Matricial Freeness and Random Matrices

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Motivations:

- unify concepts of noncommutative independence
- find and understand their relations to random matrices
- find random matrix models for various distributions
- construct a unified random matrix ensemble

Random matrix asymptotics

1. If $Y(u,n)$ is a suitable Hermitian random matrix (i.i.d. Gaussian), it converges under the trace to a semicircular operator
$$\lim_{n\to\infty} Y(u,n) \to \omega(u)$$
2. If $Y(u,n)$ is a suitable non-Hermitian random matrix (i.i.d. Gaussian), it converges under the trace to a circular operator
$$\lim_{n\to\infty} Y(u,n) \to \eta(u)$$

Aproaches to independent matrices

1. free probability and freeness
2. operator-valued free probability and freeness with amalgamation
3. matricially free probability and matricial freeness

Voiculescu's asymptotic freeness and generalizations

1. Independent Hermitian Gaussian random matrices converge to a free semicircular family

$$\{Y(u,n) : u \in \mathcal{U}\} \to \{\omega(u) : u \in \mathcal{U}\}$$

2. Independent Non-Hermitian Gaussian random matrices converge to a free circular family

$$\{Y(u,n) : u \in \mathcal{U}\} \to \{\eta(u) : u \in \mathcal{U}\}$$

3. Generalization to non-Gaussian matrices by Dykema.
4. Asymptotic freeness with amalgamation of band matrices (Gaussian independent but not identically distributed) by Schlakhtyenko.

Matriciality

1. Random matrix is a prototype of a noncommutative random variable, so it is natural to look for a matricial concept of independence.
2. Replace families of variables and subalgebras by arrays

$$\{X_i, i \in I\} \rightarrow (X_{i,j})_{(i,j)\in J}$$

$$\{\mathcal{A}_i, i \in I\} \rightarrow (\mathcal{A}_{i,j})_{(i,j)\in J}$$

3. Replace one distinguished state in a unital algebra by an array of states

$$\varphi \rightarrow (\varphi_{i,j})_{(i,j)\in J}$$

Matricial freeness

The definition of matricial freeness is based on two conditions

1. 'freeness condition'

$$\varphi_{i,j}(a_1 a_2 \ldots a_n) = 0$$

   where $a_k \in \mathcal{A}_{i_k,j_k} \cap \mathrm{Ker}\varphi_{i_k,j_k}$ and neighbors come from different algebras

2. 'matriciality condition': subalgebras are not unital, but they have internal units $1_{i,j}$, such that the unit condition

$$1_{i,j} w = w$$

   holds only if w is a 'reduced word' matricially adapted to (i,j) and otherwise it is zero.

The definition of strong matricial freeness is similar.

Benefits

This concept has allowed us to

1. unify the main notions of independence
2. give a unified approach to sums and products of independent random matrices (including Wigner, Wishart, free Bessel)
3. find a unified combinatorial description of limit distributions (non-crossing colored partitions)
4. derive explicit formulas for arbitrary mutliplicative convolutions of Marchenko-Pastur laws
5. find random matrix models for boolean independence, monotone independence for two matrices, s-freeness for two matrices (noncommutative independence defined by subordination)
6. construct a random matrix model for free Meixner laws

Decomposition

On the level of random matrices and their asymptotic operatorial realizations the idea is that of decomposition:

1. decompose random matrices $Y(u, n)$ into independent symmetric blocks
2. decompose the trace $\tau(n)$ into partial traces $\tau_j(n)$
3. decompose free semicircular (circular) families into matricial summands
4. prove that these dcompositions are in good correspondence
5. study relations between the summands (matricial freeness)

Symmetric blocks

Independent symmetric blocks are built from blocks of same color.

$$Y(u,n) = \begin{pmatrix} S_{1,1}(u,n) & S_{1,2}(u,n) & \dots & S_{1,r}(u,n) \\ S_{2,1}(u,n) & S_{2,2}(u,n) & \dots & S_{2,r}(u,n) \\ . & . & \ddots & . \\ S_{r,1}(u,n) & S_{r,2}(u,n) & \dots & S_{r,r}(u,n) \end{pmatrix}$$

If $Y(u,n)$ is Hermitian, then of course

$$S_{j,j}^{*}(u,n) = S_{j,j}(u,n) \text{ and } S_{i,j}^{*}(u,n) = S_{j,i}(u,n)$$

but we want to treat Hermitian and Non-Hermitian cases.

