Matricial R-circular Systems and Random Matrices

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Motivations

Motivations:

- describe asymptotic distributions of random blocks
- construct new random matrix models
- unify concepts of noncommutative independence

Fundamental results

1. If $Y(u,n)$ is a standard complex Hermitian Gaussian random matrix, it converges under the trace to a semicircular operator

$$\lim_{n\to\infty} Y(u,n) \to \omega(u)$$

under $\tau(n) = \mathbb{E} \circ \mathrm{Tr}(n)$

2. If $Y(u,n)$ is a standard complex non-Hermitian Gaussian random matrix, it converges to a circular operator

$$\lim_{n\to\infty} Y(u,n) \to \eta(u)$$

under $\tau(n) = \mathbb{E} \circ \mathrm{Tr}(n)$.

Different approaches

1. free probability and freeness
2. operator-valued free probability and freeness with amalgamation
3. matricially free probability and matricial freeness

Voiculescu's asymptotic freeness and generalizations

1. Complex independent Hermitian Gaussian random matrices converge to a free semicircular family

$$\{Y(u,n) : u \in \mathcal{U}\} \rightarrow \{\omega(u) : u \in \mathcal{U}\}$$

2. Complex independent Non-Hermitian Gaussian random matrices converge to a free circular family

$$\{Y(u,n) : u \in \mathcal{U}\} \rightarrow \{\eta(u) : u \in \mathcal{U}\}$$

3. Generalizations (Dykema, Schlakhtyenko, Hiai-Petz, Benaych-Georges and others)

Philosophy

By asymptotic freeness, large random matrix is a free random variable, so it is natural to

1. decompose it into large blocks
2. look for a concept of 'matricial' independence for blocks
3. find operatorial realizations for large blocks
4. reduce computations to properties of these operators

Matricial decomposition

1. decompose the set $[n] := \{1, \ldots, n\}$ into disjoint intervals
$$[n] = N_1 \cup \ldots \cup N_r$$
and set $n_j = |N_j|$,
2. use normalized partial traces
$$\tau_j(n) = \mathbb{E} \circ \mathrm{Tr}_j(n)$$
where
$$\mathrm{Tr}_j(n)(A) = \frac{n}{n_j} \mathrm{Tr}(n)(D_j A D_j)$$
and D_j is the $n_j \times n_j$ unit matrix embedded in $M_n(\mathbb{C})$ at the right place.

Matricial decomposition

1. decompose random matrices $Y(u,n)$ into independent blocks

$$S_{i,j}(u,n) = D_i Y(u,n) D_j$$

2. decompose symmetric blocks $T_{i,j}(u,n)$, built from blocks of the same color:

$$Y(u,n) = \begin{pmatrix} S_{1,1}(u,n) & S_{1,2}(u,n) & \dots & S_{1,r}(u,n) \\ S_{2,1}(u,n) & S_{2,2}(u,n) & \dots & S_{2,r}(u,n) \\ . & . & \ddots & . \\ S_{r,1}(u,n) & S_{r,2}(u,n) & \dots & S_{r,r}(u,n) \end{pmatrix}$$

Matricial freeness

In order to define a 'matricial' concept of independence,

1. replace families of variables and subalgebras by arrays

$$\{a_i, i \in I\} \to (a_{i,j})_{(i,j)\in J}$$

$$\{\mathcal{A}_i, i \in I\} \to (\mathcal{A}_{i,j})_{(i,j)\in J}$$

2. replace one distinguished state in a unital algebra by an array of states

$$\varphi \to (\varphi_{i,j})_{(i,j)\in J}$$

where we set $\varphi_{i,j} = \varphi_j$ (state 'under condition' j)

Matricial freeness

The definition of matricial freeness is based on two conditions

1. 'freeness condition'

$$\varphi_{i,j}(a_1 a_2 \ldots a_n) = 0$$

   where $a_k \in \mathcal{A}_{i_k,j_k} \cap \mathrm{Ker}\varphi_{i_k,j_k}$ and neighbors come from different algebras

2. 'matriciality condition': subalgebras are not unital, but they have internal units $1_{i,j}$, such that the unit condition

$$1_{i,j} w = w$$

   holds only if w is a 'reduced word' matricially adapted to (i,j) and otherwise it is zero.

