Matricial freeness and random pseudomatrices

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Contents

Main motivations

Main motivations:

- connect freeness (Voiculescu) with other notions of noncommutative independence
  - monotone independence (Muraki, Lu)
  - boolean independence (Bożejko, Speicher, Woroudi)
  - conditional freeness (Bożejko, Speicher)
  - conditionally monotone independence (Hasebe)
  - freeness with subordination (R.L.)
  - orthogonal independence (R.L.)
- introduce a notion of independence that would be related to random matrices

Main concepts

We propose two closely related notions of independence

- weak matricial freeness
- strong matricial freeness

Weak matricial freeness will also be called matricial freeness .

These notions of independence

- lead to unification of noncommutative independence (other than reduction to tensor independence)
- are related to subordination in free probability
- are related to random matrices

Main objects

Arrays of subalgebras and states

Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space. Instead of considering a family of subalgebras of $\mathcal{A}$, we take

1. an array $(\mathcal{A}_{i,j})$ of subalgebras of $\mathcal{A}$
2. an array of states $(\varphi_{i,j})$ on $\mathcal{A}$

Similar changes of the category can be made on the level of *-algebras and C*-algebras.

Main results

Shape of array determines independence

Under suitable assumptions on considered states, strong matricial freeness gives a correspondence between different shapes of matrices and different types of independence

- square arrays → freeness
- lower-triangular arrays → monotone independence
- upper-triangular arrays → anti-monotone independence
- diagonal arrays → boolean independence
- arrays with zeros above (below) the anti-diagonal → freeness with subordination
- one-column arrays → orthogonal independence

Main results

Generalization to conditional independence

Under slightly more general assumptions on considered states, strong matricial freeness gives a correpondence between different shapes of arrays and different types of conditional independence

- square arrays $\rightarrow$ conditional freeness
- lower-triangular arrays $\rightarrow$ conditional monotone independence
- upper-triangular arrays $\rightarrow$ conditional anti-monotone independence

Main results

Shape of array determines asymptotic independence

Under suitable assumptions on the considered states, matricial freeness gives a similar correspondence between different shapes of arrays and different types of asymptotic independence.

Magical properties

Freeness

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \Longrightarrow \{a_{1,1} + a_{1,2}, a_{2,1} + a_{2,2}\} \text{ is free}$$

Monotone independence

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \Longrightarrow (a_{1,1}, a_{2,1} + a_{2,2}) \text{ is monotone independent}$$

Boolean independence

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \Longrightarrow \{a_{1,1}, a_{2,2}\} \text{ is boolean independent}$$

Magical properties

Anti-monotone independence

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \Longrightarrow (a_{1,1} + a_{1,2}, a_{2,2}) \text{ is anti-monotone independent}$$

Subordination

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \Longrightarrow (a_{1,1} + a_{2,1}, a_{2,1}) \text{ is free with subordination}$$

Orthogonal independence

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \Longrightarrow (a_{1,1}, a_{2,1}) \text{ is orthogonally independent}$$

Main results

Asymptotics of blocks

Joint distributions of

- blocks of matricially free random variables with symmetric variances
- blocks of symmetric random matrices

agree asymptotically.

Matricially free Fock space

By the matricially free Fock space over the array of Hilbert spaces $\widehat{\mathcal{H}} = (\mathcal{H}_{i,j})$ we understand the Hilbert space direct sum

$$\mathcal{M}(\widehat{\mathcal{H}}) = \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{(i_1,i_2)\neq\ldots\neq(i_m,i_m)\\ n_1,\ldots,n_m\in\mathbb{N}}} \mathcal{H}_{i_1,i_2}^{\otimes n_1} \otimes \mathcal{H}_{i_2,i_3}^{\otimes n_2} \otimes \ldots \otimes \mathcal{H}_{i_m,i_m}^{\otimes n_m}$$

where Ω is a unit vector, with the canonical inner product.

Properties:

- freeness: neighboring pairs of indices are different
- matriciality: neighboring pairs are matricially related
- diagonal subordination: last pair is diagonal

Strongly matricially free Fock space

Strongly matricially free Fock space

By the strongly matricially free Fock space over the array of Hilbert spaces $\widehat{\mathcal{H}} = (\mathcal{H}_{i,j})$ we understand the Hilbert space direct sum

$$\mathcal{R}(\widehat{\mathcal{H}}) = \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{i_1 \neq \ldots \neq i_m \\ n_1,\ldots,n_m \in \mathbb{N}}} \mathcal{H}_{i_1,i_2}^{\otimes n_1} \otimes \mathcal{H}_{i_2,i_3}^{\otimes n_2} \otimes \ldots \otimes \mathcal{H}_{i_m,i_m}^{\otimes n_m}$$

where Ω is a unit vector, with the canonical inner product.

