Limit Theorems for Interacting Brownian Motions *

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10 June 2018 (version 2a)

Abstract

Dyson model is a stochastic particle system in one dimension $\mathbb{R}$, in which repulsive force acts between any pair of particles with strength proportional to the inverse of distance. This multivariate stochastic process is realized as the system of one-dimensional Brownian motions conditioned never to collide with each other. We can show that this many-body system is exactly solvable and of determinantal in the sense that any spatio-temporal correlation function is expressed by determinant and is controlled by a single continuous function called the correlation kernel. In this lecture, we assume the special initial configuration such that all N particles are concentrated on the origin and we discuss the limit theorems in $N \to \infty$. Wigner's semicircle law, which is extensively studied in random matrix theory and free probability, is demonstrated as the law of large numbers (LLN), which describes the density profile of particles in $\mathbb{R}$ at each time. Two kinds of limits called the bulk scaling limit and the soft-edge scaling limit are introduced in order to obtain determinantal processes with an infinite number of particles. As the central limit theorem (CLT) associated with the latter scaling limit, the Tracy–Widom distribution is discussed.

Key words: Dyson model; noncolliding Brownian motions; determinantal processes; Fredholm determinants; Wigner's semicircle law; random matrix theory; scaling limits; Tracy-Widom distribution; Painlevé II equation

Contents

*This manuscript was prepared for "Mini Workshop : Modern Theory of Stochastic Particles" in 27-28 June 2018 held at Wrocław University of Technology organized by Jacek Małecki and Piotr Graczyk.

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1 Introduction

1.1 Brownian motion

We consider the motion of a Brownian particle in one-dimensional space $\mathbb{R}$, starting from the origin 0 at time $t = 0$. At each time $t > 0$, the particle position is randomly distributed, and each realization of its path (trajectory) is denoted by ω and called a *sample path* or simply a *path*. Let Ω be the collection of all sample paths and we call it the *sample path space*. The position of the Brownian particle at time $t \geq 0$ in a path $\omega \in \Omega$ is written as $B(t, \omega)$. Usually we omit ω and simply write it as $B(t)$, $t \geq 0$.

We represent each event associated with the process by a subset of Ω, and the collection of all events is denoted by $\mathcal{F}$. The whole sample path space Ω and the empty set $\emptyset$ are in $\mathcal{F}$. For any two sets $\mathsf{A}, \mathsf{B} \in \mathcal{F}$, we assume that $\mathsf{A} \cup \mathsf{B} \in \mathcal{F}$ and $\mathsf{A} \cap \mathsf{B} \in \mathcal{F}$. If $\mathsf{A} \in \mathcal{F}$, then its complement $\mathsf{A}^{\rm c}$ is also in $\mathcal{F}$. It is closed for any infinite sum of events in the sense that, if $\mathsf{A}_n \in \mathcal{F}, n = 1, 2, \dots,$ then $\cup_{n \geq 1} \mathsf{A}_n \in \mathcal{F}$. Such a collection is said to be a *σ-field* (sigma-field). The symbol σ is for 'sum'.

A *probability measure* P is a nonnegative function defined on the σ-field $\mathcal{F}$. Since any element of $\mathcal{F}$ is given by a set as above, any input of P is a set; P is a *set function*. It satisfies the following properties: $\mathrm{P}[\mathsf{A}] \geq 0$ for all $\mathsf{A} \in \mathcal{F}$, $\mathrm{P}[\Omega] = 1$, $\mathrm{P}[\emptyset] = 0$, and if $\mathsf{A}, \mathsf{B} \in \mathcal{F}$ are disjoint, $\mathsf{A} \cap \mathsf{B} = \emptyset$, then $\mathrm{P}[\mathsf{A} \cup \mathsf{B}] = \mathrm{P}[\mathsf{A}] + \mathrm{P}[\mathsf{B}]$. In particular, $\mathrm{P}[\mathsf{A}^{\rm c}] = 1 - \mathrm{P}[\mathsf{A}]$ for all $\mathsf{A} \in \mathcal{F}$. The triplet $(\Omega, \mathcal{F}, \mathrm{P})$ is called the *probability space*.

The smallest σ-field containing all intervals on $\mathbb{R}$ is called the Borel σ-field and denoted by $\mathcal{B}(\mathbb{R})$. A *random variable* or *measurable function* is a real-valued function $f(\omega)$ on Ω such that, for every Borel set $A \in \mathcal{B}(\mathbb{R})$, $f^{-1}(A) \in \mathcal{F}$. Two events A and B are said to be *independent* if $\mathrm{P}[\mathsf{A} \cap \mathsf{B}] = \mathrm{P}[\mathsf{A}]\mathrm{P}[\mathsf{B}]$. Two random variables X and Y are *independent* if the events $\mathsf{A} = \{X : X \in A\}$ and $\mathsf{B} = \{Y : Y \in B\}$ are independent for any $A, B \in \mathcal{B}(\mathbb{R})$.

The *one-dimensional standard Brownian motion*, $\{B(t, \omega) : t \geq 0\}$, has the following three properties:

(BM1) $B(0, \omega) = 0$ with probability one.

(BM2) There is a subset of the sample path space $\widetilde{\Omega} \subset \Omega$, such that $\mathrm{P}[\widetilde{\Omega}] = 1$ and for any $\omega \in \widetilde{\Omega}$, $B(t, \omega)$ is a real continuous function of t. We say that $B(t)$ has a *continuous path* almost surely (a.s., for short).

(BM3) For an arbitrary $M \in \mathbb{N} \equiv \{1, 2, 3, \dots\}$, and for any sequence of times, $t_0 \equiv 0 < t_1 < \cdots < t_M$, the increments $B(t_m) - B(t_{m-1}), m = 1, 2, \dots, M$, are independent, and each increment is in *the normal distribution* (*the Gaussian distribution*) with mean 0 and variance $\sigma^2 = t_m - t_{m-1}$. It means that for any $1 \leq m \leq M$ and $a < b$,

$$\mathrm{P}\Big[B(t_m) - B(t_{m-1}) \in [a, b]\Big] = \int_a^b p(t_m - t_{m-1}, z|0) dz,$$

where we define for $x, y \in \mathbb{R}$

$$p(t, y|x) = \begin{cases} \dfrac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}, & \text{for } t > 0, \\ \delta(x-y), & \text{for } t = 0. \end{cases} \tag{1.1}$$

Unless otherwise noted, the one-dimensional standard Brownian motion is simply abbreviated to BM in this lecture note. The probability measure P for the BM is called the *Wiener measure*. The expectation with respect to the probability measure P is denoted by E. We write the conditional probability as $\mathrm{P}[\,\cdot\,|C]$, where C denotes the condition. The conditional expectation is similarly written as $\mathrm{E}[\,\cdot\,|C]$.

The third property **(BM3)** given above implies that for any $0 \leq s \leq t < \infty$

$$\mathrm{P}\Big[B(t) \in A \Big| B(s) = x\Big] = \int_A p(t-s, y|x) dy \tag{1.2}$$

holds, ${}^\forall A \in \mathcal{B}(\mathbb{R}), {}^\forall x \in \mathbb{R}$. Therefore the integral kernel $p(t, y|x)$ given by (1.1) is called the *transition probability density function* of Brownian motion starting from x. The probability that the BM is observed in a region $A_m \in \mathcal{B}(\mathbb{R})$ at time t_m for each $m = 1, 2, \dots, M$ is then given by

$$\mathrm{P}\Big[B(t_m) \in A_m, m = 1, 2, \dots, M\Big] = \int_{A_1} dx_1 \cdots \int_{A_M} dx_M \prod_{m=1}^{M} p(t_m - t_{m-1}, x_m | x_{m-1}), \tag{1.3}$$

where $x_0 \equiv 0$.

By **(BM3)**, we can see that, for any $c > 0$, the probability distribution of $B(c^2 t)/c$ is equivalent to that of $B(t)$ at arbitrary time $t \geq 0$. It is written as

$$\frac{1}{c} B(c^2 t) \stackrel{\mathrm{d}}{=} B(t), \quad {}^\forall c > 0,$$

where the symbol $\stackrel{\mathrm{d}}{=}$ is for *equivalence in distribution*. Moreover, (1.3) implies that, for any $c > 0$, $B(t)$, $t \geq 0$ and its time-changed process with $t \mapsto c^2 t$ multiplied by a factor $1/c$

(dilatation) follow the same probability law. This *equivalence in probability law* of stochastic processes is expressed as

$$(B(t))_{t\geq 0} \stackrel{\text{(law)}}{=} \left(\frac{1}{c}B(c^2t)\right)_{t\geq 0}, \quad {}^\forall c \geq 0, \tag{1.4}$$

and called the *scaling property of Brownian motion.*

For $a > 0$, let $T_a = \inf\{t \geq 0 : B(t) = a\}$. Then for any $t \geq 0$,

$$\mathrm{P}[T_a < t, B(t) < a] = \mathrm{P}[T_a < t, B(t) > a], \tag{1.5}$$

since the transition probability density (1.1) is a symmetric function of the increment $y - x$. This property is called the *reflection principle* of BM. For $\{\omega : B(t) > a\} \subset \{\omega : T_a < t\}$, $a > 0$, the above is equal to $\mathrm{P}[B(t) > a]$.

The formula (1.3) also means that for any fixed $s \geq 0$, under the condition that $B(s)$ is given, $\{B(t) : t \leq s\}$ and $\{B(t) : t > s\}$ are independent. This independence of the events in the future and those in the past is called the *Markov property.* A positive random variable τ is called *stopping time* (or *Markov time*), if the event $\{\omega : \tau \leq t\}$ is determined by the behavior of the process until time t and independent of that after t. For any stopping time τ, $\{B(t) : t \leq \tau\}$ and $\{B(t) : t > \tau\}$ are independent. It is called the *strong Markov property.* A stochastic process which has the strong Markov property and has a continuous path almost surely is generally called a *diffusion process.*

For each time $t \in [0, \infty)$, we write the smallest σ-field generated by the BM up to time $t \geq 0$ as $\sigma(B(s) : 0 \leq s \leq t)$ and define

$$\mathcal{F}_t \equiv \sigma(B(s) : 0 \leq s \leq t), \quad t \geq 0. \tag{1.6}$$

By definition, with respect to any event in $\mathcal{F}_t$, $B(s)$ is measurable at every $s \in [0, t]$. Then we have a nondecreasing family $\{\mathcal{F}_t : t \geq 0\}$ of sub-σ-fields of the original σ-field $\mathcal{F}$ in the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ such that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $0 \leq s < t < \infty$. We call this family of σ-fields a *filtration.*

The BM started at $x \in \mathbb{R}$, which is denoted by $B^x(t),\ t \geq 0$, is defined by

$$B^x(t) = x + B(t), \quad x \in \mathbb{R}, \quad t \geq 0. \tag{1.7}$$

We define $\mathrm{P}^x[B(t) \in \cdot\,] = \mathrm{P}[B^x(t) \in \cdot\,]$ and $\mathrm{E}^x[f(B(t))] = \mathrm{E}[f(B^x(t))]$ for any bounded measurable function f, $t \geq 0$. The stopping time τ mentioned above can be defined using the notion of filtration as follows: $\{\omega : \tau \leq t\} \in \mathcal{F}_t, {}^\forall t \geq 0$. The strong Markov property of BM is now expressed as

$$\mathrm{E}[f(B(s+t))|\mathcal{F}_s] = \mathrm{E}^{B(s)}[f(B(t))], \quad t \geq 0, \quad \text{a.s.}, \tag{1.8}$$

provided that $s \geq 0$ is any realization of a stopping time τ and f is an arbitrary measurable bounded function.

Since the probability density of increment in any time interval $t-s>0$, $p(t-s,z|0)$, has mean zero, BM satisfies the equality

$$\mathrm{E}[B(t)|\mathcal{F}_s]=B(s), \quad 0\le s<t<\infty, \quad \text{a.s.} \tag{1.9}$$

That is, the mean is constant in time, even though the variance increases in time as $\sigma^2=t$. Processes with such a property are called *martingales*. We note that for $0\le s<t<\infty$,

$$\begin{aligned}\mathrm{E}[B(t)^2|\mathcal{F}_s] &= \mathrm{E}[(B(t)-B(s))^2+2(B(t)-B(s))B(s)+B(s)^2|\mathcal{F}_s]\\ &= \mathrm{E}[(B(t)-B(s))^2|\mathcal{F}_s]+2\mathrm{E}[(B(t)-B(s))B(s)|\mathcal{F}_s]+\mathrm{E}[B(s)^2|\mathcal{F}_s].\end{aligned}$$

By the property **(BM3)** and the definition of $\mathcal{F}_s$,

$$\begin{aligned}&\mathrm{E}[(B(t)-B(s))^2|\mathcal{F}_s]=t-s,\\ &\mathrm{E}[(B(t)-B(s))B(s)|\mathcal{F}_s]=\mathrm{E}[B(t)-B(s)|\mathcal{F}_s]\,B(s)=0,\\ &\mathrm{E}[B(s)^2|\mathcal{F}_s]=B(s)^2.\end{aligned}$$

Then we have the equality

$$\mathrm{E}[B(t)^2-t|\mathcal{F}_s]=B(s)^2-s, \quad 0\le s<t<\infty, \quad \text{a.s.} \tag{1.10}$$

It means that $B(t)^2-t$ is a martingale.

For the transition probability density of BM (1.1), it should be noted that $p(\cdot,y|x)=p(\cdot,x|y)$ for any $x,y\in\mathbb{R}$, and $u_t(x)\equiv p(t,y|x)$ is a unique solution of the *heat equation* (*diffusion equation*)

$$\frac{\partial}{\partial t}u_t(x)=\frac{1}{2}\frac{\partial^2}{\partial x^2}u_t(x), \quad x\in\mathbb{R}, \quad t\ge 0 \tag{1.11}$$

with the initial condition $u_0(x)=\delta(x-y)$. The solution of (1.11) with the initial condition $u_0^f(x)=f(x),\ x\in\mathbb{R}$ is then given by

$$u_t^f(x)=\mathrm{E}^x[f(B(t))]=\int_{-\infty}^{\infty}f(y)p(t,y|x)dy, \tag{1.12}$$

if f is a measurable function satisfying the condition $\int_{-\infty}^{\infty}e^{-ax^2}|f(x)|dx<\infty$ for some $a>0$. Since $p(t,y|x)$ plays as an integral kernel in (1.12), it is also called the *heat kernel*.