Some results

The concept of matricial freeness allows to

1. unify the main notions of independence
2. give a unified approach to sums and products of a large class of independent random matrices
3. find a unified combinatorial description of limit distributions (non-crossing colored partitions)
4. derive explicit formulas for arbitrary mutliplicative convolutions of Marchenko-Pastur laws
5. find random matrix models for boolean independence, monotone independence and s-freeness (noncommutative independence defined by subordination)
6. construct a natural random matrix model for free Meixner laws

Matricially free Fock space of tracial type

Definition

By the matricially free Fock space of tracial type we understand

$$\mathcal{M} = \bigoplus_{j=1}^{r} \mathcal{M}_j,$$

where each summand is of the form

$$\mathcal{M}_j = \mathbb{C}\Omega_j \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{j_1,\ldots,j_m \\ u_1,\ldots,u_m}} \mathcal{H}_{j_1,j_2}(u_1) \otimes \ldots \otimes \mathcal{H}_{j_m,j}(u_m),$$

Creation operators

Definition

Define matricially free creation operators on $\mathcal{M}$ as partial isometries with the action onto basis vactors

$$\begin{aligned}
\wp_{i,j}(u)\Omega_j &= e_{i,j}(u) \\
\wp_{i,j}(u)(e_{j,k}(s)) &= e_{i,j}(u)\otimes e_{j,k}(s) \\
\wp_{i,j}(u)(e_{j,k}(s)\otimes w) &= e_{i,j}(u)\otimes e_{j,k}(s)\otimes w
\end{aligned}$$

for any $i,j,k\in[r]$ and $u,s\in\mathcal{U}$, where $e_{j,k}(s)\otimes w$ is a basis vector. Their actions onto the remaining basis vectors give zero.

Toeplitz-Cuntz-Krieger algebras

Relations

One square matrix of creation operators $(\wp_{i,j})$ gives an array of partial isometries satisfying relations

$$\sum_{j=1}^{r} \wp_{i,j}\wp_{i,j}^* = \wp_{k,i}^*\wp_{k,i} - \wp_i \text{ for any } k$$

$$\sum_{j=1}^{r} \wp_{k,j}^*\wp_{k,j} = 1 \text{ for any } k$$

where $\wp_i$ is the projection onto $\mathbb{C}\Omega_j$. The corresponding C^*-algebras are Toeplitz-Cuntz-Krieger algebras .

Matricially free Gaussians

Arrays of matricially free Gaussians operators

$$\omega_{i,j}(u) = \sqrt{d_j}(\wp_{i,j}(u) + \wp_{i,j}^{*}(u))$$

play the role of matricial semicircular operators

$$[\omega(u)] = \begin{pmatrix} \omega_{1,1}(u) & \omega_{1,2}(u) & \dots & \omega_{1,r}(u) \\ \omega_{2,1}(u) & \omega_{2,2}(u) & \dots & \omega_{2,r}(u) \\ . & . & \ddots & . \\ \omega_{r,1}(u) & \omega_{r,2}(u) & \dots & \omega_{r,r}(u) \end{pmatrix}$$

and generalize semicircular operators. From now on we incorporate scalaras like $\sqrt{d_j}$ or more general $\sqrt{b_{i,j}(u)}$ in the operators.

Free probability Hermitian version

Theorem [Voiculescu]

If $Y(u, n)$ are Hermitian standard Gaussian independent random matrices with complex entries, then

$$\lim_{n\to\infty} Y(u, n) = \omega(u)$$

in the sense of mixed moments under the complete trace $\tau(n)$.

Theorem

If $Y(u,n)$ are Hermitian Gaussian independent random matrices with i.b.i.d. complex entries, then

$$\lim_{n\to\infty} T_{i,j}(u,n) = \widehat{\omega}_{i,j}(u)$$

in the sense of mixed moments under partial traces $\tau_j(n)$, where

$$\widehat{\omega}_{i,j}(u) = \begin{cases} \omega_{j,j}(u) & \text{if } i=j \\ \omega_{i,j}(u)+\omega_{i,j}(u) & \text{if } i\neq j \end{cases}$$

are symmetrized Gaussian operators.