Properties:

- freeness: neighboring indices are different
- matriciality: neighboring pairs are matricially related
- diagonal subordination: last pair is diagonal

Free Fock space

In the case of square arrays, strongly matricially free Fock space is a natural generalization of the free Fock space.

Free Fock space (Voiculescu)

If the array $\widehat{\mathcal{H}}$ is square and $\mathcal{H}_{i,j} = \mathcal{H}_i$ for any i, j, then

$$\mathcal{R}(\widehat{\mathcal{H}}) \cong \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{i_1 \neq \ldots \neq i_m \\ n_1, \ldots, n_m \in \mathbb{N}}} \mathcal{H}_{i_1}^{\otimes n_1} \otimes \mathcal{H}_{i_2}^{\otimes n_2} \otimes \ldots \otimes \mathcal{H}_{i_m}^{\otimes n_m},$$

i.e. the strongly matricially free Fock space is isomorphic to the free Fock space $\mathcal{F}(\bigoplus_j \mathcal{H}_j)$.

Monotone Fock space

In the case of lower-triangular arrays, strongly matricially free Fock space is also a natural generalization of the monotone Fock space.

Monotone Fock space (Lu, Muraki)

If the array $\widehat{\mathcal{H}}$ is lower-triangular and $\mathcal{H}_{i,j} = \mathcal{H}_i$ for any $i \geqslant j$, then

$$\mathcal{R}(\widehat{\mathcal{H}}) \cong \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{i_1 > \ldots > i_m \\ n_1, \ldots, n_m \in \mathbb{N}}} \mathcal{H}_{i_1}^{\otimes n_1} \otimes \mathcal{H}_{i_2}^{\otimes n_2} \otimes \ldots \otimes \mathcal{H}_{i_m}^{\otimes n_m},$$

i.e. the strongly matricially free Fock space is isomorphic to the monotone Fock space.

Freeness (Voiculescu)

Let $\mathcal{A}$ be an algebra and let φ be a distinguished state. The family $(\mathcal{A}_i)_{i\in I}$ of unital subalgebras is free w.r.t. φ if

$$\varphi(a_1 a_2 \dots a_n) = 0$$

whenever $a_k \in \mathcal{A}_{i_k} \cap \mathrm{Ker}\varphi$ and $i_1 \neq \ldots \neq i_n$

Monotone independence (Muraki)

Let $\mathcal{A}$ be an algebra and let φ be a distinguished state. The family $(\mathcal{A}_i)_{i\in I}$ of (not assumed to be unital) subalgebras is monotone independent w.r.t. φ if

$$\varphi(a_1 \ldots a_{k-1} a_k a_{k+1} \ldots a_n) = \varphi(a_k)\varphi(a_1 \ldots a_{k-1} a_{k+1} \ldots a_n)$$

whenever $a_k \in \mathcal{A}_{i_k}$ and $i_{k-1} < i_k > i_{k+1}$ for any $1 \leqslant k \leqslant n$.

Assumptions for matricial freeness

We make the following assumptions:

1. $\mathcal{A}$ is a unital algebra with unit $1_{\mathcal{A}}$
2. $(\mathcal{A}_{i,j})$ is a diagonal-containing array of subalgebras of $\mathcal{A}$:
3. each $\mathcal{A}_{i,j}$ is equipped with an internal unit $1_{i,j}$, in general different from the unit $1_{\mathcal{A}}$
4. $(\varphi_{i,j})$ is an array of states (normalized linear functionals) on $\mathcal{A}$

Units

Matricially free array of units

The array $(1_{i,j})$ is a matricially free arrays of units if

1. it has the matricial property

$$\varphi_{p,q}(a1_{i,j}a_{i_1,j_1}\ldots a_{i_n,j_n}) = \begin{cases} \varphi_{p,q}(aa_{i_1,j_1}\ldots a_{i_n,j_n}) & \text{if } j=i_1 \\ 0 & \text{otherwise} \end{cases}$$

   for any $a_{i_k,j_k} \in \mathcal{A}_{i_k,j_k} \cap \mathrm{Ker}(\varphi_{i_k,j_k})$, $a \in \mathcal{A}$, $(i,j) \neq (i_1,j_1) \neq \ldots \neq (i_n,j_n)$ and any p,q