Decomposition of matrices

Asymptotic dimensions

For any $n \in \mathbb{N}$ we partition the set $\{1, 2, \ldots, n\}$ into disjoint nonempty subsets (intervals)

$$\{1, 2, \ldots, n\} = N_1(n) \cup \ldots \cup N_r(n)$$

where the numbers

$$\lim_{n \to \infty} \frac{|N_j(n)|}{n} = d_j \geqslant 0$$

are called asymptotic dimensions .

Decomposition of matrices

1. decomposition of independent matrices into symmetric blocks

$$Y(u,n) = \sum_{i \leqslant j} T_{i,j}(u,n)$$

2. decompose free Gaussians into matricially free Gaussians

$$\omega(u) = \sum_{i,j} \omega_{i,j}(u)$$

3. so that they correspond to each other in all mixed moments

$$\lim_{n\to\infty} T_{i,j}(u,n) \to \omega_{i,j}(u)$$

Blocks

Three types of blocks

The symmetric blocks are called

1. balanced if $d_i > 0$ and $d_j > 0$
2. unbalanced if $d_i = 0 \wedge d_j > 0$ or $d_i > 0 \wedge d_j = 0$
3. evanescent if $d_i = 0$ and $d_j = 0$

Arrays of Fock spaces

Define arrays of Fock spaces

$$\mathcal{F}_{i,j}(u) = \begin{cases} \mathcal{F}(\mathbb{C}e_{j,j}(u)) & \text{if } i = j \\ \mathcal{F}_0(\mathbb{C}e_{i,j}(u)) & \text{if } i \neq j \end{cases},$$

where $(i,j) \in \mathcal{J}$ and $u \in \mathcal{U}$, with

$$\mathcal{F}_0(\mathcal{H}) = \mathbb{C}\Omega \oplus \mathcal{H} \quad \text{and} \quad \mathcal{F}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \mathcal{H}^{\otimes m},$$

denoting boolean and free Fock spaces, respectively.

Matricially free Fock space of tracial type

Definition

By the matricially free Fock space of tracial type we understand

$$\mathcal{M} = \bigoplus_{j=1}^{r} \mathcal{M}_j,$$

where each summand is of the form

$$\mathcal{M}_j = \mathbb{C}\Omega_j \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{(i_1,i_2,u_1)\neq \ldots \neq (i_m,j,u_m)} \mathcal{F}^0_{i_1,i_2}(u_1) \otimes \ldots \otimes \mathcal{F}^0_{i_m,j}(u_m),$$

where $\mathcal{F}^0_{i,j}(u)$ is the orthocomplement of $\mathbb{C}\Omega_{i,j}(u)$ in $\mathcal{F}_{i,j}(u)$.

Creation operators

Definition

Define matricially free creation operators on $\mathcal{M}$

$$\wp_{i,j}(u) = \alpha_{i,j}(u)\tau^*\ell(e_{i,j}(u))\tau$$

where τ is the canonical embedding in the free Fock space

$$\tau : \mathcal{M} \hookrightarrow \mathcal{F}(\mathcal{H})$$

over the direct sum of Hilbert spaces

$$\mathcal{H} = \bigoplus_{i,j,u} \mathbb{C}e_{i,j}(u)$$

with the vacuum space $\bigoplus_{j=1}^{r} \mathbb{C}\Omega_j$ replacing the usual $\mathbb{C}\Omega$.