For $0\le s<t<\infty$, $\xi\in\mathbb{R}$, consider $\mathrm{E}[e^{\sqrt{-1}\xi(B(t)-B(s))}|\mathcal{F}_s]$. Using p, it is calculated as follows:

$$\begin{aligned}\int_{-\infty}^{\infty}e^{\sqrt{-1}\xi z}p(t-s,z|0)dz &= \int_{-\infty}^{\infty}e^{\sqrt{-1}\xi z}\frac{e^{-z^2/2(t-s)}}{\sqrt{2\pi(t-s)}}dz\\ &= e^{-\xi^2(t-s)/2}.\end{aligned}$$

The obtained function of $\xi\in\mathbb{R}$,

$$\mathrm{E}[e^{\sqrt{-1}\xi(B(t)-B(s))}|\mathcal{F}_s]=e^{-\xi^2(t-s)/2}, \quad 0\le s<t<\infty, \tag{1.13}$$

is called the *characteristic function* of BM.

1.2 Karlin-McGregor-Lindström-Gessel-Viennot formula

Let $N \in \mathbb{N}$ denote the spatial dimension. For $N \geq 2$, the N-dimensional BM in $\mathbb{R}^N$ starting from the position $\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ is defined by the N-dimensional vector-valued diffusion process,

$$\boldsymbol{B}^{\boldsymbol{x}}(t) = (B_1^{x_1}(t), B_2^{x_2}(t), \dots, B_N^{x_N}(t)), \quad t \geq 0, \tag{1.14}$$

where $\{B_i^{x_i}(t)\}_{i=1}^N$, $t \geq 0$ are independent one-dimensional BMs. Consider a subspace of $\mathbb{R}^N$ defined by

$$\mathbb{W}_N^{\mathrm{A}} \equiv \{\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\}, \tag{1.15}$$

which is called the *Weyl chamber of type* A_{N-1} in representation theory.

As a multidimensional extension of the absorbing Brownian motion in $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ with an absorbing wall at $x = 0$ (see, for instance, [4]), we consider the *absorbing Brownian motion* $\boldsymbol{B}^{\boldsymbol{x}}(t) = (B_1^{x_1}(t), \dots, B_N^{x_N}(t))$ in $\mathbb{W}_N^{\mathrm{A}}$. The starting point $\boldsymbol{x}$ is assumed to be in $\mathbb{W}_N^{\mathrm{A}}$. We put absorbing walls at the boundaries of $\mathbb{W}_N^{\mathrm{A}}$. When $\boldsymbol{B}^{\boldsymbol{x}}$ hits any of the walls, it is absorbed and the process is stopped. In other words, the Brownian motion $\boldsymbol{B}^{\boldsymbol{x}}$ started at $\boldsymbol{x} \in \mathbb{W}_N^{\mathrm{A}}$ is killed when it arrives at the boundaries of $\mathbb{W}_N^{\mathrm{A}}$.

We define $q_N(t, \boldsymbol{y}|\boldsymbol{x})$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N^{\mathrm{A}},\ t \geq 0$ as the probability density for the event such that this absorbing Brownian motion starting from $\boldsymbol{x}$ at time $t = 0$ 'survives' up to time t and arrives at $\boldsymbol{y}$ at the time t. Note that the boundaries of $\mathbb{W}_N^{\mathrm{A}}$ are the hyperplanes $x_i = x_j, 1 \leq i < j \leq N$ in $\mathbb{R}^N$. Then, if we interpret $\boldsymbol{x} \in \mathbb{R}^N$ as a configuration of N particles on a line $\mathbb{R}$, this absorbing Brownian motion in $\mathbb{W}_N^{\mathrm{A}}$ can be regarded as an N-particle system such that each particle executes BM when distances between neighboring particles are positive, but when any two neighboring particles collide, the process is stopped. This process is a continuum limit (diffusion scaling limit) [26, 27] of the *vicious walker model* on $\mathbb{Z}$ introduced by Fisher [9] (see also [6]).

The following is known as the *Karlin–McGregor formula* [22]. Note that the discrete analogue is known as the *Lindström–Gessel–Viennot formula* [32, 12].

Lemma 1.1 *The transition probability density of the absorbing Brownian motion in* $\mathbb{W}_N^{\mathrm{A}}$ *is given by*

$$\begin{aligned} q_N(t, \boldsymbol{y}|\boldsymbol{x}) &= \det_{1 \leq i,j \leq N} \Big[p(t, y_i|x_j) \Big] \\ &= \sum_{\sigma \in \mathcal{S}_N} \mathrm{sgn}(\sigma) \prod_{i=1}^N p(t, y_{\sigma(i)}|x_i), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N^{\mathrm{A}}, \quad t \geq 0, \end{aligned} \tag{1.16}$$

where $\mathcal{S}_N$ *denotes the set of all permutations of* N *indices* $\{1, 2, \dots, N\}$.

Proof By the property **(BM2)** and the definition of the transition probability density of BM, $p(t, y|x)$ gives the total probability mass of the Brownian path $\pi[0, t]$ from x to y with time duration t. Let $\Pi_t(x, y)$ denote the collection of all Brownian paths from $x \in \mathbb{R}$ to $y \in \mathbb{R}$ with time duration $t \geq 0$. We will interpret (1.16) as a generating function for $(N+1)$-tuples,

$(\sigma, \pi_1, \dots, \pi_N)$, where $\sigma \in \mathcal{S}_N$, $\pi_i = \pi_i[0,t] \in \Pi_t(x_i, y_{\sigma(i)})$, $1 \leq i \leq N$. Under the assumption $\boldsymbol{x} \in \mathbb{W}_N^{\rm A}$, let

$$\tau = \inf\{t > 0 : \boldsymbol{B}^{\boldsymbol{x}}(t) \notin \mathbb{W}_N^{\rm A}\}. \tag{1.17}$$

Assume that $\tau < t$ and $B_k^{x_k}(\tau) = B_\ell^{x_\ell}(\tau)$. For a pair of paths (π_k, π_ℓ), we define (π'_k, π'_ℓ) by exchanging the Brownian paths of π_k, π_ℓ after $t = \tau$:

$$\pi'_k[0,t] = \pi_k[0,\tau] \cup \pi_\ell(\tau, t], \quad \pi'_\ell[0,t] = \pi_\ell[0,\tau] \cup \pi_k(\tau, t].$$

We define $\pi'_i = \pi_i$ for $i \neq k, \ell$ and $\sigma' = \sigma \circ \sigma_{k\ell}$, where $\sigma_{k\ell}$ denotes the exchange of k and ℓ. Then the operation $(\sigma, \pi_1, \dots, \pi_N) \mapsto (\sigma', \pi'_1, \dots, \pi'_N)$ is an involution. By this operation, the absolute value of the contribution to the generating function (1.16) is not changed because of the strong Markov property (1.8) and the reflection principle of BM (1.5), but the sign is changed. So the contribution of any such pairs $\{(\sigma, \pi_1, \dots, \pi_N), (\sigma', \pi'_1, \dots, \pi'_N)\}$ is canceled out. The remaining non-zero contributions in (1.16) are from N-tuples of nonintersecting Brownian paths. Since $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N^{\rm A}$, $\sigma = \text{id}$ and so $\text{sgn}(\sigma) = \text{sgn}(\text{id}) = 1$ for nonintersecting paths. Hence (1.16) gives the total probability mass of N-tuples of nonintersecting Brownian paths from $\boldsymbol{x}$ to $\boldsymbol{y}$ with time duration t and is identified with $q_N(t, \boldsymbol{y}|\boldsymbol{x})$ for the absorbing Brownian motion in $\mathbb{W}_N^{\rm A}$. ∎

1.3 Dyson model as noncolliding Brownian motions

For an initial configuration $\boldsymbol{x} \in \mathbb{W}_N^{\rm A}$, the survival probability of the absorbing Brownian motion in $\mathbb{W}_N^{\rm A}$ up to time $t \geq 0$ is then given by

$$\mathrm{P}^{\boldsymbol{x}}[\tau > t] = \int_{\mathbb{W}_N^{\rm A}} q_N(t, \boldsymbol{y}|\boldsymbol{x}) d\boldsymbol{y}, \quad t \geq 0, \tag{1.18}$$

where $d\boldsymbol{y} = \prod_{i=1}^N dy_i$.

Now we consider an N-particle system of BMs in $\mathbb{R}$ conditioned never to collide with each other, that is, they do not collide even during the time interval (t, ∞). We simply call this process the *noncolliding Brownian motion* with N particles. The *transition probability density function* $p_N(t, \boldsymbol{y}|\boldsymbol{x})$ of this process from $\boldsymbol{x} \in \mathbb{W}_N^{\rm A}$ to $\boldsymbol{y} \in \mathbb{W}_N^{\rm A}$ with time duration $t \geq 0$ should be obtained by the following limit,

$$p_N(t, \boldsymbol{y}|\boldsymbol{x}) = \lim_{T \to \infty} \frac{q_N(t, \boldsymbol{y}|\boldsymbol{x}) \mathrm{P}^{\boldsymbol{y}}[\tau > T - t]}{\mathrm{P}^{\boldsymbol{x}}[\tau > T]}. \tag{1.19}$$

Let

$$h_N(\boldsymbol{x}) = \prod_{1 \leq i < j \leq N} (x_j - x_i). \tag{1.20}$$

Then the following are proved.

Proposition 1.2 (i) *The transition probability density of the noncolliding Brownian motion with N particles is given by*

$$p_N(t,\boldsymbol{y}|\boldsymbol{x}) = \frac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})} q_N(t,\boldsymbol{y}|\boldsymbol{x}), \quad \boldsymbol{x},\boldsymbol{y} \in \mathbb{W}_N^{\rm A}, \quad t \geq 0, \tag{1.21}$$

where q_N *is the Karlin–McGregor determinant (1.16).*

(ii) *Let* $|\boldsymbol{x}| = \sqrt{\sum_{i=1}^N x_i^2}$. *Then*

$$\begin{aligned} p_N(t,\boldsymbol{y}|\boldsymbol{0}) &\equiv \lim_{|\boldsymbol{x}|\to 0} p_N(t,\boldsymbol{y}|\boldsymbol{x}) \\ &= \frac{t^{-N(N-1)/2}}{\prod_{n=1}^{N-1} n!} h_N(\boldsymbol{y})^2 \prod_{i=1}^N p(t,y_i|0), \quad \boldsymbol{y} \in \mathbb{W}_N^{\rm A}, \quad t \geq 0. \end{aligned} \tag{1.22}$$

Proof See Section 3.3 in [24].

Denote the N-dimensional Laplacian with respect to the variables $\boldsymbol{x} = (x_1,\dots,x_N)$ by

$$\Delta^{(N)} \equiv \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}.$$

Provided $\boldsymbol{x},\boldsymbol{y} \in \mathbb{W}_N^{\rm A}$, we can verify that (1.21) satisfies the following partial differential equation (PDE),

$$\frac{\partial}{\partial t} p_N(t,\boldsymbol{y}|\boldsymbol{x}) = \frac{1}{2}\Delta^{(N)} p_N(t,\boldsymbol{y}|\boldsymbol{x}) + \sum_{\substack{1\leq i,j\leq N,\\ i\neq j}} \frac{1}{x_i - x_j}\frac{\partial}{\partial x_i} p_N(t,\boldsymbol{y}|\boldsymbol{x}) \tag{1.23}$$

with the initial condition $p_N(0,\boldsymbol{y}|\boldsymbol{x}) = \delta(\boldsymbol{y}-\boldsymbol{x}) \equiv \prod_{i=1}^N \delta(y_i - x_i)$. This PDE can be regarded as the *backward Kolmogorov equation* of the stochastic process with N particles, $\boldsymbol{X}(t) = (X_1(t),\dots,X_N(t))$, which solves the system of stochastic differential equations (SDEs),

$$dX_i(t) = dB_i^{x_i}(t) + \sum_{\substack{1\leq j\leq N,\\ j\neq i}} \frac{dt}{X_i(t) - X_j(t)}, \quad t \geq 0, \quad 1 \leq i \leq N. \tag{1.24}$$

In 1962 Dyson [8] introduced N-particle systems of *interacting Brownian motions* in $\mathbb{R}$ as a solution $\boldsymbol{X}(t) = (X_1(t), X_2(t),\dots,X_N(t))$ of the following system of SDEs: with $\beta > 0$ and the condition $\boldsymbol{x} = (x_1,x_2,\dots,x_N) \in \mathbb{W}_N^{\rm A}$ for initial positions $x_i = X_i(0), 1 \leq i \leq N$,

$$dX_i(t) = dB_i(t) + \frac{\beta}{2}\sum_{\substack{1\leq j\leq N,\\ j\neq i}} \frac{dt}{X_i(t) - X_j(t)}, \quad t \in [0, T^{\boldsymbol{x}}), \quad 1 \leq i \leq N, \tag{1.25}$$

where $\{B_i(t)\}_{i=1}^N$, $t \geq 0$ are independent BMs and

$$\begin{aligned} T_{ij}^{\boldsymbol{x}} &= \inf\{t > 0 : X_i(t) = X_j(t)\}, \quad 1 \leq i < j \leq N, \\ T^{\boldsymbol{x}} &= \min_{1 \leq i < j \leq N} T_{ij}^{\boldsymbol{x}}. \end{aligned}$$

It is called *Dyson's Brownian motion model with parameter* β [33, 11, 2, 1, 44]. We can prove that, for any $\boldsymbol{x} \in \mathbb{W}_N^{\rm A}$, $T^{\boldsymbol{x}} < \infty$ if $\beta < 1$, and $T^{\boldsymbol{x}} = \infty$ if $\beta \geq 1$ [38, 14]. [1]

The present system (1.24) is identified with Dyson's BM model with parameter $\beta = 2$. We call this special case simply *the Dyson model* [43, 34, 13, 20]. Then the equivalence between the Dyson model and the noncolliding Brownian motion is proved [13].

