which belong to A and points which do not belong to A (i.e. belong to A^c, the complement of A). Finally, the set A is *closed* if it contains its boundary.

Example 1. Let us consider several subsets of the plane $\mathbf{R}^2$. The reader is advised to check the details.

(a) any straight line in the plane is closed, is equal to its boundary, and has empty interior,

(b) the open disk $\mathcal{O}((x_0, y_0), r)$ is an open set, is equal to its interior and has empty boundary,

(c) the closed disk $\bar{\mathcal{O}}((x_0, y_0), r)$ is a closed set, its boundary is given by the circle

$$\{(x, y) : (x - x_0)^2 + (y - y_0)^2 = r^2\}$$

and its interior is given by the open disk $\mathcal{O}((x_0, y_0), r)$.

(d) the plane $\mathbf{R}^2$ is both open and closed in $\mathbf{R}^2$, is equal to its interior and has empty boundary.

It should be clear that there exist sets which are neither open nor closed. For example, the union of any open disk and any straight line is not open. The reader is advised to check the details.

For some purposes, it is enough to use open sets. However, it is sometimes necessary to use sets which have stronger properties. This leads to the definition of a region.

Definition 2. A subset A of $\mathbf{R}^2$ is called a *region*, if it is open in $\mathbf{R}^2$ and any two points of that set can be connected with a broken line contained in the set A. By a *closed region* we understand the union of a region and its boundary.

Definition 3. Any mapping $f: D_f \to \mathbf{R}$ such that $(x, y) \to f(x, y)$, where $D_f \subset \mathbf{R}^2$, is called a *function of 2 variables* and the set D_f is called its *domain*.

Usually, we assume that D_f is the maximal subset of $\mathbf{R}^2$ on which the rule given by f is defined and then we say that D_f is the *natural domain* of f. From now on, when speaking of a domain of a function f, we will assume that it is the natural domain of f unless we state otherwise.

Example 2. Of course, it is easy to give examples of functions of two variables. For instance,

$$f(x, y) = \ln(x^2 + y^2)$$

is a function of two variables x, y with the domain $D_f = \mathbf{R}^2 \setminus \{(0, 0)\}$. In turn,

$$g(x, y) = \frac{1}{xy}$$

is a function of two variables x, y with the domain $D_g = \{(x, y) : x \neq 0, y \neq 0\}$, i.e. the plane $\mathbf{R}^2$ without the x and y axes.

Definition 4. The *graph* of a function $f(x, y)$ consists of all points $(x, y, z) \in \mathbf{R}^3$ such that $(x, y) \in D_f$ and $z = f(x, y)$.

Thus, in order to draw the graph of a function of two variables, we need three-dimensional pictures. One of the ways to draw the graph of such a function is to use the so-called level curves. By a *level curve of f with value c* we understand the set of all $(x, y) \in \mathbf{R}^2$ such that $f(x, y) = c$, where c is a real number. By drawing several level curves on the plane $\mathbf{R}^2$ and then lifting them up to the values $z = c$, we get several lines which belong to the graph of f. This helps us to imagine how the graph of the function looks like.

EXAMPLE 3. Draw several level curves for the function $f(x, y) = a(x^2 + y^2)$ for $a = 1$. Then, in a three-dimensional coordinate system, lift such level curves to the height $z = c$ and sketch the graph of the function. Such functions (for any $a \neq 0$) are so-called *paraboloids of revolution*. Note that one can obtain their graphs by rotating the parabola $z = ax^2$ around the z axis (that's where the name of the surface comes from).

Let us list some other important examples:

(a) $f(x, y) = ax + by + c$ – a *plane*, with normal vector $(-a, -b, 1)$, passing through $P = (0, 0, c)$,

(b) $f(x, y) = a(x^2 - y^2)$ – a *hyperbolic paraboloid* (it looks like a "saddle"),

(c) $f(x, y) = \sqrt{R^2 - (x^2 + y^2)}$ – an *upper hemisphere* of radius R (with a minus in front of the square root we get a *lower hemisphere* of radius R),

(d) $f(x, y) = k\sqrt{x^2 + y^2}$, where $k \neq 0$, is a *conic surface*,

(e) $f(x, y) = ax^2 + b$, where $a, b \in \mathbf{R}$ and $a \neq 0$, is a *parabolic cylinder*,

EXAMPLE 4. Quite often we have to deal with "shifted" surfaces. For instance,

$$f(x, y) = 1 - \sqrt{2x - x^2 + 4y - y^2} = 1 - \sqrt{5 - ((x - 1)^2 + (y - 2)^2)}$$

is an example of a "shifted" lower hemisphere of radius $R = \sqrt{5}$ and the origin $(1, 2, 1)$.

In order to define for functions of several variables such notions as limits and continuity, we need to define the limit of a two-dimensional sequence.