Toeplitz-Cuntz-Krieger algebras

Relations

If we have one square matrix of creation operators $(\wp_{i,j})$ and $\alpha_{i,j} = 1$ for all i, j, then they are partial isometries satisfying relations

$$\sum_{j=1}^{r} \wp_{i,j}\wp_{i,j}^{*} = \wp_{k,i}^{*}\wp_{k,i} - \wp_{i} \ \text{ for any } k$$

$$\sum_{j=1}^{r} \wp_{k,j}^{*}\wp_{k,j} = 1 \ \text{ for any } k$$

where $\wp_i$ is the projection onto $\mathbb{C}\Omega_j$. The corresponding C^*-algebras are Toeplitz-Cuntz-Krieger algebras .

Matricially free Gaussians

Arrays of matricially free Gaussians operators

$$\omega_{i,j}(u) = \wp_{i,j}(u) + \wp_{i,j}^{*}(u)$$

play the role of matricial semicircular operators

$$[\omega(u)] = \begin{pmatrix} \omega_{1,1}(u) & \omega_{1,2}(u) & \dots & \omega_{1,r}(u) \\ \omega_{2,1}(u) & \omega_{2,2}(u) & \dots & \omega_{2,r}(u) \\ . & . & \ddots & . \\ \omega_{r,1}(u) & \omega_{r,2}(u) & \dots & \omega_{r,r}(u) \end{pmatrix}$$

and generalize semicircular operators.

The corresponding arrays of distributions in the states $\{\Psi_1, \ldots, \Psi_r\}$ from which we build the array $(\Psi_{i,j})$ by setting $\Psi_{i,j} = \Psi_j$:

$$[\sigma(u)] = \left(\begin{array}{cccc} \sigma_{1,1}(u) & \kappa_{1,2}(u) & \ldots & \kappa_{1,r}(u) \\ \kappa_{2,1}(u) & \sigma_{2,2}(u) & \ldots & \kappa_{2,r}(u) \\ . & . & \ddots & . \\ \kappa_{r,1}(u) & \kappa_{r,2}(u) & \ldots & \sigma_{r,r}(u) \end{array} \right)$$

where $\sigma_{j,j}(u)$ is a semicircle law and $\kappa_{i,j}(u)$ is a Bernoulli law.

Symmetrized Gaussian operators

Symmetrized Gaussian operators

We still need to symmetrize matricially free Gaussians and define the ensemble of symmetrized Gaussian operators

$$\widehat{\omega}_{i,j}(u) = \begin{cases} \omega_{j,j}(u) & \text{if } i = j \\ \omega_{i,j}(u) + \omega_{i,j}(u) & \text{if } i \neq j \end{cases}$$

which give Fock space realizations of limit distributions.

Theorem

Under natural assumptions (block-identical variances), the Hermitian Gaussian Symmetric Block Ensemble converges in moments to the ensemble of symmetrized Gaussian operators

$$\lim_{n\to\infty} \tau_j(n)(T_{i_1,j_1}(u_1,n)\ldots T_{i_m,j_m}(u_m,n)) =$$

$$\Psi_j(\widehat{\omega}_{i_1,j_1}(u_1)\ldots\widehat{\omega}_{i_m,j_m}(u_m))$$

where $u_1,\ldots,u_m \in \mathcal{U}$, and $\tau_j(n)$ denotes the normalized partial trace over the set of basis vectors $\{e_k : k \in N_q\}$ composed with classical expectation.

Free probability version

Theorem [Voiculescu]

Under natural assumptions, the Hermitian Gaussian Ensemble converges in moments to the ensemble of free Gaussian operators

$$\lim_{n\to\infty} \tau(n)(Y(u_1, n) \ldots Y(u_m, n)) = \Phi(\omega(u_1) \ldots \omega(u_m))$$

where $u_1, \ldots, u_m \in \mathcal{U}$, $\tau(n)$ denotes the normalized trace composed with classical expectation and Φ is the vacuum vector.