Theorem 1.3 *The noncolliding Brownian motion* $\boldsymbol{X}(t), t \geq 0$ *is equivalent in probability law with the Dyson model. Its transition probability is given by*

$$\begin{aligned} \mathrm{Prob}(\boldsymbol{X}(t) \in d\boldsymbol{y} | \boldsymbol{X}(s) = \boldsymbol{x}) &= p_N(t-s, \boldsymbol{y}|\boldsymbol{x}) d\boldsymbol{y} \\ &= \frac{h_N(\boldsymbol{y})}{h_N(\boldsymbol{x})} \det_{1 \leq i,j \leq N} \Big[p(t-s, y_i|x_j) \Big] d\boldsymbol{y}, \end{aligned} \tag{1.26}$$

for $0 < s < t < \infty$, $\boldsymbol{x} = (x_1, \dots, x_N), \boldsymbol{y} = (y_1, \dots, y_N) \in \mathbb{W}_N^{\rm A}$.

We can see that

$$\Delta^{(N)} h_N(\boldsymbol{x}) = 0. \tag{1.27}$$

and

$$h_N(\boldsymbol{x}) > 0, \text{ if } \boldsymbol{x} \in \mathbb{W}_N^{\rm A}, \quad \text{and} \quad h_N(\boldsymbol{x}) = 0, \text{ if } \boldsymbol{x} \in \partial \mathbb{W}_N^{\rm A}. \tag{1.28}$$

Proposition 1.2 (i) states that the Dyson model (1.26) is the h-transformation of the absorbing Brownian motion in $\mathbb{W}_N^{\rm A}$ [7, 13], in which the harmonic function is given by

$$h_N(\boldsymbol{x}) = \prod_{1 \leq i < j \leq N} (x_j - x_i).$$

Therefore, at any positive time $t > 0$ the configuration is an element of $\mathbb{W}_N^{\rm A}$,

$$\boldsymbol{X}(t) = (X_1(t), X_2(t), \dots, X_N(t)) \in \mathbb{W}_N^{\rm A}, \quad t > 0, \tag{1.29}$$

and hence there are no multiple points at which coincidence of particle positions, $X_i(t) = X_j(t), i \neq j$, occurs.

We can consider, however, the Dyson model starting from initial configurations with multiple points. In order to describe configurations with multiple points, we represent each particle configuration by a sum of delta measures in the form

$$\xi(\cdot) = \sum_{i \in \mathbb{I}} \delta_{x_i}(\cdot) \tag{1.30}$$

[1] The existence of a strong and pathwise unique noncolliding solution of SDEs (1.25) for general initial conditions $x_1 \leq x_2 \leq \cdots \leq x_N$ was conjectured by Rogers and Shi [38]. It was proved by Cépa and Lépingle [5] using multivalued SDE theory and by Graczyk and Małecki [15] by classical Itô calculus. See also [16, 17].

with a sequence of points in $\mathbb{R}$, $\boldsymbol{x} = (x_i)_{i\in\mathbb{I}}$, where $\mathbb{I}$ is a countable index set. Here for $y \in \mathbb{R}$, $\delta_y(\cdot)$ denotes the delta measure such that $\delta_y(\{x\}) = 1$ for $x = y$ and $\delta_y(\{x\}) = 0$ otherwise. Then, for (1.30) and $A \subset \mathbb{R}$, $\xi(A) = \int_A \xi(dx) = \sum_{i\in\mathbb{I}:x_i\in A} 1 = \sharp\{x_i, x_i \in A\}$.

The measures of the form (1.30) satisfying the condition $\xi(K) < \infty$ for any compact subset $K \subset \mathbb{R}$ are called the *nonnegative integer-valued Radon measures* on $\mathbb{R}$ and we denote the space they form by $\mathfrak{M}$. The set of configurations without multiple points is denoted by $\mathfrak{M}_0 = \{\xi \in \mathfrak{M} : \xi(\{x\}) \leq 1, {}^\forall x \in \mathbb{R}\}$. There is a trivial correspondence between $\mathbb{W}_N^{\rm A}$ and $\mathfrak{M}_0$. We call $\boldsymbol{x} \in \mathbb{R}^N$ a *labeled configuration* and $\xi \in \mathfrak{M}$ an *unlabeled configuration.*

We consider the Dyson model as an $\mathfrak{M}$-valued diffusion process,

$$\Xi(t, \cdot) = \sum_{i=1}^N \delta_{X_i(t)}(\cdot), \quad t \geq 0, \tag{1.31}$$

starting from the initial configuration

$$\xi = \sum_{i=1}^N \delta_{x_i}, \tag{1.32}$$

where $\boldsymbol{X}(t) = (X_1(t), \cdots, X_N(t))$ is the solution of (1.24) under the initial configuration $\boldsymbol{x} = (x_1, \dots, x_N) \in \overline{\mathbb{W}_N^{\rm A}}$. We write the process as $(\Xi, \mathbb{P}^\xi)$ and express the expectation with respect to the probability law $\mathbb{P}^\xi$ of the Dyson model by $\mathbb{E}^\xi[\,\cdot\,]$. We introduce a filtration $\{(\mathcal{F}_\Xi)_t\}_{t\in[0,\infty)}$ on the space of continuous paths $\mathrm{C}([0,\infty) \to \mathfrak{M})$ defined by $(\mathcal{F}_\Xi)_t = \sigma(\Xi(s), s \in [0,t])$, where σ denotes the smallest σ-field.

In order to characterize the process, we consider the Laplace transformations of the multitime joint distribution functions of $(\Xi, \mathbb{P}^\xi)$. For any integer $M \in \mathbb{N}$, a sequence of times $\boldsymbol{t} = (t_1, \dots, t_M) \in [0,\infty)^M$ with $0 \leq t_1 < \cdots < t_M < \infty$, and a sequence of functions $\boldsymbol{f} = (f_{t_1}, \dots, f_{t_M}) \in \mathrm{C}_{\rm c}(\mathbb{R})^M$, let

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] \equiv \mathbb{E}^\xi\left[\exp\left\{\sum_{m=1}^M \int_{\mathbb{R}} f_{t_m}(x)\Xi(t_m, dx)\right\}\right]. \tag{1.33}$$

By (1.31), if we set test functions as

$$\chi_{t_m}(\cdot) = e^{f_{t_m}(\cdot)} - 1, \quad 1 \leq m \leq M, \tag{1.34}$$

we can rewrite (1.33) in the form

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \mathbb{E}^\xi\left[\prod_{m=1}^M \prod_{i=1}^N \{1 + \chi_{t_m}(X_i(t_m))\}\right]. \tag{1.35}$$

We expand this with respect to test functions and define the spatio-temporal correlation functions $\{\rho_\xi\}$ as coefficients,

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \sum_{\substack{0\leq N_m\leq N,\\ 1\leq m\leq M}} \int_{\prod_{m=1}^M \mathbb{W}_{N_m}^{\rm A}} \prod_{m=1}^M d\boldsymbol{x}_{N_m}^{(m)} \prod_{i=1}^{N_m} \chi_{t_m}\Big(x_i^{(m)}\Big)\rho_\xi\Big(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\Big), \tag{1.36}$$

where $\boldsymbol{x}_{N_m}^{(m)}$ denotes $(x_1^{(m)}, \dots, x_{N_m}^{(m)})$ and $d\boldsymbol{x}_{N_m}^{(m)} = \prod_{i=1}^{N_m} dx_i^{(m)}$, $1 \leq m \leq M$. Here the empty products equal 1 by convention and the term with $N_m = 0, 1 \leq \forall m \leq M$ is considered to be 1. The function $\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}]$ is a *generating function of correlation functions*.

1.4 Fredholm determinants and determinantal processes

Given an integral kernel, $\boldsymbol{K}(s, x; t, y)$, $(s, x), (t, y) \in [0, \infty) \times \mathbb{R}$, and a sequence of functions $(\chi_{t_1}, \dots, \chi_{t_M}) \in \mathrm{C_c}(\mathbb{R})^M$, $M \in \mathbb{N}$, the *Fredholm determinant* associated with $\boldsymbol{K}$ and $(\chi_{t_m})_{m=1}^M$ is defined as

$$\begin{aligned} &\mathop{\mathrm{Det}}_{\substack{(s,t)\in\{t_1,\dots,t_M\}^2,\\(x,y)\in\mathbb{R}^2}} \Big[\delta_{st}\delta(x-y) + \boldsymbol{K}(s,x;t,y)\chi_t(y)\Big] \\ &= \sum_{\substack{0\leq N_m\leq N,\\1\leq m\leq M}} \int_{\prod_{m=1}^M \mathbb{W}_{N_m}^{\mathrm{A}}} \prod_{m=1}^M d\boldsymbol{x}_{N_m}^{(m)} \prod_{k=1}^{N_m} \chi_{t_m}\Big(x_k^{(m)}\Big) \det_{\substack{1\leq i\leq N_m, 1\leq j\leq N_n,\\1\leq m,n\leq M}} \Big[\boldsymbol{K}(t_m, x_i^{(m)}; t_n, x_j^{(n)})\Big]. \end{aligned} \tag{1.37}$$

If we consider the simplest case where $M = 1$ and $t_1 = t \in [0, \infty)$ in (1.37), we have

$$\mathop{\mathrm{Det}}_{(x,y)\in\mathbb{R}^2} \Big[\delta(x-y) + \boldsymbol{K}(t,x;t,y)\chi_t(y)\Big] = \sum_{N'=0}^{N} \int_{\mathbb{W}_{N'}^{\mathrm{A}}} d\boldsymbol{x}_{N'} \prod_{k=1}^{N'} \chi_t(x_k) \det_{1\leq i,j\leq N'} [\boldsymbol{K}(t, x_i; t, x_j)].$$

Given $\boldsymbol{v} = (v_1, \dots, v_N) \in \mathbb{W}_N^{\mathrm{A}}$, put $\chi_t(x) = \sum_{\ell=1}^N \widehat{\chi}_\ell \delta_{v_\ell}(x)$ with $\widehat{\chi}_\ell \in \mathbb{R}, 1 \leq \ell \leq N$. In this case the above is equal to

$$\sum_{N'=0}^{N} \sum_{\mathbb{J}\subset\mathbb{I}_N, \sharp\mathbb{J}=N'} \prod_{k\in\mathbb{J}} \widehat{\chi}_k \det_{i,j\in\mathbb{J}} [K_{ij}]$$

with $K_{ij} = \boldsymbol{K}(t, v_i; t, v_j), 1 \leq i, j \leq N$, where we write $\mathbb{J} \subset \mathbb{I}_N$, $\sharp\mathbb{J} = N'$, if $\mathbb{J} = \{j_1, \dots, j_{N'}\}, 1 \leq j_1 < \cdots < j_{N'} \leq N$. This is obtained as the *Fredholm expansion formula* of $\det_{1\leq i,j\leq N}[\delta_{ij} + K_{ij}\widehat{\chi}_j]$, that is,

$$\begin{aligned} \det_{1\leq i,j\leq N} \Big[\delta_{ij} + K_{ij}\widehat{\chi}_j\Big] &= \sum_{N'=0}^{N} \sum_{\mathbb{J}\subset\mathbb{I}_N, \sharp\mathbb{J}=N'} \prod_{k\in\mathbb{J}} \widehat{\chi}_k \det_{i,j\in\mathbb{J}} [K_{ij}] \\ &= 1 + \sum_{1\leq i\leq N} \widehat{\chi}_i K_{ii} + \sum_{1\leq i<j\leq N} \widehat{\chi}_i\widehat{\chi}_j \begin{vmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{vmatrix} \\ &\quad + \sum_{1\leq i<j<k\leq N} \widehat{\chi}_i\widehat{\chi}_j\widehat{\chi}_k \begin{vmatrix} K_{ii} & K_{ij} & K_{ik} \\ K_{ji} & K_{jj} & K_{jk} \\ K_{ki} & K_{kj} & K_{kk} \end{vmatrix} + \cdots . \end{aligned} \tag{1.38}$$

For this reason, (1.37) is called the Fredholm determinant. See, for instance, Chapter 21 in [33], Chapter 9 in [11], and Chapter 3 in [2] for more details of Fredholm determinants.

Definition 1.4 *If any moment generating function (1.33) is given by a Fredholm determinant, the process* $(\Xi, \mathbb{P}^{\xi})$ *is said to be determinantal. In this case all spatio-temporal correlation functions are given by determinants as*

$$\rho_{\xi}\Big(t_1, \boldsymbol{x}^{(1)}_{N_1}; \dots; t_M, \boldsymbol{x}^{(M)}_{N_M}\Big) = \det_{\substack{1 \le i \le N_m, 1 \le j \le N_n, \\ 1 \le m, n \le M}} \Big[\mathbb{K}_{\xi}(t_m, x^{(m)}_i; t_n, x^{(n)}_j)\Big], \qquad (1.39)$$

$0 \le t_1 < \cdots < t_M < \infty$, $1 \le N_m \le N$, $\boldsymbol{x}^{(m)}_{N_m} \in \mathbb{R}^{N_m}, 1 \le m \le M \in \mathbb{N}$. *Here the integral kernel,* $\mathbb{K}_{\xi} : ([0,\infty) \times \mathbb{R})^2 \mapsto \mathbb{R}$, *is a function of the initial configuration* ξ *and is called the correlation kernel.*

Remark 1.1 If the process $(\Xi, \mathbb{P}^{\xi})$ is determinantal, then, for each specified time $0 \le t < \infty$, all spatial correlation functions are given by determinants as

$$\rho_{\xi}(\boldsymbol{x}_{N'}) = \det_{1 \le i,j \le N'} [\mathrm{K}(x_i, x_j)], \quad 1 \le N' \le N, \qquad (1.40)$$

with $\mathrm{K}(x,y) = \mathbb{K}_{\xi}(t,x;t,y)$. In general a random integer-valued Radon measure in $\mathfrak{M}$ (resp. $\mathfrak{M}_0$) is called a *point process* (resp. *simple point process*). A simple point process is said to be a *determinantal point process* (DPP) (or *Fermion point process*) with kernel K, if its spatial correlation functions exist and are given in the form (1.40). When K is symmetric, *i.e.*, $\mathrm{K}(x,y) = \mathrm{K}(y,x)$, $x,y \in \mathbb{R}$, Soshnikov [42] and Shirai and Takahashi [41] gave sufficient conditions for K to be a correlation kernel of a determinantal point process (see also [19, 2, 1]). The notion of determinantal process given by Definition 1.4 is a dynamical extension of the determinantal point process [3, 29].