Types of blocks

The symmetric blocks are called

1. balanced if $d_i > 0$ and $d_j > 0$
2. unbalanced if $d_i = 0 \wedge d_j > 0$ or $d_i > 0 \wedge d_j = 0$
3. evanescent if $d_i = 0$ and $d_j = 0$

where the numbers

$$\lim_{n\to\infty} \frac{|N_j|}{n} = d_j \geqslant 0$$

are called asymptotic dimensions .

Special cases

Special cases

In the general formula for mixed moments, we get

1. $T_{i,j}(u,n) \to \widehat{\omega}_{i,j}(u)$ if block is balanced
2. $T_{i,j}(u,n) \to \omega_{i,j}(u)$ if block is unbalanced, $j = 0 \wedge i > 0$
3. $T_{i,j}(u,n) \to \omega_{j,i}(u)$ if block is unbalanced, $j > 0 \wedge i = 0$
4. $T_{i,j}(u,n) \to 0$ if block is evanescent

Free probability non-Hermitian version

Theorem [Voiculescu]

If $Y(u,n)$ are standard complex non-Hermitian Gaussian independent random matrices, then

$$\lim_{n\to\infty} Y(u,n) = \eta(u)$$

in the sense of moments under $\tau(n)$, where each $\eta(u)$ is circular.

Non-Hermitian case

Theorem

If $Y(u,n)$ are non-Hermitian Gaussian independent random matrices with i.b.i.d. complex entries, then

$$\lim_{n\to\infty} T_{i,j}(u,n) = \eta_{i,j}(u)$$

where

$$\eta_{i,j}(u) = \widehat{\wp}_{i,j}(2u-1) + \widehat{\wp}^{*}_{i,j}(2u)$$

are called matricial circular operators and $\widehat{\wp}_{i,j}(s)$ are symmetrizations of $\wp_{i,j}(s)$ (two partial isometries for each $\eta_{i,j}(u)$).

Matricial realization of partial isometries

Lemma

We can identify partial isometries $\wp_{i,j}(u)$ with

$$\wp_{i,j}(u) \quad = \quad \ell(i,j,u) \otimes e(i,j) \in M_r(\mathcal{A})$$

where

1. the family $\{\ell(i,j,u) : i,j \in [r], u \in \mathcal{U}\}$ is a system of *-free creation operators w.r.t. state φ on $\mathcal{A}$
2. $(e_{i,j})$ is a system of matrix units in $M_r(\mathbb{C})$
3. $\Psi_j = \varphi \otimes \psi_j$, where ψ_j is the state associated with $e(j) \in \mathbb{C}^r$.

Matricial realization - Hermitian case

Lemma

Consequently,

$$\begin{aligned}\omega_{i,j}(u) &= \ell(i,j,u)\otimes e(i,j)+\ell(i,j,u)^{*}\otimes e(j,i)\\ \widehat{\omega}_{i,j}(u) &= \begin{cases} g(i,j,u)\otimes e(i,j)+g(i,j,u)^{*}\otimes e(j,i) & \text{if } i<j\\ f(i,u)\otimes e(i,i) & \text{if } i=j\end{cases}\end{aligned}$$

where

$$\{g(i,j,u): i,j\in[r], u\in\mathcal{U}\}, \qquad \{f(i,u): i\in[r], u\in\mathcal{U}\}$$

are *-free generalized circular and semicircular systems, respectively.

Matricial realization - non-Hermitian case

Lemma

We can also identify matricial circular operators with

$$\eta_{i,j}(u) = \begin{cases} g(i,j,u) \otimes e(i,j) + g(j,i,u) \otimes e(j,i) & \text{if } i < j \\ g(i,i,u) \otimes e(i,i) & \text{if } i = j \end{cases}$$

Asymptotic realization of blocks

Theorem

If $Y(u,n)$ are complex Gaussian independent random matrices with i.b.i.d. entries, then

$$\lim_{n\to\infty} S_{i,j}(u,n) = \zeta_{i,j}(u)$$

where

$$\zeta_{i,j}(u) = g(i,j,u)\otimes e(i,j)$$

are called matricial R-circular operators .