Asymptotic distribution

Symbolically

Under the partial traces and under the trace, we have

$$\lim_{n\to\infty} T_{i,j}(u,n) = \widehat{\omega}_{i,j}(u)$$

which is a block refinement of

$$\lim_{n\to\infty} Y(u,n) = \omega(u)$$

under the trace in free probability.

Asymptotic distribution

Symbolically

The general formula reduces to

1. $T_{i,j}(u,n) \to \widehat{\omega}_{i,j}(u)$ if block is balanced
2. $T_{i,j}(u,n) \to \omega_{i,j}(u)$ if block is unbalanced, $j = 0 \wedge i > 0$
3. $T_{i,j}(u,n) \to \omega_{j,i}(u)$ if block is unbalanced, $j > 0 \wedge i = 0$
4. $T_{i,j}(u,n) \to 0$ if block is evanescent

Colored non-crossing pair partition

We color blocks $\pi_1, \ldots, \pi_m$ of a non-crossing pair partition π by numbers from the set $\{1, 2, \ldots, r\}$. If we denote the coloring function by f, we get

$$(\pi, f) = \{(\pi_1, f), \ldots, (\pi_m, f)\}$$

the collection of colored blocks. We add the imaginary block and we also color that block.

Combinatorics

Let a real-valued matrix $B(u) = (b_{i,j}(u))$ be given for any $u \in [t]$. Limit mixed moments can be expressed in terms of products

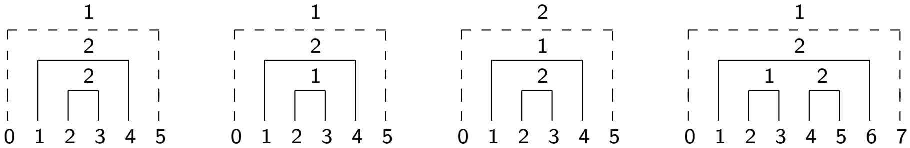
$$b_q(\pi, f) = b_q(\pi_1, f) \ldots b_q(\pi_k, f)$$

where b_q is defined on the set of blocks as

$$b_q(\pi_k, f) = b_{i,j}(u),$$

whenever block $\pi_k = \{r, s\}$ is colored by i, its nearest outer block $o(\pi_k)$ is colored by j and $u_r = u_s = u$, where we assume that the imaginary block is colored by q.

Examples



(π, f_1) (π, f_2) (π, f_3) (σ, g)

$b_{2,2}b_{2,1}$ $b_{1,2}b_{2,1}$ $b_{2,1}b_{1,2}$ $b_{1,2}b_{2,2}b_{2,1}$

Convolutions of matrices

Limit distributions can be described in terms of convolutions.

Definition

Convolve matricial semicircle laws

$$[\sigma] = [\sigma(1)] \boxplus [\sigma(2)] \boxplus \ldots \boxplus [\sigma(m)]$$

according to the rule

$$[\mu] \boxplus [\nu] = \begin{cases} \mu_{j,j} \boxplus \nu_{j,j} & \text{if } i = j \\ \mu_{i,j} \uplus \nu_{i,j} & \text{if } i \neq j \end{cases}$$

where $\uplus$ denotes the Boolean convolution.

Non-Hermitian case

Symbolically

In the case when the matrices $Y(u,n)$ are non-Hermitian, variances of $Y_{i,j}(u,n)$ are block-identical and symmetric, then

$$\lim_{n\to\infty} T_{i,j}(u,n) = \eta_{i,j}(u)$$

which is a block refinement of

$$\lim_{n\to\infty} Y(u,n) = \eta(u)$$

under the trace in free probability, where $\eta(u)$ are circular operators.