The following has been established in [30, 23, 24].

Theorem 1.5 *For any finite and fixed initial configuration* $\xi \in \mathfrak{M}, \xi(\mathbb{R}) = N \in \mathbb{N}$, *the Dyson model is determinantal.*

2 Wigner's Semicircle Law as LLN

2.1 Hermite orthonormal functions and BM

For BM, we perform the following transformation with parameter $\alpha \in \mathbb{C} \equiv \{z = x + \sqrt{-1}y : x, y \in \mathbb{R}\}$, $B \mapsto \check{B}_{\alpha}$,

$$\check{B}_{\alpha}(t) = \frac{e^{\alpha B(t)}}{\mathrm{E}[e^{\alpha B(t)}]}, \quad t \ge 0, \qquad (2.1)$$

which is called the *Esscher transformation*. It is easy to see that

$$\mathrm{E}[e^{\alpha B(t)}] = \int_{-\infty}^{\infty} e^{\alpha x} p(t, x|0) dx = e^{\alpha^2 t/2}, \quad t \ge 0.$$

Then the above is written as

$$\check{B}_\alpha(t) = G_\alpha(t, B(t)), \quad t \geq 0$$

with

$$G_\alpha(t,x) = e^{\alpha x - \alpha^2 t/2}. \tag{2.2}$$

For $0 < s < t$,

$$\begin{aligned} \mathrm{E}[G_\alpha(t,B(t))|\mathcal{F}_s] &= \frac{\mathrm{E}[e^{\alpha B(t)}|\mathcal{F}_s]}{\mathrm{E}[e^{\alpha B(t)}]} \\ &= \frac{\mathrm{E}[e^{\alpha B(s)} e^{\alpha(B(t)-B(s))}|\mathcal{F}_s]}{\mathrm{E}[e^{\alpha B(s)} e^{\alpha(B(t)-B(s))}]}. \end{aligned}$$

By the definition of $\mathcal{F}_s$ and independence of increment of BM (the property **(BM3)**), the numerator is equal to $e^{\alpha B(s)} \mathrm{E}[e^{\alpha(B(t)-B(s))}]$, and the denominator is equal to $\mathrm{E}[e^{\alpha B(s)}] \times \mathrm{E}[e^{\alpha(B(t)-B(s))}]$. Hence the above equals $e^{\alpha B(s)}/\mathrm{E}[e^{\alpha B(s)}] = G_\alpha(s, B(s))$. This implies that $G_\alpha(t, B(t))$ is a martingale:

$$\mathrm{E}[G_\alpha(t,B(t))|\mathcal{F}_s] = G_\alpha(s, B(s)), \quad 0 \leq s \leq t. \tag{2.3}$$

The function (2.2) is expanded as

$$G_\alpha(t,x) = \sum_{n=0}^{\infty} m_n(t,x) \frac{\alpha^n}{n!} \tag{2.4}$$

with

$$m_n(t,x) = \left(\frac{t}{2}\right)^{n/2} H_n\left(\frac{x}{\sqrt{2t}}\right), \quad n \in \mathbb{N}_0 \equiv \{0,1,2,\dots\}. \tag{2.5}$$

Here $\{H_n(x)\}_{n \in \mathbb{N}_0}$ are the *Hermite polynomials* of degrees $n \in \mathbb{N}_0$,

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n} && (2.6) \\ &= \sum_{k=0}^{[n/2]} (-1)^k \frac{n!}{k!(n-2k)!} (2x)^{n-2k}, && (2.7) \end{aligned}$$

where for $a \geq 0$, $[a]$ denotes the largest integer which is not larger than a.

Lemma 2.1 *The functions* $\{m_n(t,x)\}_{n \in \mathbb{N}_0}$ *satisfy the following.*
(i) *They are monic polynomials of degrees* $n \in \mathbb{N}_0$ *with time-dependent coefficients:*

$$m_n(t,x) = x^n + \sum_{k=0}^{n-1} c_n^{(k)}(t) x^k, \quad t \geq 0.$$

(ii) *For* $0 \le k \le n-1$, $c_n^{(k)}(0) = 0$. *That is,*

$$m_n(0, x) = x^n, \quad n \in \mathbb{N}_0.$$

(iii) *If we set* $x = B(t)$, *they provide martingales:*

$$\mathrm{E}[m_n(t, B(t))|\mathcal{F}_s] = m_n(s, B(s)), \quad 0 \le s \le t, \quad n \in \mathbb{N}_0. \tag{2.8}$$

Proof By the definition (2.5) with (2.7), (i) and (ii) are obvious. Note that when n is even (resp. odd), $c_n^{(k)}(t) \equiv 0$ for odd (resp. even) k. Since $G_\alpha(t, B(t))$, $t \ge 0$ was shown to be a martingale for any $\alpha \in \mathbb{C}$, the expansion (2.4) implies (iii). ∎

Concerning the Hermite polynomials, we can prove the following. See, for instance, Exercises 1.2-1.4 in [24]

Lemma 2.2 (i) *The Hermite polynomials* $\{H_n(x)\}_{n \in \mathbb{N}_0}$ *have the orthogonality property*

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n n! \sqrt{\pi} \delta_{nm}, \quad n, m \in \mathbb{N}_0. \tag{2.9}$$

(ii) *The formula*

$$\sum_{n=0}^{\infty} H_n(z) \frac{s^n}{n!} = e^{2sz - s^2} \tag{2.10}$$

is established. This implies the following contour integral representations of the Hermite polynomials,

$$H_n(z) = \frac{n!}{2\pi\sqrt{-1}} \oint_{C(\delta_0)} d\eta \, \frac{e^{2\eta z - \eta^2}}{\eta^{n+1}}, \quad n \in \mathbb{N}_0, \tag{2.11}$$

where $C(\delta_0)$ *is a closed contour on the complex plane* $\mathbb{C}$ *encircling the origin 0 once in the positive direction.*
(iii) *The Hermite polynomials satisfy the following recurrence relations:*

$$H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z) = 0, \tag{2.12}$$

$$H_n'(z) = 2nH_{n-1}(z), \quad n \in \mathbb{N}, \tag{2.13}$$

where $H_n'(z) = dH_n(z)/dz$. *From (2.12) and (2.13), the equations*

$$H_n''(z) - 2zH_n'(z) + 2nH_n(z) = 0, \quad n \in \mathbb{N}_0$$

are derived. That is, $\{H_n(z)\}_{n \in \mathbb{N}_0}$ *satisfy the differential equation*

$$u'' - 2zu' + 2nu = 0. \tag{2.14}$$

This is known as the Hermite differential equation.

By the orthogonality (2.9) if we put

$$\varphi_n(x) = \frac{1}{\sqrt{\sqrt{\pi} 2^n n!}} H_n(x) e^{-x^2/2}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}_0, \tag{2.15}$$

then we have the equalities,

$$\int_{\mathbb{R}} dx\, \varphi_n(x) \varphi_m(x) = \delta_{nm}, \quad n, m \in \mathbb{N}_0. \tag{2.16}$$

The functions $\{\varphi_n(x)\}_{n \in N_0}$ are called the *Hermite orthonormal functions* on $\mathbb{R}$. The following expression for the transition probability density (1.1) of BM is known as *Mehler's formula*,

$$p(s-t, x|y) = \frac{e^{-x^2/4s}}{e^{-y^2/4t}} \frac{1}{\sqrt{2s}} \sum_{n=0}^{\infty} \left(\frac{t}{s}\right)^{n/2} \varphi_n\left(\frac{x}{\sqrt{2s}}\right) \varphi_n\left(\frac{y}{\sqrt{2t}}\right), \quad 0 < s < t, \quad x, y \in \mathbb{R}. \tag{2.17}$$

See, for instance, Exercise 3.15 in [24]

2.2 Dyson model starting from $\xi = N\delta_0$ and Hermite kernel

From now on, we will consider the Dyson model starting from the following special initial configuration,

$$\xi = N\delta_0. \tag{2.18}$$

That is, the initial configuration ξ is the state such that all N points are concentrated on an origin. In this case, the correlation kernel for the Dyson model is given by [34, 3, 29]

$$\begin{aligned} \boldsymbol{K}^{(N)}_{\text{Hermite}}(s, x; t, y) &\equiv \mathbb{K}_{N\delta_0}(s, x; t, y) \\ &= \begin{cases} \displaystyle \frac{1}{\sqrt{2s}} \sum_{n=0}^{N-1} \left(\frac{t}{s}\right)^{n/2} \varphi_n\left(\frac{x}{\sqrt{2s}}\right) \varphi_n\left(\frac{y}{\sqrt{2t}}\right) & \text{for } s \le t, \\ \displaystyle -\frac{1}{\sqrt{2s}} \sum_{n=N}^{\infty} \left(\frac{t}{s}\right)^{n/2} \varphi_n\left(\frac{x}{\sqrt{2s}}\right) \varphi_n\left(\frac{y}{\sqrt{2t}}\right) & \text{for } s > t. \end{cases} \end{aligned} \tag{2.19}$$

This kernel is called the *extended Hermite kernel.*

Denote the equal-time correlation kernel as

$$\begin{aligned} \mathrm{K}^{(N,t)}_{\text{Hermite}}(x, y) &\equiv \boldsymbol{K}^{(N)}_{\text{Hermite}}(t, x; t, y) \\ &= \frac{1}{\sqrt{2t}} \sum_{n=0}^{N-1} \varphi_n\left(\frac{x}{\sqrt{2t}}\right) \varphi_n\left(\frac{y}{\sqrt{2t}}\right), \quad 0 < t < \infty. \end{aligned}$$

By the recurrence formulas (2.12), we can see that this has the following expression,

$$\mathrm{K}^{(N,t)}_{\mathrm{Hermite}}(x,y) = \sqrt{\frac{N}{2}} \frac{\varphi_N(x/\sqrt{2t})\varphi_{N-1}(y/\sqrt{2t}) - \varphi_{N-1}(x/\sqrt{2t})\varphi_N(y/\sqrt{2t})}{x-y}, \quad \text{if } x \neq y, \tag{2.20}$$

$$\mathrm{K}^{(N,t)}_{\mathrm{Hermite}}(x,x) = \frac{1}{\sqrt{2t}} \left[N \left\{ \varphi_N\left(\frac{x}{\sqrt{2t}}\right) \right\}^2 - \sqrt{N(N+1)}\varphi_{N-1}\left(\frac{x}{\sqrt{2t}}\right)\varphi_{N+1}\left(\frac{x}{\sqrt{2t}}\right) \right], \tag{2.21}$$

$0 < t < \infty$. This spatial correlation kernel is a special case of the *Christoffel–Dorboux kernel* (see, for instance, Chapter 9 in [11] and Chapter 3 in [2]). It is called the *Hermite kernel* and defines the determinantal point process [42, 41] on $\mathbb{R}$ such that a spatial correlation function is given by

$$\rho^{(N,t)}_{\mathrm{Hermite}}(\boldsymbol{x}_{N'}) = \det_{1 \leq i,j \leq N'} \left[\mathrm{K}^{(N,t)}_{\mathrm{Hermite}}(x_i, x_j) \right] \tag{2.22}$$

for any $1 \leq N' \leq N$ and $\boldsymbol{x}_{N'} = (x_1, \dots, x_{N'}) \in \mathbb{R}^{N'}$, $t > 0$. We write the probability measure of this DPP as $\mathrm{P}^{(N,t)}_{\mathrm{Hermite}}$.

2.3 Wigner's semicircle law

The density function at time $t \geq 0$ and position $x \in \mathbb{R}$ is given by

$$\begin{aligned} \rho^{(N)}_{\mathrm{Hermite}}(t,x) &= \mathrm{K}^{(N,t)}_{\mathrm{Hermite}}(x,x) \\ &= \frac{1}{\sqrt{2t}} \sum_{n=0}^{N-1} \varphi_n\left(\frac{x}{\sqrt{2t}}\right)^2 \\ &= \frac{1}{\sqrt{2t}} \left[N\varphi_N\left(\frac{x}{\sqrt{2t}}\right)^2 - \sqrt{N(N+1)}\varphi_{N-1}\left(\frac{x}{\sqrt{2t}}\right)\varphi_{N+1}\left(\frac{x}{\sqrt{2t}}\right) \right]. \end{aligned} \tag{2.23}$$

It is easy to verify that

$$\int_{-\infty}^{\infty} \rho^{(N)}_{\mathrm{Hermite}}(t,x)dx = N$$

by the orthonormality (2.16) of $\{\varphi_n(x)\}_{n \in \mathbb{N}_0}$.