Example

Consider one off-diagonal $\zeta_{i,j}$ of the form

$$\zeta_{i,j} = (\ell_1 + \ell_2^*) \otimes e(i,j)$$

where $i \neq j$ and ℓ_1, ℓ_2 are free creation operators with covariances γ_1 and γ_2, respectively. We have

$$\Psi_j(\zeta_{i,j}^* \zeta_{i,j} \zeta_{i,j}^* \zeta_{i,j}) = \varphi(\ell_1^* \ell_1 \ell_1^* \ell_1) + \varphi(\ell_1^* \ell_2^* \ell_2 \ell_1) = \gamma_1^2 + \gamma_1 \gamma_2$$

whereas all remaining *-moments of $\zeta_{i,j}$ in the state Ψ_j vanish.

Matricial freeness

Theorem

The array of *-subalgebras $(\mathcal{M}_{i,j})$ of $M_r(\mathcal{A})$, each generated by

$$\{\ell(i,j,u)\otimes e(i,j) : u \in \mathcal{U}\},$$

with the unit

$$1_{i,j} = t(i,j)\otimes e(i,i) + 1\otimes e(j,j)$$

where

$$t(i,j) = \begin{cases} \sum_{u\in\mathcal{U}} \ell(i,j,u)\ell(i,j,u)^* & \text{if } \; i\neq j \\ 0 & \text{if } \; i=j \end{cases}$$

is matricially free with respect to $(\Psi_{i,j})$, where $\Psi_{j,j} = \varphi\otimes\psi_j$.

Symmetric matricial freeness

Theorem

The array $(\mathcal{M}_{i,j})$ of *-subalgebras of $M_r(\mathcal{A})$, each generated by

$$\{\ell(i,j,u)\otimes e(i,j), \ell(j,i,u)\otimes e(j,i) : u \in \mathcal{U}\}$$

and the symmetrized unit

$$1_{i,j} = 1 \otimes e(i,i) + 1 \otimes e(j,j),$$

is symmetrically matricially free with respect to $(\Psi_{i,j})$.

Symmetric matricial freeness

Corollary

The array $(\mathcal{M}_{i,j})$ of *-subalgebras of $M_r(\mathcal{A})$ such that one of the following cases holds:

1. each $\mathcal{M}_{i,j}$ is generated by $\{\widehat{\omega}_{i,j}(u) : u \in \mathcal{U}\}$,
2. each $\mathcal{M}_{i,j}$ is generated by $\{\eta_{i,j}(u) : u \in \mathcal{U}\}$,
3. each $\mathcal{M}_{i,j}$ is generated by $\{\zeta_{i,j}(u) : u \in \mathcal{U}\}$,

is symmetrically matricially free with respect to $(\Psi_{i,j})$.

Adapted partitions

If $a_k = c_k \otimes e(i_k, j_k) \in M_r(\mathcal{A})$ for $k = 1, \ldots, m$, where $c_k \in \mathcal{A}$, we will denote by

$$\mathcal{NC}_m(a_1, \ldots, a_m)$$

the set of non-crossing partitions of $[m]$ which are adapted to $(e(i_1, j_1), \ldots, e(i_m, j_m))$ which means that this tuple is cyclic

$$j_1 = i_2, j_2 = i_3, \ldots, j_n = i_1$$

together with all tuples associated with the blocks of π. Its subset consisting of pair partitions will be denoted $\mathcal{NC}_m^2(a_1, \ldots, a_m)$.