Other results

Using the Gaussian Symmetric Block Ensemble and matricial freeness, we can

1. find limit distributions of Wishart matrices $B(n)B^*(n)$ for rectangular $B(n)$
2. prove asymptotic freeness of independent Wishart matrices
3. find limit distributions of $B(n)B^*(n)$, where $B(n)$ is a sum or a product of independent rectangular random matrices
4. find a random matrix model for boolean independence, monotone independence and s-freeness
5. find a random matrix model for free Bessel laws (and generalize that result)
6. produce explicit expressions for moments of free multiplicative convolutions of Marchenko-Pastur laws

Embedding products of random matrices

In order to study products of independent random matrices, we embed them as symmetric blocks $T_{j,j+1}(n)$ of one matrix

$$Y(n) = \begin{pmatrix} 0 & S_{1,2} & 0 & \dots & 0 & 0 \\ S_{2,1} & 0 & S_{2,3} & \dots & 0 & 0 \\ 0 & S_{3,2} & 0 & \dots & 0 & 0 \\ . & . & . & \ddots & . & . \\ 0 & 0 & 0 & \dots & 0 & S_{p-1,p} \\ 0 & 0 & 0 & \dots & S_{p,p-1} & 0 \end{pmatrix}$$

built from $S_{j,j+1}(n)$ and $S_{j+1,j}(n)$, where $S_{j,k} = S_{j,k}(n)$.

Limit distribution of products

Theorem

Under the assumptions of identical block variances of symmetric blocks and for any $p \in \mathbb{N}$, let

$$B(n) = T_{1,2}(n) T_{2,3}(n) \ldots T_{p,p+1}(n)$$

for any $n \in \mathbb{N}$. Then, for any $k \in \mathbb{N}$,

$$\lim_{n \to \infty} \tau_1(n) \left((B(n)B^*(n))^k \right) = P_k(d_1, d_2, \ldots, d_{p+1})$$

where $d_1, d_2, \ldots, d_{p+1}$ are asymptotic dimensions and P_k's are some multivariate polynomials.

Multivariate Fuss-Narayana polynomials

Theorem

The polynomials P_k have the form

$$P_k(d_1, \ldots, d_{p+1}) = \sum_{j_1+\ldots+j_{p+1}=pk+1} N(k, j_1, \ldots, j_{p+1})\, d_1^{j_1} d_2^{j_2} \ldots d_{p+1}^{j_{p+1}}$$

and are called multivariate Fuss-Narayana polynomials since their coefficients are given by

$$N(k, j_1, \ldots, j_{p+1}) = \frac{1}{k}\binom{k}{j_1+1}\binom{k}{j_2}\ldots\binom{k}{j_p}.$$

If $p = 1$, we get so-called Narayana polynomials .

Marchenko-Pastur law

Marchenko-Pastur law

The special case of $p = 1$ corresponds to Wishart matrices and the Marchenko-Pastur law with shape parameter $t > 0$, namely

$$\rho_t = \max\{1-t, 0\}\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x}\mathbb{1}_{[a,b]}(x)dx$$

where $a = (1-\sqrt{t})^2$ and $b = (1+\sqrt{t})^2$.

Free convolution of Marchenko-Pastur laws

Corollary 2

If $d_1/d_2 = t_1, d_2/d_3 = t_2, \ldots, d_{p-1}/d_p = t_{p-1}, d_{p+1}/d_p = t_p$, then the moments of the n-fold free convolution of Marchenko-Pastur laws

$$\rho_{t_1} \boxtimes \rho_{t_2} \boxtimes \ldots \boxtimes \rho_{t_n}$$

are given by

$$C_k P_k(d_1, d_2, \ldots, d_{p+1})$$

where $k \in \mathbb{N}$ and C_k's are multiplicative constants.

Consider now the special case of the matricially free Fock space

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2,$$

where

$$\begin{aligned} \mathcal{M}_1 &= \mathbb{C}\Omega_1 \oplus \bigoplus_{k=0}^{\infty} (\mathcal{H}_2^{\otimes k} \otimes \mathcal{H}_1), \\ \mathcal{M}_2 &= \mathbb{C}\Omega_2 \oplus \bigoplus_{k=1}^{\infty} \mathcal{H}_2^{\otimes k}, \end{aligned}$$

and Ω_1, Ω_2 are unit vectors, $\mathcal{H}_j = \mathbb{C}e_j$ for $j \in \{1, 2\}$, where e_1, e_2 are unit vectors.