We will obtain estimations for the asymptotics at $N \to \infty$. The following formulas are derived from Theorem 8.22.9 (a) and (b) in Chapter VIII of [40]. Let ε and ω be a fixed positive numbers. We have

$$
\text{(i)} \qquad \varphi_N\Big(\sqrt{2N+1}\cos\phi\Big) = \frac{1}{\sqrt{\pi\sin\phi}}\left(\frac{2}{N}\right)^{1/4}
$$
$$
\times\left\{\sin\left[\left(\frac{N}{2}+\frac{1}{4}\right)(\sin 2\phi - 2\phi) + \frac{3}{4}\pi\right] + \mathcal{O}\left(\frac{1}{N}\right)\right\}, \quad \varepsilon \le \phi \le \pi - \varepsilon \tag{2.24}
$$

$$
\text{(ii)} \qquad \varphi_N\Big(\sqrt{2N+1}\cosh\phi\Big) = \frac{1}{\sqrt{2\pi\sinh\phi}}\left(\frac{1}{2N}\right)^{1/4}
$$
$$
\times\exp\left[\left(\frac{N}{2}+\frac{1}{4}\right)(2\phi - \sinh 2\phi) + \frac{3}{4}\pi\right]\left\{1 + \mathcal{O}\left(\frac{1}{N}\right)\right\}, \quad \varepsilon \le \phi \le \omega. \tag{2.25}
$$

Now we apply these asymptotic estimates to the Christoffel–Darboux formula for the Hermite functions,

$$
\sum_{n=0}^{N-1}\Big(\varphi_n(x)\Big)^2 = N\Big(\varphi_N(x)\Big)^2 - \sqrt{N(N+1)}\varphi_{N+1}(x)\varphi_{N-1}(x). \tag{2.26}
$$

Case 1: $x = \sqrt{2N+1}\cos\phi, \varepsilon \le \phi \le \pi - \varepsilon$

In order to calculate (2.26), first we derive the asymptotics for $\varphi_{N-1}(x)$ and $\varphi_{N+1}(x)$ from (2.24). We see

$$
\begin{aligned}
\varphi_{N-1}\Big(\sqrt{2N+1}\cos\phi\Big) &= \varphi_{N-1}\left(\sqrt{2(N-1)+1}\times\sqrt{\frac{2N+1}{2N-1}}\cos\phi\right) \\
&= \varphi_{N-1}\Big(\sqrt{2(N-1)+1}\cos(\phi+\eta)\Big),
\end{aligned} \tag{2.27}
$$

where we have set

$$
\cos(\phi+\eta) = \sqrt{\frac{2N+1}{2N-1}}\cos\phi.
$$

Since

$$
\sqrt{\frac{2N+1}{2N-1}} = 1 + \frac{1}{2N} + \mathcal{O}\left(\frac{1}{N^2}\right),
$$

and

$$
\cos(\phi+\eta) = \cos\phi - \eta\sin\phi + \mathcal{O}(\eta^2),
$$

we have

$$
\eta = -\frac{1}{2N}\frac{\cos\phi}{\sin\phi} + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{2.28}
$$

Applying (2.24) to (2.27), we obtain

$$\varphi_{N-1}\Big(\sqrt{2N+1}\,\cos\phi\big) = \frac{1}{\sqrt{\pi\sin\phi}}\left(\frac{2}{N}\right)^{1/4} \times\left\{\sin\left[\left(\frac{N-1}{2}+\frac{1}{4}\right)\Big\{\sin(2(\phi+\eta))-2(\phi+\eta)\Big\}+\frac{3}{4}\pi\right]+\mathcal{O}\left(\frac{1}{N}\right)\right\}. \tag{2.29}$$

Here, if we use (2.28), then

$$\begin{aligned}\sin(2(\phi+\eta))-2(\phi+\eta) &\simeq \sin 2\phi-2\phi+2\eta(\cos 2\phi-1)\\ &\simeq \sin 2\phi-2\phi-\frac{1}{N}\frac{\cos\phi}{\sin\phi}(\cos 2\phi-1)\\ &= \sin 2\phi-2\phi+\frac{2}{N}\sin\phi\cos\phi\\ &= \sin 2\phi-2\phi+\frac{1}{N}\sin 2\phi,\end{aligned}$$

and hence

$$\left(\frac{N-1}{2}+\frac{1}{4}\right)\Big\{\sin(2(\phi+\eta))-2(\phi+\eta)\Big\}\simeq\left(\frac{N}{2}+\frac{1}{4}\right)(\sin 2\phi-2\phi)+\phi+\mathcal{O}\left(\frac{1}{N}\right).$$

Therefore, by (2.29), we have the estimate

$$\varphi_{N-1}\Big(\sqrt{2N+1}\,\cos\phi\Big) = \frac{1}{\sqrt{\pi\sin\phi}}\left(\frac{2}{N}\right)^{1/4} \times\left\{\sin\left[\left(\frac{N}{2}+\frac{1}{4}\right)\Big\{\sin 2\phi-2\phi\Big\}+\frac{3}{4}\pi+\phi\right]+\mathcal{O}\left(\frac{1}{N}\right)\right\}. \tag{2.30}$$

Similarly, we put

$$\begin{aligned}\varphi_{N+1}\Big(\sqrt{2N+1}\cos\phi\Big) &= \varphi_{N+1}\left(\sqrt{2(N+1)+1}\times\sqrt{\frac{2N+1}{2N+3}}\,\cos\phi\right)\\ &= \varphi_{N+1}\Big(\sqrt{2(N+1)+1}\cos(\phi+\widetilde{\eta})\Big),\end{aligned} \tag{2.31}$$

with

$$\cos(\phi+\widetilde{\eta})=\sqrt{\frac{2N+1}{2N+3}}\cos\phi.$$

Then

$$\widetilde{\eta}=\frac{1}{2N}\frac{\cos\phi}{\sin\phi}+\mathcal{O}\left(\frac{1}{N^2}\right),$$

and we obtain the estimate

$$\varphi_{N+1}\Big(\sqrt{2N+1}\,\cos\phi\Big) = \frac{1}{\sqrt{\pi\sin\phi}}\left(\frac{2}{N}\right)^{1/4} \times\left\{\sin\left[\left(\frac{N}{2}+\frac{1}{4}\right)\Big\{\sin 2\phi-2\phi\Big\}+\frac{3}{4}\pi-\phi\right]+\mathcal{O}\left(\frac{1}{N}\right)\right\}. \tag{2.32}$$

Inserting them into (2.26) gives

$$\sum_{n=0}^{N-1}\Big(\varphi_n(\sqrt{2N+1}\cos\phi)\Big)^2 = \frac{\sqrt{2N}}{\pi\sin\phi}\Bigg\{\sin^2\left[\left(\frac{N}{2}+\frac{1}{4}\right)(\sin 2\phi-2\phi)+\frac{3}{4}\pi\right] - \sin\left[\left(\frac{N}{2}+\frac{1}{4}\right)(\sin 2\phi-2\phi)+\frac{3}{4}\pi+\phi\right] \times \sin\left[\left(\frac{N}{2}+\frac{1}{4}\right)(\sin 2\phi-2\phi)+\frac{3}{4}\pi-\phi\right]\Bigg\} + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).$$

Put

$$A = \left(\frac{N}{2}+\frac{1}{4}\right)(\sin 2\phi-2\phi)+\frac{3}{4}\pi.$$

Then we see

$$\begin{aligned}
\Big\{\cdots\Big\} &= \sin^2 A - \sin(A+\phi)\sin(A-\phi)\\
&= \sin^2 A - (\sin A\cos\phi+\cos A\sin\phi)(\sin A\cos\phi-\cos A\sin\phi)\\
&= \sin^2 A - (\sin^2 A\cos^2\phi-\cos^2 A\sin^2\phi)\\
&= \sin^2 A(1-\cos^2\phi)+\cos^2 A\sin^2\phi\\
&= \sin^2 A\sin^2\phi+\cos^2 A\sin^2\phi\\
&= \sin^2\phi
\end{aligned}$$

and hence

$$\begin{aligned}
\sum_{n=0}^{N-1}\Big(\varphi_n(\sqrt{2N+1}\cos\phi)\Big)^2 &= \frac{\sqrt{2N}}{\pi\sin\phi}\sin^2\phi+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)\\
&= \frac{\sqrt{2N}}{\pi}\sin\phi+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right).
\end{aligned} \tag{2.33}$$

Since

$$\begin{aligned}
\sin\phi &= \sqrt{1-\cos^2\phi}\simeq\sqrt{1-\frac{x^2}{2N}}\\
&= \frac{1}{\sqrt{2N}}\sqrt{2N-x^2},
\end{aligned}$$

we obtain in $N\to\infty$,

$$\sum_{n=0}^{N-1}\Big(\varphi_n(x)\Big)^2 \simeq \frac{1}{\pi}\sqrt{2N-x^2}, \quad -\sqrt{2N}<x<\sqrt{2N}.$$

Case 2: $x = \sqrt{2N+1}\cosh\phi, \varepsilon \le \phi \le \omega$

We put

$$\begin{aligned}\varphi_{N-1}\Big(\sqrt{2N+1}\cosh\phi\Big) &= \varphi_{N-1}\left(\sqrt{2(N-1)+1}\times\sqrt{\frac{2N+1}{2N-1}}\cosh\phi\right)\\ &= \varphi_{N-1}\Big(\sqrt{2(N-1)-1}\,\cosh(\phi+\eta)\Big),\end{aligned}$$

where

$$\begin{aligned}\cosh(\phi+\eta) &= \sqrt{\frac{2N+1}{2N-1}}\cosh(\phi+\eta)\\ &\simeq \left(1+\frac{1}{2N}+\mathcal{O}\left(\frac{1}{N^2}\right)\right)\cosh\phi.\end{aligned}$$

Since

$$\cosh(\phi+\eta) = \cosh\phi + \eta\sinh\phi + \mathcal{O}(\eta^2),$$

we have

$$\eta = \frac{1}{2N}\frac{\cosh\eta}{\sinh\eta} + \mathcal{O}\left(\frac{1}{N^2}\right).$$

Insert the above into (2.25). Then we have

$$\begin{aligned}\varphi_{N-1}\Big(\sqrt{2N+1}\cosh\phi\Big) &= \frac{1}{\sqrt{2\pi\sinh\phi}}\left(\frac{1}{2N}\right)^{1/4}\\ &\times\exp\left[\left(\frac{N-1}{2}+\frac{1}{4}\right)\Big\{2(\phi+\eta)-\sinh 2(\phi+\eta)\Big\}\right]\left\{1+\mathcal{O}\left(\frac{1}{N}\right)\right\}.\end{aligned}$$

For

$$\begin{aligned}&2(\phi+\eta)-\sinh 2(\phi+\eta) \simeq 2\phi - \sinh 2\phi + 2\eta(1-\cosh 2\phi)\\ \simeq\ & 2\phi - \sinh 2\phi + \frac{1}{N}\frac{\cosh\phi}{\sinh\phi}(-2\sinh^2\phi) = 2\phi - \sinh 2\phi - \frac{1}{N}2\sinh\phi\cosh\phi\\ =\ & 2\phi - \sinh 2\phi - \frac{1}{N}\sinh 2\phi,\end{aligned}$$

the above is written as

$$\begin{aligned}\varphi_{N-1}\Big(\sqrt{2N+1}\cosh\phi\Big) &= \frac{1}{\sqrt{2\pi\sinh\phi}}\left(\frac{1}{2N}\right)^{1/4}\\ &\times\exp\left[\left(\frac{N}{2}+\frac{1}{4}\right)(2\phi-\sinh 2\phi)-\phi\right]\left\{1+\mathcal{O}\left(\frac{1}{N}\right).\right\}\end{aligned}$$

Similarly, we obtain

$$\varphi_{N+1}\Big(\sqrt{2N+1}\cosh\phi\Big) = \frac{1}{\sqrt{2\pi\sinh\phi}}\left(\frac{1}{2N}\right)^{1/4}$$
$$\times \exp\left[\left(\frac{N}{2}+\frac{1}{4}\right)(2\phi-\sinh 2\phi)+\phi\right]\left\{1+\mathcal{O}\left(\frac{1}{N}\right)\right\}.$$

Inserting the above into (2.26) implies that, in the limit $N \to \infty$,

$$\begin{aligned}
&\sum_{n=0}^{N-1}\Big(\phi_n\Big(\sqrt{2N+1}\cosh\phi\Big)\Big)^2 \\
\simeq\ & \frac{N}{2\pi\sinh\phi}\frac{1}{\sqrt{2N}}\Big\{\exp\left[(N+1/2)(2\phi-\sinh 2\phi)\right] \\
&\qquad -\exp\left[(N/2+1/4)(2\phi-\sinh 2\phi)-\phi\right] \\
&\qquad \times\exp\left[(N/2+1/4)(2\phi-\sinh 2\phi)+\phi\right]\Big\}\left\{1+\mathcal{O}\left(\frac{1}{N}\right)\right\} \\
\simeq\ & \frac{1}{2\pi\sinh\phi}\sqrt{\frac{N}{2}}\mathcal{O}\left(\frac{1}{N}\right) = \mathcal{O}(N^{-1/2}) \quad \to \quad 0.
\end{aligned}$$

Using the above evaluations, we have the asymptotics of the density profile at $N \to \infty$,

$$\rho^{(N)}_{\mathrm{Hermite}}(t,x) \simeq \begin{cases} \dfrac{1}{\pi\sqrt{2t}}\sqrt{2N-\dfrac{x^2}{2t}}, & \text{if } -2\sqrt{Nt} \le x \le 2\sqrt{Nt}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.34}$$

The distribution of N particles has a finite support, whose interval $\propto \sqrt{N}$, and thus $\rho^{(N)}_{\mathrm{Hermite}}(t,x) \sim \sqrt{N} \to \infty$ as $N \to \infty$ for fixed $0 < t < \infty$, when $x \in (-2\sqrt{Nt}, 2\sqrt{Nt})$. If we set $x = 2\sqrt{Nt}\xi$, we see that

$$\lim_{N\to\infty}\frac{1}{N}\rho^{(N)}_{\mathrm{Hermite}}(t,2\sqrt{Nt}\xi)dx = \begin{cases} \dfrac{2}{\pi}\sqrt{1-\xi^2}\,d\xi, & \text{if } -1 \le \xi \le 1, \\ 0, & \text{otherwise,} \end{cases} \tag{2.35}$$

which is known as *Wigner's semicircle law* [33]. Here we regard this as the *law of large numbers* (LLN) for $\mathrm{P}^{(N,t)}_{\mathrm{Hermite}}$ with fixed $t \in (0,\infty)$.

See, for instance, Section 5 in [2] for the importance of Wigner's semicircle law in free probability.

3 Scaling Limits and Infinite Particle Systems

3.1 Bulk scaling limit and sine kernel

First we consider the central region $x \simeq 0$ in the semicircle-shaped profile of particle density in the scaling limit

$$t \simeq \frac{N}{\pi^2} \to \infty. \tag{3.1}$$

In this limit the system becomes homogeneous also in space with a constant density $\rho = 1$. We call this the *bulk scaling limit*.

