CALCULUS II

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Lecture 7

Continuity and first partial derivatives

In this lecture we are going to study continuity of functions of several variables and the so-called first partial derivatives. We begin with the definition of continuity of $f(x,y)$ at $(x_0, y_0) \in D_f$, which is similar to the analogous definition of the continuity in the one-dimensional case.

Definition 1. Suppose that $f(x,y)$ is defined in $\mathcal{O}((x_0,y_0),r)$ or some $r>0$. Then $f(x,y)$ is *continuous at* (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Theorem 1. *The sum, product, quotient and composition of continuous functions are continuous on their domains.*

The above theorem follows easily from Definition 1 and appropriate theorems on the 2-dimensional limits. One has to remember that the statement of the theorem holds on appropriate domains. For instance, the quotient f/g of two continuous functions f, g is continuous at every point $(x_0,y_0) \in D_{f/g}$ but note that $D_{f/g}$ is the subset of $D_f \cap D_g$ where $g(x_0,y_0) \neq 0$. A similar restriction applies to the composition of two functions.

Example 1. We will show that the function

$$f(x,y) = \begin{cases} \frac{\sin(xy)}{xy} & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$

is continuous everywhere.

First, let us notice that i $x_0y_0 \neq 0$, then there exists an open neighborhood about (x_0y_0) in which $f(x,y) = \sin(xy)/(xy)$ and therefore is a continuous function as a quotient of two continuous functions. Therefore, let us assume that $x_0y_0 = 0$. Without loss of generality (the reader is advised to justify that) we can consider two cases: (a) $(x,y) \to (x_0,y_0)$ and $xy \neq 0$ for all x, y, (b) $(x,y)\to(x_0,y_0)$ and $xy = 0$ for all x, y. In case (a), we get

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} \frac{\sin(xy)}{xy} = \lim_{z\to 0}\frac{\sin z}{z} = 1 = f(x_0,y_0)$$

In case (b), we get

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} 1 = 1 = f(x_0,y_0)$$

Therefore, the function f is continuous at every $(x_0,y_0) \in \mathbf{R}^2$.

EXAMPLE 2. Let us find the set of all points at which the function

$$f(x,y) = \begin{cases} x+y & x>0 \\ \sqrt{x^2+y^2} & x \le 0 \end{cases}$$

is continuous.

If $x_0 \neq 0$, then the continuity at (x_0,y_0) for any y_0 follows from the continuity of the functions $(x,y) \to x+y$ and $(x,y) \to \sqrt{x^2+y^2}$ (cf. Example 1). Now, let us assume that $x_0 = 0$. Again, without loss of generality, as in Example 1, it is enough to compute the limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ for (x,y) such that $x>0$ and (x,y) such that $x \le 0$ separately (the reader is advised to justify that). If $x>0$ (i.e. we approach the point (x_0,y_0) from the right open half-plane), we get

$$\lim_{(x,y)\to(0,y_0)} f(x,y) = \lim_{(x,y)\to(0,y_0)} (x+y) = 0 + y_0 = y_0.$$

In turn, if $x \le 0$ (i.e. we approach the point (x_0,y_0) from the left closed half-plane), we get

$$\lim_{(x,y)\to(0,y_0)} f(x,y) = \lim_{(x,y)\to(0,y_0)} \sqrt{x^2+y^2} = \sqrt{y_0^2} = |y_0|.$$

Now, we have $f(0,y_0) = |y_0|$, so we can see that both limits computed above are equal to $|y_0|$ only for $y_0 \ge 0$. Therefore, the function is continuous at every point which does not lie on the negative y-axis.

Let us turn to the definition of the first partial derivatives. The first partial derivative with respect to x, denoted f_x or $\partial f/\partial x$ is the rate of change of $f(x,y)$ with y fixed (treated as a constant). In turn, the first partial derivative with respect to y, denoted f_y or $\partial f/\partial y$ is the rate of change of $f(x,y)$ with x fixed. This is usually considered at some point (x_0,y_0) where the first partial derivatives can be defined, unless they are treated as functions of x and y. In other words, in order to compute $\partial f/\partial x$ at (x_0,y_0) we keep $y = y_0$ fixed (treat as a constant) and differentiate $f(x,y_0)$ with respect to x. A similar prescription holds for $\partial f/\partial y$. A formal definition looks as follows.

DEFINITION 2. Suppose $f(x,y)$ is defined in some open neighborhood of (x_0,y_0). Then the *first partial derivatives with respect to x and y* are given by

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{\Delta x\to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x},$$

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{\Delta y\to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0,y_0)}{\Delta y},$$

respectively, if these limits exist. Another notation is also used: $f_x(x_0, y_0)$, $f_y(x_0, y_0)$, respectively. In a similar way on defines partial first derivatives for functions of more variables, in particular, $f(x, y, z)$.

Remark. It is clear from this definition that one can use the usual rules of differentiation when calculating partial derivatives. In particular, the rules of differentiating the sum, product, quotient and composition hold.

EXAMPLE 3. Let us compute the first partial derivatives of the function

$$f(x, y) = e^{x\mathrm{sin}y}$$

We obtain

$$\frac{\partial f}{\partial x} = e^{x\mathrm{sin}y}\mathrm{sin}y, \quad \frac{\partial f}{\partial y} = e^{x\mathrm{sin}y}x\mathrm{cos}y$$

EXAMPLE 4. Let us compute $f_z(1, \pi, 1)$ for the function

$$f(x, y, z) = \sin\left(\frac{xy}{z}\right)$$

We get

$$f_z(x, y, z) = \cos\left(\frac{xy}{z}\right) \cdot \left(-\frac{xy}{z^2}\right)$$

and hence

$$f_z(1, \pi, 1) = -\pi\mathrm{cos}\pi = \pi$$

EXAMPLE 5. Let us compute the first partial derivatives of the function

$$f(x, y) = \begin{cases} xy/(x^2 + y^2) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

The situation is more complicated than in Examples 3-4 since the function $f(x, y)$ is "glued" at $(0, 0)$ and we need to use Definition 2 to find partial derivatives. We have

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0.$$

Similarly,

$$\frac{\partial f}{\partial y}(0, 0) = 0.$$

Remark. Note that the function $f(x, y)$ of Example 5 is not continuous at $(0, 0)$ (the reader is advised to check that). Therefore, even if a function has first partial derivatives at some point, it doesn't have to be continuous at that point.

EXAMPLE 6. Let us compute the first partial derivatives of the function $f(x, y) = \sqrt{x^2 + y^2}$ at $(0, 0)$. We need to use Definition 2 again and then we get

$$\frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{(\Delta x)^2} - \sqrt{0}}{\Delta x} = \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$$

but this limit does not exist since if we take $\Delta x < 0$, we get the left limit equal to -1, and if we take $\Delta x > 0$, we get the right limit equal to 1. Hence $f_x(0, 0)$ doesn't exist. In a similar way one can show that $f_y(0, 0)$ doesn't exist. Note that f is continuous at $(0, 0)$ (that continuity is not strong enough to imply existence of partial derivatives is less surprising).