2. it is normalized according to

$$\varphi_{i,j}(1_{k,l}) = \delta_{j,l}$$

   for any i,j,k,l

Matricial freeness and strong matricial freeness

Matricial freeness

The array $(\mathcal{A}_{i,j})_{i,j\in I}$ is matricially free w.r.t. $(\varphi_{i,j})$ if

1. $\varphi(a_1 a_2 \ldots a_n) = 0$ whenever $a_k \in \mathcal{A}_{i_k,j_k} \cap \mathrm{Ker}\varphi_{i_k,j_k}$ and $(i_1, j_1) \neq \ldots \neq (i_n, j_n)$
2. the array $(1_{i,j})$ is a matricially free array of units.

Strong matricial freeness

The array $(\mathcal{A}_{i,j})_{i,j\in I}$ is strongly matricially free w.r.t. $(\varphi_{i,j})$ if

1. $\varphi(a_1 a_2 \ldots a_n) = 0$ whenever $a_k \in \mathcal{A}_{i_k,j_k} \cap \mathrm{Ker}\varphi_{i_k,j_k}$ and $(i_1, j_1) \neq \ldots \neq (i_n, j_n)$
2. the array $(1_{i,j})$ is a strongly matricially free array of units.

Theorem on independence

Assumptions for the theorem on independence

Suppose that we have arrays

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ . & . & \ddots & . \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varphi & \varphi_2 & \dots & \varphi_n \\ \varphi_1 & \varphi & \dots & \varphi_n \\ . & . & \ddots & . \\ \varphi_1 & \varphi_2 & \dots & \varphi \end{pmatrix}$$

and assume that

- $(a_{i,j})$ is strongly matricially free under $(\varphi_{i,j})$
- $(a_{i,j})$ is row-identically distributed under $(\varphi_{i,j})$

Theorem on independence

Shape of array determines independence

Under the above assumptions, the shape of an array determines independence, namely

- the family $\{a_j := \sum_k a_{j,k} : 1 \leqslant j \leqslant n\}$ is free
- the family $\{b_j := \sum_{k \leqslant j} a_{j,k} : 1 \leqslant j \leqslant n\}$ is monotone
- the family $\{c_j := \sum_{k \geqslant j} a_{j,k} : 1 \leqslant j \leqslant n\}$ is anti-monotone
- the family $\{d_j := a_{j,j} : 1 \leqslant j \leqslant n\}$ is boolean

under the state φ.

Two-dimensional matrices

Shape of array determines independence

In the case of two dimensions, the variables

- $a_1 := a_{1,1} + a_{1,2}$ and $a_2 = a_{2,1} + a_{2,2}$ are free
- $b_1 := a_{1,1}$ and $b_2 = a_{2,1} + a_{2,2}$ are monotone
- $c_1 := a_{1,1} + a_{1,2}$, $c_2 := a_{2,2}$ are anti-monotone
- $d_1 := a_{1,1}$ and $d_2 := a_{2,2}$ are boolean
- $e_1 := a_{1,1}$ and $e_2 := a_{2,1}$ are orthogonal

Theorem on convolutions

Shape of array determines convolution

Under the above assumptions, if the variables $(a_{i,j})$ have distributions $(\mu_{i,j})$ under $(\varphi_{i,j})$, then

- $\mathcal{D}(a_{1,1} + a_{1,2}, \varphi) = \mathcal{D}(a_{1,1}, \varphi) := \mu_1$
- $\mathcal{D}(a_{2,1} + a_{2,2}, \varphi) = \mathcal{D}(a_{2,2}, \varphi) := \mu_2$
- $\mathcal{D}(a_{1,1} + a_{2,2}, \varphi) = \mu_1 \uplus \mu_2$ (boolean additive)
- $\mathcal{D}(a_{1,1} + a_{2,1} + a_{2,2}, \varphi) = \mu_1 \rhd \mu_2$ (monotone additive)
- $\mathcal{D}(a_{1,1} + a_{2,1}, \varphi) = \mu_1 \vdash \mu_2$ (orthogonal additive)
- $\mathcal{D}(\sum_{i,j} a_{i,j}, \varphi) = \mu_1 \boxplus \mu_2$ (free additive)