Proposition 3.1 *For any $M \in \mathbb{N}$, any sequence $\{N_m\}_{m=1}^M$ of positive integers, and any strictly increasing sequence $\{s_m\}_{m=1}^M$ of positive numbers,*

$$\begin{aligned}
&\lim_{N\to\infty} \rho_{\rm Hermite}^{(N)} \left(\frac{N}{\pi^2} + 2s_1, \boldsymbol{x}_{N_1}^{(1)}; \cdots ; \frac{N}{\pi^2} + 2s_M, \boldsymbol{x}_{N_M}^{(M)} \right) \\
&\quad = \det_{\substack{1\leq i\leq N_m, 1\leq j\leq N_n, \\ 1\leq m,n\leq M}} \left[\boldsymbol{K}_{\sin}(s_m, x_i^{(m)}; s_n, x_j^{(n)}) \right] \\
&\quad \equiv \rho_{\sin}\Big(s_1, \boldsymbol{x}_{N_1}^{(1)}; \cdots ; s_M, \boldsymbol{x}_{N_M}^{(M)} \Big),
\end{aligned} \tag{3.2}$$

where

$$\boldsymbol{K}_{\sin}(s,x;t,y) = \begin{cases} \displaystyle\int_0^1 du\, e^{-\pi^2 u^2 (s-t)} \cos\{\pi u(x-y)\}, & \textit{if } t > s, \\ \mathrm{K}_{\sin}(x,y), & \textit{if } t = s, \\ \displaystyle -\int_1^\infty du\, e^{-\pi^2 u^2 (s-t)} \cos\{\pi u(x-y)\}, & \textit{if } t < s, \end{cases} \tag{3.3}$$

with

$$\mathrm{K}_{\sin}(x,y) = \int_0^1 du\, \cos\{\pi u(x-y)\} = \frac{\sin\{\pi(x-y)\}}{\pi(x-y)}, \quad x,y \in \mathbb{R}. \tag{3.4}$$

Proof For any $u \in \mathbb{R}$, the formulas

$$\begin{aligned}
&\lim_{\ell\to\infty} (-1)^\ell \ell^{1/4} \varphi_{2\ell} \left(\frac{u}{2\sqrt{\ell}} \right) = \frac{1}{\sqrt{\pi}} \cos u, \\
&\lim_{\ell\to\infty} (-1)^\ell \ell^{1/4} \varphi_{2\ell+1} \left(\frac{u}{2\sqrt{\ell}} \right) = \frac{1}{\sqrt{\pi}} \sin u
\end{aligned} \tag{3.5}$$

are known (see Eq. (8.22.8) in Chapter VIII of [40]). We note that

$$\begin{aligned}
\left(\frac{t_n}{t_m}\right)^{\alpha} &= \left(\frac{N/\pi^2+2s_n}{N/\pi^2+2s_m}\right)^{\alpha} \\
&= \left\{ \left(1+\frac{2\pi^2 s_n}{N}\right)^{N} \left(1+\frac{2\pi^2 s_m}{N}\right)^{-N} \right\}^{\alpha/N} \\
&\simeq e^{-2\pi^2\alpha(s_m-s_n)/N}
\end{aligned}$$

for $N \gg 1$ with a fixed number α. Then (2.19) with $s = t_m = N/\pi^2 + 2s_m \leq t = t_n = N/\pi^2 + 2s_n$ is evaluated at $N \to \infty$ as

$$\begin{aligned}
&\boldsymbol{K}^{(N)}_{\rm Hermite}(t_m, x; t_n, y) \simeq \frac{1}{N} \sum_{\ell=0}^{[N/2-1]} e^{-2\pi^2\ell(s_m-s_n)/N} \\
&\times \sqrt{\frac{N}{2\ell}} \left\{ \cos\left(\pi\sqrt{\frac{2\ell}{N}}\,x\right) \cos\left(\pi\sqrt{\frac{2\ell}{N}}\,y\right) + \sin\left(\pi\sqrt{\frac{2\ell}{N}}\,x\right) \sin\left(\pi\sqrt{\frac{2\ell}{N}}\,y\right) \right\} \\
&\simeq \frac{1}{2}\int_0^1 \frac{d\lambda}{\sqrt{\lambda}}\, e^{-\pi^2\lambda(s_m-s_n)} \Big\{ \cos(\pi\sqrt{\lambda}x)\cos(\pi\sqrt{\lambda}y) + \sin(\pi\sqrt{\lambda}x)\sin(\pi\sqrt{\lambda}y) \Big\} \\
&= \int_0^1 du\, e^{-\pi^2 u^2(s_m-s_n)} \cos\{\pi u(x-y)\}.
\end{aligned}$$

In particular, when $t_m = t_n$, *i.e.*, $s_n - s_m = 0$, the integration is readily performed to have $\int_0^1 du\, \cos\{\pi u(x-y)\} = \sin\{\pi(x-y)\}/\pi(x-y)$. A similar evaluation at $N \to \infty$ can be done also for (2.19) with $s = t_m > t = t_n$. ∎

Remark 3.1 The correlation kernel (3.3) is called the *extended sine kernel*. Since it is a function of $s-t$ and $x-y$, the determinantal process obtained by the bulk scaling limit is a temporally and spatially homogeneous process with an infinite number of particles, which we write as $(\Xi, \mathbf{P}_{\sin})$. Let $\mathrm{P}_{\sin}$ be a stationary probability measure on $\mathbb{R}$, which is a determinantal point process [42, 41] such that the spatial correlation function is given by

$$\rho_{\sin}(\boldsymbol{x}_N) = \det_{1\leq i,j\leq N}\Big[\mathrm{K}_{\sin}(x_i, x_j)\Big] \tag{3.6}$$

for any $N \in \mathbb{N}, \boldsymbol{x}_N = (x_1, \dots, x_N) \in \mathbb{R}^N$, where $\mathrm{K}_{\sin}$ is given by (3.4). The determinantal process $(\Xi, \mathbf{P}_{\sin})$ is reversible with respect to $\mathrm{P}_{\sin}$.

3.2 Soft-edge scaling limit and Airy kernel

Next we consider the scaling limit

$$t \simeq N^{1/3} \quad \text{and} \quad x \simeq 2N^{2/3}. \tag{3.7}$$

Since (3.7) gives $x^2/2t \simeq 2N$, this scaling limit allows us to zoom into the right edge of the semicircle-shaped profile (2.34), and we obtain a spatially inhomogeneous infinite particle system. Following random matrix theory [33], we call (3.7) the *soft-edge scaling limit*.

In order to describe the limit, we introduce the *Airy function*

$$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^\infty dk \, \cos\left(\frac{k^3}{3} + kx\right). \tag{3.8}$$

It is the solution of the equation

$$\frac{d^2}{dx^2}\mathrm{Ai}(x) = x\mathrm{Ai}(x), \tag{3.9}$$

which obeys the asymptotics given by

$$\begin{aligned} \mathrm{Ai}(x) &\simeq \frac{1}{2\sqrt{\pi}x^{1/4}} \exp\left(-\frac{2}{3}x^{3/2}\right), \\ \mathrm{Ai}(-x) &\simeq \frac{1}{\sqrt{\pi}x^{1/4}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right) \quad \text{as } x \to \infty. \end{aligned} \tag{3.10}$$

In the proof of the following theorem, we will use the formula

$$\lim_{\ell\to\infty} 2^{-1/4}\ell^{1/12}\varphi_\ell\left(\sqrt{2\ell} + \frac{x}{\sqrt{2}}\ell^{-1/6}\right) = \mathrm{Ai}(x) \quad \text{for } x \in \mathbb{R}, \tag{3.11}$$

which is obtained from Theorem 8.22.9 (c) in Chapter VIII of [40]. Let

$$a_N(s) = 2N^{2/3} + N^{1/3}s - \frac{s^2}{4}, \tag{3.12}$$

and $a_N(s) + \boldsymbol{x}_{N'} = (a_N(s) + x_1, a_N(s) + x_2, \cdots, a_N(s) + x_{N'})$.

Proposition 3.2 *For any $M \in \mathbb{N}$, any sequence $\{N_m\}_{m=1}^M$ of positive integers, and any strictly increasing sequence $\{s_m\}_{m=1}^M$ of positive numbers*

$$\begin{aligned} &\lim_{N\to\infty} \rho^{(N)}_{\text{Hermite}}\Big(N^{1/3} + s_1, a_N(s_1) + \boldsymbol{x}^{(1)}_{N_1}; \cdots; N^{1/3} + s_M, a_N(s_M) + \boldsymbol{x}^{(M)}_{N_M}\Big) \\ &\quad = \det_{\substack{1\le i\le N_m, 1\le j\le N_n, \\ 1\le m,n\le M}} \Big[\boldsymbol{K}_{\text{Airy}}(s_m, x^{(m)}_i; s_n, x^{(n)}_j)\Big] \\ &\quad \equiv \rho_{\text{Airy}}\Big(s_1, \boldsymbol{x}^{(1)}_{N_1}; \cdots; s_M, \boldsymbol{x}^{(M)}_{N_M}\Big), \end{aligned} \tag{3.13}$$

where

$$\boldsymbol{K}_{\text{Airy}}(s, x; t, y) = \begin{cases} \displaystyle\int_0^\infty du\, e^{u(s-t)/2}\mathrm{Ai}(x+u)\mathrm{Ai}(y+u), & \text{if } t \ge s, \\ \displaystyle -\int_0^\infty du\, e^{u(s-t)/2}\mathrm{Ai}(x+u)\mathrm{Ai}(y+u), & \text{if } t < s. \end{cases} \tag{3.14}$$

Proof Replacing the summation index in (2.19) by $N-p-1$ for the case where $m \le n$, we have

$$\begin{aligned}
&\boldsymbol{K}^{(N)}_{\mathrm{Hermite}}(t_m, x; t_n, y) \\
&\quad = \left(\frac{t_n}{t_m}\right)^{(N-1)/2} \frac{1}{\sqrt{2t_m}} \sum_{p=0}^{N-1} \left(\frac{t_n}{t_m}\right)^{-p/2} \varphi_{N-p-1}\left(\frac{x}{\sqrt{2t_m}}\right) \varphi_{N-p-1}\left(\frac{y}{\sqrt{2t_n}}\right).
\end{aligned}$$

When we set $t_m = N^{1/3} + s_m$, we see that

$$\frac{a_N(s_m)+x}{\sqrt{2t_m}} = \sqrt{2N} + \frac{x}{\sqrt{2}} N^{-1/6} + \mathcal{O}(N^{-1/2}), \tag{3.15}$$

and we can use the formula (3.11):

$$\begin{aligned}
\varphi_{N-p-1}\left(\frac{a_N(s_m)+x}{\sqrt{2t_m}}\right) &\simeq \varphi_{N-p-1}\left(\sqrt{2N} + \frac{x}{\sqrt{2}} N^{-1/6}\right) \\
&\simeq \varphi_{N-p-1}\left(\sqrt{2(N-p-1)} + \frac{1}{\sqrt{2}}(N-p-1)^{-1/6}\left\{x + \frac{p}{N^{1/3}}\right\}\right) \\
&\simeq 2^{1/4} N^{-1/12} \mathrm{Ai}\left(x + \frac{p}{N^{1/3}}\right).
\end{aligned}$$

For

$$\left(\frac{t_n}{t_m}\right)^{-p/2} = \left[\left(\frac{1+s_n/N^{1/3}}{1+s_m/N^{1/3}}\right)^{N^{1/3}/2}\right]^{-p/N^{1/3}} \simeq e^{p(s_m-s_n)/2N^{1/3}} \quad \text{as } N \to \infty,$$

we have, for $n \ge m$,

$$\begin{aligned}
&\boldsymbol{K}^{(N)}_{\mathrm{Hermite}}(N^{1/3}+s_m, a_N(s_m)+x; N^{1/3}+s_n, a_N(s_n)+y) \\
&\quad \sim \frac{1}{N^{1/3}} \sum_{p=0}^{N-1} e^{p(s_m-s_n)/2N^{1/3}} \mathrm{Ai}\left(x+\frac{p}{N^{1/3}}\right) \mathrm{Ai}\left(y+\frac{p}{N^{1/3}}\right) \\
&\quad \simeq \int_0^\infty du\, e^{u(s_m-s_n)/2} \mathrm{Ai}(x+u)\mathrm{Ai}(y+u) \quad \text{as } N \to \infty.
\end{aligned}$$

Note that the factor $(t_n/t_m)^{(N-1)/2}$ was omitted in the second line in the above equations, since it is irrelevant in calculating determinants. A similar evaluation at $N \to \infty$ of (2.19) can be done also for $m > n$. ∎

The infinite system obtained by the soft-edge scaling limit (3.7) is temporally homogeneous, but spatially inhomogeneous as shown by the correlation kernel $\boldsymbol{K}_{\mathrm{Airy}}$, (3.14). We call $\boldsymbol{K}_{\mathrm{Airy}}$ the *extended Airy kernel* [35, 25] and write this stationary determinantal process as $(\Xi, \mathbf{P}_{\mathrm{Airy}})$. Prähofer and Spohn [36] and Johansson [21] studied the rightmost path in the present system and called it the *Airy process* $(A(t))_{t \ge 0}$.

Remark 3.2 Let $\mathrm{P}_{\mathrm{Airy}}$ be the stationary probability measure on $\mathbb{R}$, which is a determinantal point process [42, 41] such that the spatial correlation function is given by

$$\rho_{\mathrm{Airy}}(\boldsymbol{x}_N) = \det_{1 \leq i,j \leq N} \Big[\mathrm{K}_{\mathrm{Airy}}(x_i, x_j) \Big] \tag{3.16}$$

for any $N \in \mathbb{N}, \boldsymbol{x}_N = (x_1, \dots, x_N) \in \mathbb{R}^N$, where

$$\begin{aligned} \mathrm{K}_{\mathrm{Airy}}(x,y) &= \boldsymbol{K}_{\mathrm{Airy}}(t,x;t,y) \\ &= \int_0^\infty du\, \mathrm{Ai}(x+u)\mathrm{Ai}(y+u). \end{aligned} \tag{3.17}$$

The *Airy kernel* $\mathrm{K}_{\mathrm{Airy}}$ is also written as

$$\mathrm{K}_{\mathrm{Airy}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}, \quad x \neq y, \tag{3.18}$$

$$\mathrm{K}_{\mathrm{Airy}}(x,x) = \mathrm{Ai}'(x)^2 - x\mathrm{Ai}(x)^2, \tag{3.19}$$

where $\mathrm{Ai}'(x) = d\mathrm{Ai}(x)/dx$.