Example

For $(e(i,j), e(j,i), e(i,j), e(j,i))$, where $i \neq j$, there are three non-crossing partitions adapted to it:

$$\{\{1,2,3,4\}\}, \ \ \{\{1,4\},\{2,3\}\}, \ \ \{\{1,2\},\{3,4\}\}.$$

In turn, the partitions

$$\{\{1,2,5\},\{3,4\}\}, \ \{\{1,2,3,4,5\}\} \ \ \{\{1,2,3\},\{4,5\}\}$$

are the only non-crossing partitions adapted to the tuple $(e(i,j), e(j,k), e(k,i), e(i,m), e(m,i))$, where $i \neq j \neq k \neq i \neq m$.

Combinatorics of mixed *-moments

Lemma

With the above notations, let $a_k = \zeta_{i_k,j_k}^{\epsilon_k}(u_k)$, where $i_k, j_k \in [r]$, $u_k \in \mathcal{U}$, and $\epsilon_k \in \{1, *\}$ for $j \in [m]$ and $m \in \mathbb{N}$. Then

$$\Psi_j(a_1 \ldots a_m) = \sum_{\pi \in \mathcal{NC}_m^2(a_1,\ldots,a_m)} b_j(\pi, f)$$

where j is equal to the second index of last matrix unit (associated with a_m) and f is the unique coloring of π adapted to $(a_1, \ldots, a_m)$,

$$b_j(\pi, f) = \prod_k b_j(\pi_k, f)$$

with $b_j(\pi_k) = b_{c(k),c(o(k))}(u)$, where $c(k)$, $c(o(k))$, j are colors assigned to π_k, its nearest outer block $\pi_{o(k)}$ and to the imaginary block. In the remaining cases, the moment vanishes.

Definition

A family of multilinear functions $\kappa_\pi[\ .\ ; j]$, where $j \in [r]$, of matricial variables $a_k = c_k \otimes e(i_k, j_k) \in M_r(\mathcal{A})$ is cyclically multiplicative over the blocks $\pi_1, \ldots, \pi_s$ of $\pi \in \mathcal{NC}_m(a_1, \ldots, a_m)$ if

$$\kappa_\pi[a_1, \ldots, a_m; q] = \prod_{k=1}^{s} \kappa(\pi_k)[a_1, \ldots, a_m; j(k)],$$

for any $a_1, \ldots, a_m$, where

$$\kappa(\pi_k)[a_1, \ldots, a_m; j(k)] = \kappa_s(a_{q_1}, \ldots, a_{q_s}; j_{q_s})$$

for the block $\pi_k = (q_1 < \ldots < q_n)$, where $\{\kappa_s(.;j) : s \geqslant 1, j \in [r]\}$ is a family of multilinear functions. If $\pi \notin \mathcal{NC}_m(a_1, \ldots, a_m)$, we set $\kappa_\pi[a_1, \ldots, a_m; j] = 0$ for any j.

Cyclic cumulants

Definition

By the cyclic cumulants we shall understand the family of multilinear cyclically multiplicative functionals over the blocks of non-crossing partitions

$$\pi \to \kappa_{\pi}[\,.\,;j],$$

defined by r moment-cumulant formulas

$$\Psi_j(a_1 \ldots a_m) = \sum_{\pi \in \mathcal{NC}_m(a_1,\ldots,a_m)} \kappa_{\pi}[a_1,\ldots,a_m;j],$$

where $j \in [r]$ and $\Psi_j = \varphi \otimes \psi_j$.

Example

Take

$$\zeta_{i,j} = (\ell_1 + \ell_2^*) \otimes e(i,j) \ \text{ and } \ \zeta_{i,j}^* = (\ell_1^* + \ell_2) \otimes e(j,i),$$

where $i \neq j$. The non-trivial second order cumulants are

$$\begin{aligned} \kappa_2(\zeta_{i,j}^*, \zeta_{i,j}; j) &= \Psi_j(\zeta_{i,j}^* \zeta_{i,j}) = \varphi(\ell_1^* \ell_1) = \gamma_1 \\ \kappa_2(\zeta_{i,j}, \zeta_{i,j}^*; i) &= \Psi_i(\zeta_{i,j} \zeta_{i,j}^*) = \varphi(\ell_2^* \ell_2) = \gamma_2 \end{aligned}$$