Theorem on conditional independence

Shape of array determines conditional independence

More generally, if only the off-diagonal variables are row-identically distributed, then

- the family $\{a_j := \sum_k a_{j,k} : 1 \leqslant j \leqslant n\}$ is c-free
- the family $\{b_j := \sum_{k \leqslant j} a_{j,k} : 1 \leqslant j \leqslant n\}$ is c-monotone
- the family $\{c_j := \sum_{k \geqslant j} a_{j,k} : 1 \leqslant j \leqslant n\}$ is c-anti-monotone

under (φ, ψ), where ψ is any state which agrees with φ_j on the off-diagonal $\mathcal{A}_{i,j} = \mathrm{alg}(a_{i,j}, 1_{i,j})$.

Analytic subordination in free probability

Free convolutions have the so-called analytic subordination property In the case of the free additive convolution it takes the following form.

Analytic subordination property (Voiculescu, Biane)

The free additive convolution of probability measures on the real line $\mu_1, \mu_2 \in \mathcal{M}_{\mathbb{R}}$ has the subordination property

$$F_{\mu_1 \boxplus \mu_2}(z) = F_{\mu_1}(F_2(z))$$

in terms of the reciprocal Cauchy transforms.

Theorem on operatorial subordination

Operatorial subordination property

1. The sum of free random variables X_1, X_2 with distributions μ_1 and μ_2 has the decomposition
$$X_1 + X_2 = x + Y,$$
where the pair (x, Y) is monotone independent under φ.
2. The corresponding distributions satisfy the equation
$$\mu_1 \boxplus \mu_2 = \mu_1 \rhd (\mu_2 \boxplus\!\!\!\!\!\!\;\mid\;\, \mu_1),$$
where $\mathcal{D}(x, \varphi) = \mu_1$ and $\mathcal{D}(Y, \varphi) = \mu_2 \boxplus\!\!\!\!\!\!\;\mid\;\, \mu_1$ is the s-free additive convolution of μ_1 and μ_2 associated with s-freeness .

Operatorial subordination versus strong matricial freeness

Subordination property in terms of strong matricial freeness

In the case of two dimensions, the variables

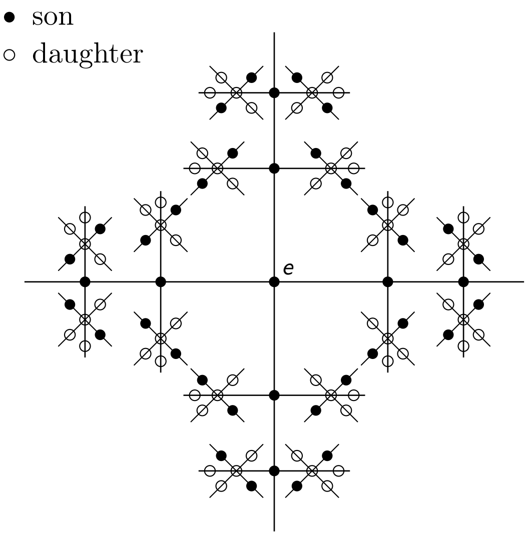
- $u_1 := a_{1,1} + a_{1,2}$ and $u_2 := a_{2,1}$ are s-free
- $t_1 := a_{2,2} + a_{2,1}$ and $t_2 := a_{1,2}$ are s-free
- $\mathcal{D}(u_1 + u_2, \varphi) = \mu_1 \boxminus \mu_2$ (s-free additive convolution)
- $\mathcal{D}(t_1 + t_2, \varphi) = \mu_2 \boxminus \mu_1$ (s-free additive convolution)

Decomposition of the free additive convolution

The following decomposition holds:

$$\mu_1 \boxplus \mu_2 = (\mu_1 \boxminus \mu_2) \uplus (\mu_2 \boxminus \mu_1)$$

Matricially free analog of homogenous tree $\mathbb{H}_4$



Random pseudomatrices

Random pseudomatrix

Let $(X_{i,j}(n))_{1\leqslant i,j\leqslant n}$ be arrays of self-adjoint random variables in unital *-algebras $\mathcal{A}(n)$ which are matricially free with respect to the array $(\varphi_{i,j}(n))$ of states. Then the sum of the form

$$S(n) = \sum_{i,j=1}^{n} X_{i,j}(n)$$

will be called a random pseudomatrix .