4 Tracy–Widom Distribution as CLT

4.1 Distribution function of the maximum position of particles

Consider a determinantal point process (Ξ, P) with an infinite number of particles on $\mathbb{R}$ such that $\Xi = \sum_{i \in \mathbb{N}} \delta_{X_i} \in \mathfrak{M}_0$. Assume that the correlation kernel is given by $\mathrm{K}(x,y), (x,y) \in \mathbb{R}^2$ [42, 41]. Two examples, $(\Xi, \mathrm{P}_{\sin})$ with the sine kernel $\mathrm{K}_{\sin}$ and $(\Xi, \mathrm{P}_{\mathrm{Airy}})$ with the Airy kernel $\mathrm{K}_{\mathrm{Airy}}$, were given in the previous section. With a test function $\chi \in \mathrm{C}_{\mathrm{c}}(\mathbb{R})$ the generating function of spatial correlation functions defined by

$$\Psi[\chi] = \mathrm{E}\left[\prod_{i \in \mathbb{N}} \{1 + \chi(X_i)\} \right], \tag{4.1}$$

is expressed by a Fredholm determinant

$$\Psi[\chi] = \mathop{\mathrm{Det}}_{(x,y) \in \mathbb{R}^2} \Big[\delta(x-y) + \mathrm{K}(x,y)\chi(y) \Big]. \tag{4.2}$$

Let $\mathbf{1}_{(\omega)}$ be the indicator function of condition ω; $\mathbf{1}_{(\omega)} = 1$ if ω is satisfied, and $\mathbf{1}_{(\omega)} = 0$ otherwise. If we set $\chi(x) = -\mathbf{1}_{(x \geq s)}$ with a parameter $s \in \mathbb{R}$, (4.1) becomes

$$\begin{aligned} \Psi[-\mathbf{1}_{(\cdot \geq s)}] &= \mathrm{E}\left[\prod_{i \in \mathbb{N}} \{1 - \mathbf{1}_{(X_i \geq s)}\} \right] = \mathrm{E}\left[\prod_{i \in \mathbb{N}} \mathbf{1}_{(X_i < s)} \right] \\ &= \mathrm{P}\Big[X_i < s, {}^\forall i \in \mathbb{N} \Big] = \mathrm{P}\left[\max_{i \in \mathbb{N}} X_i < s \right]. \end{aligned}$$

This is the distribution function of the *maximum position of particles*, and by (4.2), it has the Fredholm determinantal expression

$$\mathrm{P}\left[\max_{i\in\mathbb{N}} X_i < s\right] = \operatorname*{Det}_{(x,y)\in\mathbb{R}^2}\Big[\delta(x-y) - \mathrm{K}_s(x,y)\Big], \tag{4.3}$$

where

$$\mathrm{K}_s(x,y) = \mathrm{K}(x,y)\mathbf{1}_{(y\geq s)}, \quad x,y,s\in\mathbb{R}. \tag{4.4}$$

For integrable functions $f_i(x,y)$, $i\in\mathbb{N}$, $(x,y)\in\mathbb{R}^2$, we use the following notations,

$$[f_1 f_2 \cdots f_n](x_1, x_{n+1}) = \int_{\mathbb{R}^{n-1}} f_1(x_1,x_2) f_2(x_2,x_3)\cdots f_n(x_n,x_{n+1}) dx_2 \dots dx_n,$$

$n\in\{2,3,\dots\}$, $x_1, x_{n+1}\in\mathbb{R}$. We regard $f_1 f_2\cdots f_n$ as an operator such that its (x,y)-element is given by $[f_1 f_2\cdots f_n](x,y), (x,y)\in\mathbb{R}^2$. The trace of an operator f is defined by

$$\mathrm{Tr} f = \int_{\mathbb{R}} f(x,x)dx, \tag{4.5}$$

and if $\mathrm{Tr} f < \infty$, f is said to be a *trace class operator* [39]. Put $1(x,y) = \delta(x-y), (x,y)\in\mathbb{R}^2$. The *resolvent* of K_a is defined by

$$\rho_a = \sum_{n=0}^{\infty} \mathrm{K}_a^n \equiv (1-\mathrm{K}_a)^{-1}. \tag{4.6}$$

Let

$$\mathrm{R}_a \equiv \rho_a \mathrm{K}_a = \sum_{n=0}^{\infty} \mathrm{K}_a^{n+1}, \tag{4.7}$$

and

$$r(a) = \mathrm{R}_a(a,a) \equiv \lim_{y\to x} \mathrm{R}_a(x,y)\Big|_{x=a}, \quad a\in\mathbb{R}. \tag{4.8}$$

The correlation kernels K of determinantal point processes are trace class operators and the following exponential expression for (4.3) is proved (see, for instance, Lemmas 2.1 and 2.2 in [41]).

Lemma 4.1 *If r is integrable,*

$$\mathrm{P}\left[\max_{i\in\mathbb{N}} X_i < s\right] = \exp\left(-\int_s^{\infty} da\, r(a)\right), \quad s\in\mathbb{R}. \tag{4.9}$$

Proof The explicit expression of (4.3) is

$$\operatorname*{Det}_{(x,y)\in\mathbb{R}^2}\Big[\delta_x(\{y\}) - \mathrm{K}_s(x,y)\Big] = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} I_n(s)$$

with

$$I_n(s) = \int_{\mathbb{R}^n} d\boldsymbol{x}_n \det_{1\le i,j\le n}[\mathrm{K}_s(x_i, x_j)].$$

Using the Maclaurin expansion

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \tag{4.10}$$

we can show that

$$\log\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} I_n(s)\right) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\, \mathrm{K}_s^n. \tag{4.11}$$

We rewrite this as

$$\begin{aligned} \mathrm{Tr}\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{K}_s^n\right) &= \int_s^{\infty} da \frac{\partial}{\partial a} \mathrm{Tr}\left(\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{K}_a^n\right) \\ &= \int_s^{\infty} da\, \mathrm{Tr}\left(\sum_{n=1}^{\infty} \mathrm{K}_a^{n-1} \frac{\partial \mathrm{K}_a}{\partial a}\right) = \int_s^{\infty} da\, \mathrm{Tr}\left(\rho_a \frac{\partial \mathrm{K}_a}{\partial a}\right). \end{aligned}$$

Here

$$\begin{aligned} \frac{\partial \mathrm{K}_a}{\partial a}(x,y) &= \frac{\partial}{\partial a}\left\{\mathrm{K}(x,y)\mathbf{1}_{(y\ge a)}\right\} \\ &= \mathrm{K}(x,y)\frac{\partial}{\partial a}\mathbf{1}_{(y\ge a)} = -\mathrm{K}(x,y)\delta(y-a), \end{aligned}$$

and hence

$$\begin{aligned} \mathrm{Tr}\left(\rho_a \frac{\partial \mathrm{K}_a}{\partial a}\right) &= \int_{\mathbb{R}} dx \left[\rho_a \frac{\partial \mathrm{K}_a}{\partial a}\right](x,x) \\ &= \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy\, \rho_a(x,y) \frac{\partial \mathrm{K}_a}{\partial a}(y,x) \\ &= -\int_{\mathbb{R}} dy\, \rho_a(a,y)\mathrm{K}(y,a) = -\mathrm{R}_a(a,a). \end{aligned}$$

The proof is completed. ∎

For $(x,y) \in \mathbb{R}^2$, as $a \to \infty$, $\mathrm{K}_a(x,y) = \mathrm{K}(x,y)\mathbf{1}_{(y\ge a)} \to 0$. Then the definition (4.7) gives $\mathrm{R}_a(x,y) \simeq \mathrm{K}_a(x,y) = \mathrm{K}(x,y)\mathbf{1}_{(y\ge a)}$ in $a \to \infty$ for $(x,y) \in \mathbb{R}^2$. If we put $x = y = a$, we have

$$r(a) \simeq \mathrm{K}(a,a) \quad \text{as } a \to \infty. \tag{4.12}$$

Note that, by definitions (4.6) and (4.7), $\rho_a = 1 + \mathrm{R}_a$ and hence

$$\rho_a(x,y) = \delta(x-y) + \mathrm{R}_a(x,y), \quad (x,y) \in \mathbb{R}^2. \tag{4.13}$$

4.2 Integrals involving resolvent of correlation kernel

Let $N \in \mathbb{N}, t \in (0,\infty)$. Now we assume

$$\mathrm{K}(x,y) = \mathrm{K}^{(N,t)}_{\mathrm{Hermite}}(x,y), \quad (x,y) \in \mathbb{R}^2. \tag{4.14}$$

The formula (2.20) is simply written as

$$\mathrm{K}(x,y) = \frac{A(x)B(y) - B(x)A(y)}{x-y}, \quad x \neq y, \tag{4.15}$$

with

$$A(x) = \left(\frac{N}{2}\right)^{1/4} \varphi_N\left(\frac{x}{\sqrt{2t}}\right), \quad B(x) = \left(\frac{N}{2}\right)^{1/4} \varphi_{N-1}\left(\frac{x}{\sqrt{2t}}\right). \tag{4.16}$$

We can prove the following.

Lemma 4.2 *Let*

$$\begin{aligned} P_a(x) &= \int_{\mathbb{R}} dz\, \rho_a(x,z) B(z) = [\rho_a B](x), \\ Q_a(x) &= \int_{\mathbb{R}} dz\, \rho_a(x,z) A(z) = [\rho_a A](x). \end{aligned} \tag{4.17}$$

Then

$$r(a) = \left[\frac{dQ_a(x)}{dx} P_a(x) - \frac{dP_a(x)}{dx} Q_a(x)\right]_{x=a}. \tag{4.18}$$

Using (4.16) and (4.17), we define the following integrals,

$$w(a) = \int_{\mathbb{R}} dx\, P_a(x) \mathbf{1}_{(x\geq a)} B(x) = \int_a^\infty dx\, P_a(x) B(x), \tag{4.19}$$

$$u(a) = \int_{\mathbb{R}} dx\, Q_a(x) \mathbf{1}_{(x\geq a)} A(x) = \int_a^\infty dx\, Q_a(x) A(x). \tag{4.20}$$

Then the following is proved.

Lemma 4.3 *The following equations hold:*

$$\begin{aligned} \frac{dP_a(x)}{dx} &= \frac{x}{2t} P_a(x) - \left(\sqrt{\frac{N}{t}} + \frac{w(a)}{t}\right) Q_a(x) + \mathrm{R}_a(x,a) P_a(a), \\ \frac{dQ_a(x)}{dx} &= -\frac{x}{2t} Q_a(x) + \left(\sqrt{\frac{N}{t}} - \frac{u(a)}{t}\right) P_a(x) + \mathrm{R}_a(x,a) Q_a(a), \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} \mathrm{R}_a(x,x) = &\left\{ -\frac{x}{t} P_a(x) Q_a(x) + \left(\sqrt{\frac{N}{t}} - \frac{u(a)}{t}\right) P_a(x)^2 + \left(\sqrt{\frac{N}{t}} + \frac{w(a)}{t}\right) Q_a(x)^2 \right. \\ &\left. + \mathrm{R}_a(x,a)\{Q_a(a)P_a(x) - Q_a(x)P_a(a)\} \right\} \mathbf{1}_{(x\geq a)}. \end{aligned} \tag{4.22}$$

4.3 Nonlinear third-order differential equation

Let

$$p(a) = P_a(a), \quad q(a) = Q_a(a), \quad a \in \mathbb{R}. \tag{4.23}$$

By the definition (4.8), (4.22) gives

$$r(a) = -\frac{a}{t}p(a)q(a) + \left(\sqrt{\frac{N}{t}} - \frac{u(a)}{t}\right)p(a)^2 + \left(\sqrt{\frac{N}{t}} + \frac{w(a)}{t}\right)q(a)^2. \tag{4.24}$$

Its derivative is

$$\begin{aligned} r'(a) &= -\frac{1}{t}p(a)q(a) - \frac{a}{t}p'(a)q(a) - \frac{a}{t}p(a)q'(a) \\ &- \frac{u'(a)}{t}p(a)^2 + 2\left(\sqrt{\frac{N}{t}} - \frac{u(a)}{t}\right)p(a)p'(a) \\ &+ \frac{w'(a)}{t}q(a)^2 + 2\left(\sqrt{\frac{N}{t}} + \frac{w(a)}{t}\right)q(a)q'(a). \end{aligned} \tag{4.25}$$

We find the following system of differential equations.

Lemma 4.4 *For $a \in \mathbb{R}$,*

$$p'(a) = \frac{a}{2t}p(a) - \left(\sqrt{\frac{N}{t}} + \frac{w(a)}{t}\right)q(a), \tag{4.26}$$

$$q'(a) = -\frac{a}{2t}q(a) + \left(\sqrt{\frac{N}{t}} - \frac{u(a)}{t}\right)p(a), \tag{4.27}$$

$$w'(a) = -p(a)^2, \tag{4.28}$$

$$u'(a) = -q(a)^2. \tag{4.29}$$

Inserting (4.26)–(4.29) into (4.25) gives a remarkably simple equation,

$$r'(a) = -\frac{1}{t}p(a)q(a). \tag{4.30}$$

Moreover, Tracy and Widom derived the following result [45, 46].