whereas all higher order cumulants vanish. For instance,

$$\begin{aligned} \kappa_2(\zeta_{i,j}^*, \zeta_{i,j}, \zeta_{i,j}^*, \zeta_{i,j}; j) &= \Psi_j(\zeta_{i,j}^* \zeta_{i,j} \zeta_{i,j}^* \zeta_{i,j}) \\ &- \kappa_2(\zeta_{i,j}^*, \zeta_{i,j}; j)\kappa_2(\zeta_{i,j}^*, \zeta_{i,j}; j) \\ &\quad -\kappa_2(\zeta_{i,j}^*, \zeta_{i,j}; j)\kappa_2(\zeta_{i,j}, \zeta_{i,j}^*; i) \\ &= \gamma_1^2 + \gamma_1\gamma_2 - \gamma_1^2 - \gamma_1\gamma_2 = 0, \end{aligned}$$

Cyclic R-transforms

Let $\zeta := (\zeta_{i,j})$ and $\zeta^* := (\zeta^*_{i,j})$. Cyclic R-transforms of this pair of arrays are formal power series in $z := (z_{i,j})$ and $z^* := (z^*_{i,j})$ of the form

$$R_{\zeta,\zeta^*}(z, z^*; j)$$

$$= \sum_{n=1}^{\infty} \sum_{\substack{i_1,\ldots,i_n \\ j_1,\ldots,j_n}} \sum_{\epsilon_1,\ldots,\epsilon_n} \kappa_n(\zeta^{\epsilon_1}_{i_1,j_1}, \ldots, \zeta^{\epsilon_n}_{i_n,j_n}; j) z^{\epsilon_1}_{i_1,j_1} \ldots z^{\epsilon_n}_{i_n,j_n},$$

where $i_1, \ldots, i_n, j, j_1, \ldots j_n \in [r]$ and $\epsilon_1, \ldots, \epsilon_n \in \{1, *\}$.

Cyclic R-transforms of R-circular systems

Theorem

If $\zeta := (\zeta_{i,j})$ is the square array of matricial R-circular operators and $\zeta^* := (\zeta^*_{i,j})$, then its cyclic R-transforms are of the form

$$R_{\zeta,\zeta*}(z, z^*; j) = \sum_{i=1}^{r} (b_{i,j} z^*_{i,j} z_{i,j} + b_{i,j} z_{j,i} z^*_{j,i})$$

where $j \in [r]$, where $b_{i,j} = \kappa_2(\xi^*_{i,j}, \xi_{i,j}; j) = \kappa_2(\xi_{j,i}, \xi^*_{j,i}; j)$.

Theorem

If $\zeta := \zeta_{i,j}$, where $i \neq j$, then non-trivial cyclic R-transforms of the pair $\{\zeta, \zeta^*\}$ are of the form

$$\begin{aligned} R_{\zeta,\zeta^*}(z, z^*; j) &= b_{i,j} z^*_{i,j} z_{i,j} \\ R_{\zeta,\zeta^*}(z, z^*; i) &= b_{j,i} z_{i,j} z^*_{i,j} \end{aligned}$$

and if $\zeta := \zeta_{j,j}$, then

$$R_{\zeta,\zeta^*}(z, z^*; j) = b_{j,j}(z^*_{j,j} z_{j,j} + z_{j,j} z^*_{j,j})$$

Cyclic R-transforms of single R-circular operators

Corollary

If $b_{i,j} = d_i$ for any i, j, then $c := \sum_{i,j} \zeta_{i,j}$ is circular with respect to $\Psi = \sum_{j=1}^{r} d_j \Psi_j$ and the corresponding R-transform takes the form

$$R_{c,c*}(z_1, z_2) = \sum_{j=1}^{r} d_j R_{\zeta,\zeta*}(z, z^*; j) = z_1 z_2 + z_2 z_1,$$

where $R_{\zeta,\zeta*}(z, z^*; j)$ are the cyclic R-transforms considered above in which all entries of arrays z and z^* are identified with z_1 and z_2, respectively.