Assumptions

We shall consider two types of arrays depending on n. Let $\varphi(n)$ be a distinguished state on $\mathcal{A}$ and let $(\varphi_j(n))_{j\in I}$ be a family of additional states on $\mathcal{A}(n)$ called conditions .

Case 1

We say that the array $(\varphi_{i,j}(n))$ is defined by the state $\varphi(n)$ and the family $(\varphi_j(n))_{j\in I}$ if

$$\varphi_{j,j}(n) = \varphi(n) \quad \text{and} \quad \varphi_{i,j}(n) = \varphi_j(n) \quad \text{for} \quad i \neq j$$

Case 2

We say that the array $(\varphi_{i,j}(n))$ is defined by the family $(\varphi_j(n))_{j\in I}$ if

$$\varphi_{i,j}(n) = \varphi_j(n) \quad \text{for} \;\; \text{any} \quad (i,j)$$

Assumptions

Assumptions for central limit theorem:

1. Each set $[n] := \{1, 2, \ldots, n\}$ is partitioned into disjoint non-empty intervals,

$$[n] = N_1 \cup N_2 \cup \ldots \cup N_r$$

   where $r \in \mathbb{N}$, such that $|N_j|/n \to d_j$ as $n \to \infty$,
2. $(X_{i,j}(n))$ is matricially free with respect to $(\varphi_{i,j}(n))$,
3. the variables have zero expectations:

$$\varphi_{i,j}(n)(X_{i,j}(n)) = 0,$$

4. their variances are block-identical:

$$\varphi_{i,j}(n)(X_{i,j}^2(n)) = \frac{u_{p,q}}{n} \ \text{for} \ (i,j) \in N_p \times N_q,$$

5. their moments are uniformly bounded.

Continued (multi)fractions

Limit laws can be expressed in terms of continued (multi)fractions

Lemma

For given matrix $B \in M_r(\mathbb{R})$ with nonnegative entries, continued (multi)fractions of the form

$$K_{i,j}(z) = \cfrac{b_{i,j}}{z - \sum_k \cfrac{b_{k,i}}{z - \sum_p \cfrac{b_{p,k}}{z - \ldots}}}$$

where $1 \leqslant i, j \leqslant r$, converge uniformly on the compact subsets of $\mathbb{C}^+$ to the K-transforms of some $\mu_{i,j} \in \mathcal{M}_{\mathbb{R}}$ with compact supports.

Central Limit Theorem

Under the assumptions stated above, the $\varphi(n)$-distributions of $S(n)$ converge weakly to the distribution

$$\mu_0 = \mu_{1,1} \uplus \mu_{2,2} \uplus \ldots \uplus \mu_{r,r}$$

where $\mu_{j,j}$ is the distribution defined by $K_{j,j}$ for each j with $B = DU$ (block variance matrix times the diagonal dimension matrix).

Tracial Central Limit Theorem

Under the above assumptions, the distributions of random pseudomatrices $S(n)$ under the states

$$\psi(n) = \frac{1}{n}\sum_{j=1}^{n} \varphi_j(n)$$

converge weakly to the convex linear combination

$$\mu = \sum_{j=1}^{r} d_j \mu_j$$

where $\mu_j = \mu_{1,j} \uplus \mu_{2,j} \uplus \ldots \uplus \mu_{r,j}$ for each $j = 1, \ldots, r$ and $\mu_{i,j}$ is the distribution defined by $K_{i,j}$ for any i, j.

Boolean compressions

Boolean compressions

1. boolean compression of μ is $T_t\mu$ defined by $K_{T_t\mu}(z) = tK_\mu(z)$, where

$$K_\mu(z) = z - F_\mu(z) = z - \frac{1}{G_\mu(z)}$$

2. boolean compression of the semicircle law is $\sigma_{\alpha,\beta} = T_t\sigma_\alpha$, where $t = \beta^2/\alpha^2$, with the Cauchy transform

$$G_{\sigma_{\alpha,\beta}}(z) = \frac{(2\alpha^2 - \beta^2)z - \beta^2\sqrt{z^2 - 4\alpha^2}}{(2\alpha^2 - 2\beta^2)z^2 + 2\beta^4}$$