Use simplified notation

$$\wp_1 = \wp_{2,1}, \quad \wp_2 = \wp_{2,2}$$

for the creation operators associated with constants β_1 and β_2 (squares of previously used $\alpha_{i,j}$) Let

$$\omega_1 = \omega_{2,1}, \quad \omega_2 = \omega_{2,2}$$

be the associated Gaussians.

Moments of free Meixner laws

Theorem

If μ is the free Meixner law corresponding to $(\alpha_1, \alpha_2, \beta_1, \beta_2)$, where $\beta_1 \neq 0$ and $\beta_2 \neq 0$, then its m-th moment is given by

$$M_m(\mu) = \Psi_1((\omega + \gamma)^m),$$

where

$$\omega = \omega_1 + \omega_2$$

and

$$\gamma = (\alpha_2 - \alpha_1)(\beta_1^{-1}\wp_1\wp_1^* + \beta_2^{-1}\wp_2\wp_2^*) + \alpha_1,$$

and Ψ_1 is the state defined by the vector Ω_1.

Free Meixner laws

Consider the sequence of Gaussian Hermitian random matrices $Y(n)$ of the block form

$$Y(n) = \begin{pmatrix} A(n) & B(n) \\ C(n) & D(n) \end{pmatrix}$$

where

1. the sequence $(D(n))$ is *balanced*,
2. the sequence of symmetric blocks built from $(B(n))$ and $(C(n))$ is *unbalanced*,
3. the sequence $(A(n))$ is *evanescent*,

Theorem

Let $\tau_1(n)$ be the partial normalized trace over the set of first N_1 basis vectors and let $\beta_1 = v_{2,1} > 0$ and $\beta_2 = v_{2,2} > 0$ be the variances. Then

$$\lim_{n\to\infty} \tau_1(n)\left((M(n))^m\right) = \Psi_1((\omega+\gamma)^m)$$

where

$$M(n) = Y(n) + \alpha_1 I_1(n) + \alpha_2 I_2(n)$$

for any $n \in \mathbb{N}$, where $I(n) = I_1(n) + I_2(n)$ is the decomposition of the $n \times n$ unit matrix induced by $[n] = N_1 \cup N_2$.

Asymptotic conditional freeness

Theorem

The Free Meixner Ensemble

$$\{M(u,n) : u \in \mathcal{U}, n \in \mathbb{N}\}$$

is asymptotically conditionally free with respect to the pair of partial traces $(\tau_1(n), \tau_2(n))$.

Bibliography

M. Anshelevich, Free martingale polynomials, *J. Funct. Anal.* **201**(2003), 228-261.

F. Benaych-Georges, Rectangular random matrices, related convolution, *Probab. Theory Relat. Fields* **144** (2009), 471-515.

M. Capitaine, M. Casalis, Asymptotic freeness by generalized moments for Gaussian and Wishart matrices. Applications to beta random matrices. *Indiana Univ. Math. J.* **53** (2004), 397-431.

D. Shlyakhtenko, Random Gaussian band matrices and freeness with amalgamation, *Int. Math. Res. Notices* **20** (1996), 1013-1025.

D. Voiculescu, Limit laws for random matrices and free products, *Invent. Math.* **104** (1991), 201-220.

Papers relevant to this talk

R.L., Matricially free random variables, *J. Funct. Anal.* **258** (2010), 4075-4121.

R.L., Asymptotic properties of random matrices and pseudomatrices, *Adv. Math.* **228** (2011), 2403-2440.

R.L., Rafał Sałapata, Multivariate Fuss-Narayana polynomials in random matrix theory, . *Electron. J. Combin.* **20**, Issue 2 (2013).

R.L., Limit distributions of random matrices, *arXiv* (2012).

R.L., Random matrix model for free Meixner laws *arXiv* (2013).