DEFINITION 5. A *two-dimensional* sequence is a mapping $g : \mathbf{N} \to \mathbf{R}^2$ such that $n \to g(n) = (x_n, y_n)$. The limit of such a sequence is given by

$$\lim_{n\to\infty} (x_n, y_n) = (\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n)$$

EXAMPLE 5. As Definition 4 shows, the computation the limit of a two-dimensional sequence reduces to the computation of the limits of the two corresponding one-dimensional sequences. For instance,

$$\lim_{n\to\infty} (n^{1/n}, \frac{\sin n}{n}) = (1, 0)$$

We choose to define the limit of a function of two variables using sequences. This sequential definition is equivalent to the (ϵ, δ)– definition, which is more elegant from the mathematical point of view, but is more abstract and therefore will be omitted.

Definition 6. Suppose f is defined on $\mathcal{S} = \mathcal{O}((x_0, y_0), r) \setminus \{(x_0, y_0)\}$, i.e. on some open disk about (x_0, y_0), except perhaps at (x_0, y_0). Then we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = g$$

where $-\infty \leq g \leq \infty$ if for every sequence $(x_n, y_n) \in S$ such that $(x_n, y_n) \neq (x_0, y_0)$ and $\lim_{n\to\infty}(x_n, y_n) = (x_0, y_0)$ we have

$$\lim_{n\to\infty} f(x_n, y_n) = g$$

Remarks. [1]. In the above definition, it is very important that we need to get the same limit g *for every* sequence (x_n, y_n) converging to (x_0, y_0) and not hitting (x_0, y_0) (at that point the function f does not have to be even defined). Therefore, we cannot take our "favorite" sequence to show that the limit exists.

[2]. Rules for computing limits of sums, products and quotients are the same as in the case of functions of one variable, which can be proved using Definition 5.

Example 6. Using the definition of the limit, show that

$$\lim_{(x,y)\to(1,2)} \frac{2x_n - y_n}{x_n^2 + y_n^2} = 0$$

Let $(x_n, y_n) \to (0,0)$, $(x_n, y_n) \neq (0,0)$. Then

$$\lim_{n\to\infty} \frac{2x_n - y_n}{x_n^2 + y_n^2} = \frac{2\lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n}{(\lim_{n\to\infty} x_n)^2 + (\lim_{n\to\infty} y_n)^2} = \frac{2\cdot 1 - 2}{1^2 + 2^2} = 0$$

Of course, we used Remark 2, but note that we kept the needed generality from the beginning to the end, i.e. we did not choose for (x_n, y_n) any specific example. A common mistake in such a problem would be to take $(x_n, y_n) = (1/n, 1/n)$, for instance, do the calculations for this sequence, and claim that the limit is zero.

Example 7. Show that this limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

Here, the situation is different. We want to show that the limit *doesn't exist*. For that purpose, it is enough to find two sequences, (x_n, y_n) and (x'_n, y'_n), both converging to (x_0, y_0), not hitting (x_0, y_0) and such that

$$\lim_{n\to\infty} f(x_n, y_n) = g \neq g' = \lim_{n\to\infty} f(x'_n, y'_n)$$

One has to choose these two sequences in a clever way. The first can be arbitrary. Of course, we want a sequence, for which we can easily calculate the limit or show that the limit does not exist (in the second case, we do not even need to look for the second sequence, so it seems nice but it is not always possible to find a sequence for which the limit does not exist). Let us take $(x_n, y_n) = (1/n, 0)$. Then $(x_n, y_n) \to (0,0)$ and $(x_n, y_n) \neq (0,0)$ for all $n \in \mathbf{N}$. We obtain

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} \frac{1/n \cdot 0}{1/n^2 + 0} = \lim_{n\to\infty} \frac{0}{1/n^2} = 0$$

Let us look for another seqeunce which will give a different limit. For instance, take $(x'_n, y'_n) = (1/n, 1/n)$. Clearly, $(x'_n, y'_n) \to (0,0)$ and $(x'_n, y'_n) \neq (0,0)$ for all $n \in \mathbf{N}$. We obtain

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} \frac{1/n \cdot 1/n}{1/n^2 + 1/n^2} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

and since $0 \neq 1/2$, we conclude that the investigated limit does not exist.

EXAMPLE 8. In general, it is more difficult to calculate the limit which exists than to show that a limit doesn't exist, unless it is some limit to which the usual rules can be applied like in Example 6. In the first case, cancellations are welcome as the following example shows:

$$\lim_{(x,y)\to(1,1)} \frac{x^3 - y^3}{y - x} = \lim_{(x,y)\to(1,1)} (-x^2 - xy - y^2) = -1 + 1 - 1 = -1$$

EXAMPLE 9. It is sometimes convenient to use the Three-function Theorem which holds for multidimensional limits, too. For instance, to show that

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2) \cos \frac{1}{xy} = 0$$

it is enough to use the inequality

$$0 \leq |(x^2 + y^2) \cos \frac{1}{xy}| \leq x^2 + y^2$$

and notice that the first and the third functions tend to zero as $(x, y) \to (0,0)$ and thus the middle one does, from which we infer that the investigated limit is also equal to zero.