Two-dimensional semi-circle laws

Two-dimensional limit laws

If the variances in a 2 by 2 matrix $\alpha^2, \beta^2, \gamma^2, \delta^2$ do not vanish, then the diagonal measures have the form

$$\begin{aligned} \mu_{1,1} &= T_{1/t}(\sigma_{\alpha,\beta} \boxplus \sigma_{\delta,\gamma}) \\ \mu_{2,2} &= T_{1/s}(\sigma_{\delta,\gamma} \boxplus \sigma_{\alpha,\beta}) \end{aligned}$$

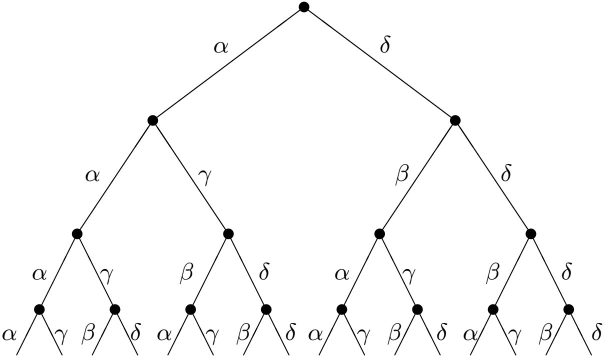
and the off-diagonal measures are given by

$$\begin{aligned} \mu_{1,2} &= \sigma_{\alpha,\beta} \boxplus \sigma_{\delta,\gamma} \\ \mu_{2,1} &= \sigma_{\delta,\gamma} \boxplus \sigma_{\alpha,\beta} \end{aligned}$$

where $t = (\beta/\alpha)^2$ and $s = (\gamma/\delta)^2$.

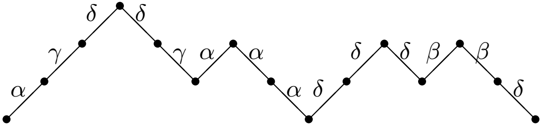
Weighted binary tree

In the 2-dimensional case, the moments of μ_0 are given by counting weighted root-to-root paths in the binary tree.



Weighted Catalan path

Another realization can be given in terms of weighted Catalan paths.



Asymptotic freeness and monotone independence

Asymptotic freeness

If the array of variances is r-dimensional, square and has identical non-zero variances in each row, then

$$\mu_j = \sigma_{\alpha_1} \boxplus \sigma_{\alpha_2} \boxplus \ldots \boxplus \sigma_{\alpha_r}$$

for each $1 \leqslant j \leqslant r$, and they all coincide with μ and μ_0.

Asymptotic monotone independence

If the array of variances is r-dimensional, lower-triangular and has identical non-zero variances in each row, then

$$\mu_j = \sigma_{\alpha_j} \rhd \sigma_{\alpha_{j+1}} \rhd \ldots \rhd \sigma_{\alpha_r}$$

for each $1 \leqslant j \leqslant r$. Moreover, $\mu_0 = \mu_1$ and $\mu = \sum_j d_j \mu_j$.

Boolean and free Fock spaces

We want to find a Hilbert space realization of the limit laws. For that purpose let us recall the definition of free and boolean Fock spaces.

Boolean and free Fock spaces

Recall that by the boolean and free Fock spaces over the Hilbert space $\mathcal{H}$, respectively, we understand the direct sums

$$\mathcal{F}_0(\mathcal{H}) = \mathbb{C}\xi \oplus \mathcal{H} \quad \text{and} \quad \mathcal{F}(\mathcal{H}) = \mathbb{C}\xi \oplus \bigoplus_{m=1}^{\infty} \mathcal{H}^{\otimes m},$$

where ξ is a unit vector, endowed with the canonical inner products.

We shall take a suitable product of an array of such Fock spaces.

Matricially free product of Hilbert spaces

The product of Hilbert spaces in our theory is called the matricially free product of $(\mathcal{H}_{i,j}, \xi_{i,j})$ and is related the matricially free Fock space.

Matricially free product of Hilbert spaces

Let $(\mathcal{H}_{i,j}, \xi_{i,j})$ be an array of Hilbert spaces with distinguished unit vectors. By the matricially free product of $(\mathcal{H}_{i,j}, \xi_{i,j})$ we understand the pair $(\mathcal{H}, \xi)$, where

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{(i_1,i_2)\neq \ldots \neq (i_m,i_m)} \mathcal{H}^0_{i_1,i_2} \otimes \mathcal{H}^0_{i_2,i_3} \otimes \ldots \otimes \mathcal{H}^0_{i_m,i_m},$$

with $\mathcal{H}^0_{i,j} = \mathcal{H}_{i,j} \ominus \mathbb{C}\xi_{i,j}$ and ξ being a unit vector, with the canonical inner product. We denote it $(\mathcal{H}, \xi) = *^M_{i,j}(\mathcal{H}_{i,j}, \xi_{i,j})$.

Matricially free-boolean Fock space

By the matricially free-boolean Fock space over the array $\widehat{\mathcal{H}} = (\mathcal{H}_{i,j})$ we shall understand the matricially free product

$$(\mathcal{F}, \xi) = *_{i,j}^{M}(\mathcal{F}_{i,j}, \xi_{i,j}), \quad \text{where} \quad \mathcal{F}_{i,j} = \begin{cases} \mathcal{F}(\mathcal{H}_{j,j}) & \text{if} \;\; i = j \\ \mathcal{F}_0(\mathcal{H}_{i,j}) & \text{if} \;\; i \neq j \end{cases}$$

and $\xi_{i,j}$ denotes the distinguished unit vector in $\mathcal{F}_{i,j}$.

Matricially free Gaussian operators

Matricially free Gaussian operators

Let $A = (\alpha_{i,j})$ be a diagonal-containing array of positive real numbers and let $(\mathcal{H}_{i,j}) = (\mathbb{C}e_{i,j})$ be the associated array of Hilbert spaces. By the matricially free creation operators associated with A we understand operators of the form

$$\varsigma_{i,j} = \alpha_{i,j}\tau^{*}\ell(e_{i,j})\tau,$$

where $\tau : \mathcal{F} \to \mathcal{F}(\bigoplus_{i,j}\mathcal{H}_{i,j})$ is the canonical embedding and the $\ell(e_{i,j})$'s denote the canonical free creation operators.

Further operators on $\mathcal{F}$

- matricially free Gaussian operators: $\zeta_{i,j} = \varsigma_{i,j} + \varsigma_{i,j}^{*}$
- truncated matricially free creation operators: $\wp_{i,j} = \varsigma_{i,j} P$, where P is the projection onto $\mathcal{F} \ominus \Omega$
- truncated matricially free Gaussian operators: $\omega_{i,j} = \wp_{i,j} + \wp_{i,j}^{*}$,

Blocks

Blocks of matricially free random variables

Suppose that the array $(X_{i,j}(n))$ is decomposed into blocks according to the partition $[n] = N_1 \cup N_2 \cup \ldots \cup N_r$ into disjoint non-empty subsets. The sums of the form

$$S_{p,q}(n) = \sum_{(i,j)\in N_p \times N_q} X_{i,j}(n)$$

will be called blocks of the pseudomatrix $S(n)$.

Fock space realization of central limit joint distributions

Joint limit distributions in the Central Limit Theorem can be realized on the matricially free Fock space as

$$\lim_{n\to\infty} \varphi(n)(S_{p_1,q_1}(n)\ldots S_{p_m,q_m}(n)) = \varphi(\zeta_{p_1,q_1}\ldots\zeta_{p_m,q_m})$$

where φ is the vacuum state on $B(\mathcal{F}$.

Fock space realization of tracial central limit joint distributions

Joint limit distributions in the Tracial Central Limit Theorem can be realized on the matricially free Fock space as

$$\lim_{n\to\infty}\psi(n)(S_{p_1,q_1}(n)\ldots S_{p_m,q_m}(n)) = \psi(\omega_{p_1,q_1}\ldots\omega_{p_m,q_m})$$

where $\psi = \sum_j d_j\psi_j$ and ψ_j is the state on $B(\mathcal{F})$ associated with the vector $e_{j,j}$.

Asymptotics of symmetric blocks

Agreement with symmetric random matrices

Symmetric blocks of random pseudomatrices given by symmetric blocks

$$Z_{p,q}(n) = S_{p,q}(n) + S_{q,p}(n)$$

have the same asymptotics under $\psi(n)$ as symmetric blocks of random matrices under classical expectation composed with normalized trace in the approach of Voiculescu (Gaussian case) and Dykema (non-Gaussian case).

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