Proposition 4.5 *The function $r(a)$ solves the following nonlinear third-order differential equation,*

$$r'''(a) - \left(\frac{a^2}{t^2} - \frac{4N}{t}\right)r'(a) + \frac{a}{t^2}r(a) + 6r'(a)^2 = 0. \tag{4.31}$$

Proof By (4.26) and (4.27), we have

$$\begin{aligned}(p(a)q(a))' &= p'(a)q(a) + p(a)q'(a) \\ &= \left(\sqrt{\frac{N}{t}} - \frac{u(a)}{t}\right) p(a)^2 - \left(\sqrt{\frac{N}{t}} + \frac{w(a)}{t}\right) q(a)^2. \qquad (4.32)\end{aligned}$$

On the other hand, (4.28) and (4.29) give

$$\begin{aligned}&\left(\sqrt{\frac{N}{t}}(u(a) - w(a)) + \frac{1}{t}u(a)w(a)\right)' \\ &= \sqrt{\frac{N}{t}}(u'(a) - w'(a)) + \frac{1}{t}(u'(a)w(a) + u(a)w'(a)) \\ &= \left(\sqrt{\frac{N}{t}} - \frac{u(a)}{t}\right) p(a)^2 - \left(\sqrt{\frac{N}{t}} + \frac{w(a)}{t}\right) q(a)^2. \qquad (4.33)\end{aligned}$$

Then, we find the equality

$$(p(a)q(a))' = \left(\sqrt{\frac{N}{t}}(u(a) - w(a)) + \frac{1}{t}u(a)w(a)\right)'. \qquad (4.34)$$

For finite x, $\mathbf{1}_{(x \geq a)} \to 0$ as $a \to \infty$, and $A(a) \to 0, B(a) \to 0$ as $a \to \infty$. Therefore $p(a), q(a), w(a)$ and $u(a)$ all become zero as $a \to \infty$. By integrating both sides of (4.34) from a to ∞, we obtain the equality

$$p(a)q(a) = \sqrt{\frac{N}{t}}(u(a) - w(a)) + \frac{1}{t}u(a)w(a). \qquad (4.35)$$

If we use (4.32), the derivative of (4.30) is written as

$$r''(a) = -\frac{1}{t}\left\{\left(\sqrt{\frac{N}{t}} - \frac{u(a)}{t}\right) p(a)^2 - \left(\sqrt{\frac{N}{t}} + \frac{w(a)}{t}\right) q(a)^2\right\},$$

and then

$$\begin{aligned}r'''(a) &= -\frac{2}{t^2}p(a)^2q(a)^2 - \frac{a}{t^2}\left\{\left(\sqrt{\frac{N}{t}} - \frac{u(a)}{t}\right) p(a)^2 + \left(\sqrt{\frac{N}{t}} + \frac{w(a)}{t}\right) q(a)^2\right\} \\ &\quad + \frac{4}{t}p(a)q(a)\left[\frac{N}{t} - \frac{1}{t}\left\{\sqrt{\frac{N}{t}}(u(a) - w(a)) + \frac{1}{t}u(a)w(a)\right\}\right], \qquad (4.36)\end{aligned}$$

where (4.26)–(4.29) are used. By (4.24) and (4.35), (4.36) is rewritten as

$$r'''(a) = -\frac{a}{t^2}r(a) - \left(\frac{a^2}{t^3} - \frac{4N}{t^2}\right) p(a)q(a) - \frac{6}{t^2}(p(a)q(a))^2.$$

By combining it with (4.30), we obtain (4.31). This completes the proof. ∎

4.4 Soft-edge scaling limit

For $N \in \mathbb{N}, t \in (0, \infty)$, we perform the variable transformation $a \to u$ by

$$a = 2\sqrt{Nt} + \sqrt{t}N^{-1/6}u \quad \Longleftrightarrow \quad u = (a - 2\sqrt{Nt})t^{-1/2}N^{1/6}. \tag{4.37}$$

Since $\partial/\partial a = t^{-1/2}N^{1/6}\partial/\partial u$, if we set $\widetilde{r}(u) = t^{1/2}N^{-1/6}r(a)$ with (4.37), (4.31) is transformed into

$$\widetilde{r}'''(u) - 4u\widetilde{r}'(u) + 2\widetilde{r}(u) + 6\widetilde{r}'(u)^2 - N^{-2/3}u\{u\widetilde{r}'(u) - \widetilde{r}(u)\} = 0.$$

On the other hand, (4.9) is written as

$$\exp\left(- \int_{(s-2\sqrt{Nt})t^{-1/2}N^{1/6}}^{\infty} du\, \widetilde{r}(u) \right).$$

Let $x = (s - 2\sqrt{Nt})t^{-1/2}N^{1/6}$. Then

$$\max_{1 \le i \le N} X_i(t) < s \quad \Longleftrightarrow \quad \frac{\max_{1 \le i \le N} X_i(t) - 2\sqrt{Nt}}{t^{1/2}N^{-1/6}} < x.$$

Therefore, we have the following limit,

$$\lim_{N \to \infty} \mathrm{P}_{\mathrm{Hermite}}^{(N,t)} \left[\frac{\max_{1 \le i \le N} X_i(t) - 2\sqrt{Nt}}{t^{1/2}N^{-1/6}} < x \right] = \exp\left(- \int_x^{\infty} du\, \widetilde{r}(u) \right), \tag{4.38}$$

where $\widetilde{r}(u)$ solves the equation

$$\widetilde{r}'''(u) - 4u\widetilde{r}'(u) + 2\widetilde{r}(u) + 6\widetilde{r}'(u)^2 = 0. \tag{4.39}$$

With (4.37), the BM scaling variable $a/\sqrt{2t}$ behaves as

$$\frac{a}{\sqrt{2t}} = \sqrt{2N} + \frac{1}{\sqrt{2}}N^{-1/6}u,$$

which is the same as (3.15). Then the present limit $N \to \infty$ realizes the *soft-edge scaling limit* discussed in Section 3.2. By Proposition 3.2, we can conclude that the left-hand side of (4.38) is equal to [10]

$$\mathrm{P}_{\mathrm{Airy}} \left[\max_{1 \le i \le N} X_i < x \right] = \mathop{\mathrm{Det}}_{(u,v) \in \mathbb{R}^2} \left[\delta_u(\{v\}) - \mathrm{K}_{\mathrm{Airy}}(u,v)\mathbf{1}_{(v \ge x)} \right], \quad x \in \mathbb{R},$$

where the Airy kernel, $\mathrm{K}_{\mathrm{Airy}}$, is given by (3.18). Since $\mathrm{P}_{\mathrm{Airy}}$ is a stationary probability measure, this distribution obtained in the limit (4.38) does not depend on time $t \in (0, \infty)$.

4.5 Painlevé II and limit theorem of Tracy and Widom

Let

$$\widetilde{r}(u) = \int_u^\infty dv\, f(v)^2. \tag{4.40}$$

Then (4.39) is written as

$$f'(u)^2 + f(u)f''(u) - 2uf(u)^2 - \int_u^\infty dv\, f(v)^2 - 3f(u)^4 = 0.$$

If we differentiate this equation by u, we obtain

$$\left\{ f(u)\frac{d}{du} + 3f'(u) \right\} \left\{ f''(u) - uf(u) - 2f(u)^3 \right\} = 0.$$

Here we consider the equation

$$f''(u) = uf(u) + 2f(u)^3, \tag{4.41}$$

which is a special case of the *Painlevé II equation* (see, for instance, Chapter 21 and Appendix A.45 in [33], Chapter 8 in [11], Chapter 3 in [2], and Chapter 9 in [1]). Since (4.40) gives

$$\begin{aligned} \int_x^\infty du\, \widetilde{r}(u) &= \int_x^\infty du \int_u^\infty dv\, f(v)^2 \\ &= \int_x^\infty dv\, f(v)^2 \int_x^v du = \int_x^\infty dv\, (v-x) f(v)^2, \end{aligned}$$

the RHS of (4.38) is written as $\exp(-\int_x^\infty dv\, (v-x)f(v)^2)$.

By (4.12), in the soft-scaling limit, we find

$$\widetilde{r}(u) \simeq \mathrm{K}_{\mathrm{Airy}}(u,u) \quad \text{as } u \to \infty.$$

We note that the integral representation of $\mathrm{K}_{\mathrm{Airy}}$ (3.17) gives

$$\mathrm{K}_{\mathrm{Airy}}(u,u) = \int_0^\infty dw\, \mathrm{Ai}(u+w)^2 = \int_u^\infty dv\, \mathrm{Ai}(v)^2.$$

Comparing this with (4.40), we can conclude that

$$f(u) \simeq \mathrm{Ai}(u) \quad \text{as } u \to \infty. \tag{4.42}$$

Hastings and McLeod [18] proved that the Painlevé II equation (4.41) has a unique solution $f_{\mathrm{HM}}(u)$, which satisfies (4.42).

Now we arrive at the following limit theorem for the maximum position of particles of the Dyson model with an infinite number of particles.

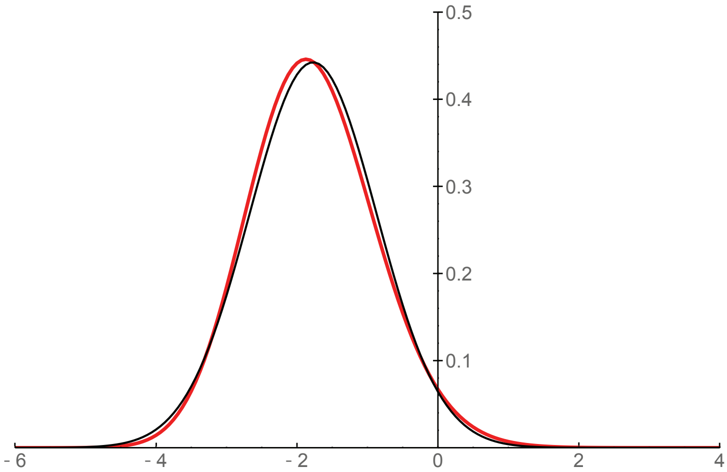

Figure 1: The probability density function of the Tracy–Widom distribution (4.46) is shown by a red curve. The black curve shows the probability density function of the Gaussian distribution (4.48) with the same values of mean and variance as the Tracy–Widom distribution given by (4.47) ($\mu = \mu_{\rm TW}$, $\sigma^2 = \sigma^2_{\rm TW}$).

Theorem 4.6 *For any $t \in (0, \infty)$, the probability*

$$\lim_{N \to \infty} \mathrm{P}^{(N,t)}_{\rm Hermite} \left[\frac{\displaystyle\max_{1 \le i \le N} X_i(t) - 2\sqrt{Nt}}{t^{1/2} N^{-1/6}} < x \right] = \mathrm{P}_{\rm Airy} \left[\max_{i \in \mathbb{N}} X_i < x \right], \quad x \in \mathbb{R}, \tag{4.43}$$

has the following two expressions,

$$F_{\rm TW}(x) = \mathop{\mathrm{Det}}_{(u,v) \in \mathbb{R}^2} \left[\delta_u(\{v\}) - \mathrm{K}_{\rm Airy}(u,v) \mathbf{1}_{(v \ge x)} \right] \tag{4.44}$$

$$= \exp\left(- \int_x^\infty dv\, (v-x) f_{HM}(v)^2 \right), \quad x \in \mathbb{R}. \tag{4.45}$$

Here the former is the Fredholm determinantal expression, and the latter is the expression in terms of the Hastings–McLeod solution f_{HM} of the Painlevé II equation (4.41).

We regard (4.43) as the *central limit theorem* (CLT) of $\mathrm{P}^{(N,t)}_{\rm Hermite}$ in $N \to \infty$ at the (right) soft edge in which the mean value is given by $2\sqrt{Nt}$. The exponent of the CLT is $-1/6$, which is very different from the classical exponent $1/2$ in the Gaussian classical CLT.

The probability distribution function (4.45) is called the *Tracy–Widom distribution* [45, 46]. It has the probability density function

$$p_{\rm TW}(x) = \frac{dF_{\rm TW}(x)}{dx}, \quad x \in \mathbb{R}. \tag{4.46}$$

Numerical values of the mean, variance, skewness, and kurtosis are the following (see [47],

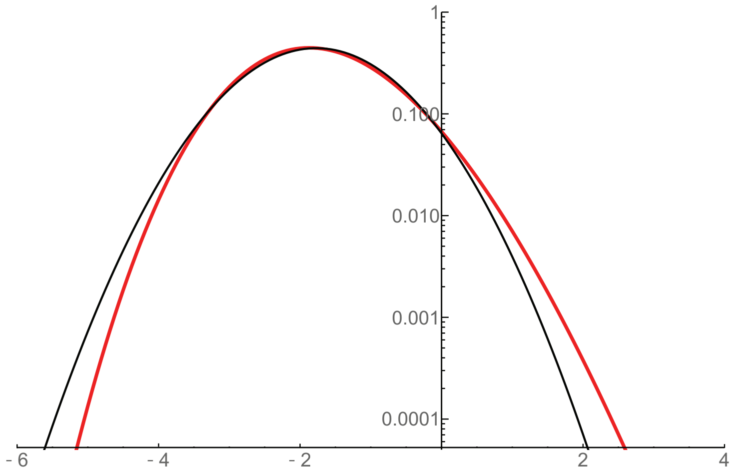

Figure 2: The semi-log plots of the probability density function of the Tracy–Widom distribution (4.46) (the red curve) and that of the Gaussian distribution (4.48) with the same values of mean and variance (the black curve).

Section 9.4.2 in [11], and [37]),

$$\begin{aligned}\mu_{\rm TW} &= \int_{-\infty}^{\infty} x p_{\rm TW}(x) dx = -1.771086807, \\ \sigma_{\rm TW}^2 &= \int_{-\infty}^{\infty} (x-\mu_{\rm TW})^2 p_{\rm TW}(x) dx = 0.813194792,\end{aligned}$$

$$\begin{aligned}S_{\rm TW} &= \int_{-\infty}^{\infty} \left(\frac{x-\mu_{\rm TW}}{\sigma_{\rm TW}} \right)^3 p_{\rm TW}(x) dx = 0.224084203, \\ K_{\rm TW} &= \int_{-\infty}^{\infty} \left(\frac{x-\mu_{\rm TW}}{\sigma_{\rm TW}} \right)^4 p_{\rm TW}(x) dx - 3 = 0.093448087. \end{aligned} \tag{4.47}$$

Figure 1 shows the comparison between $p_{\rm TW}(x)$ and the probability density function of the Gaussian distribution

$$p_{\rm G}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2}, \quad x \in \mathbb{R}, \tag{4.48}$$

with the same values of mean and variance as the Tracy–Widom distribution given by (4.47) ($\mu = \mu_{\rm TW}$, $\sigma^2 = \sigma_{\rm TW}^2$). The difference between $p_{\rm TW}$ and $p_{\rm G}$ can be shown better, if we represent them in the semi-log plots as given by Fig. 2.

Acknowledgements This manuscript was prepared for "Mini Workshop : Modern Theory of Stochastic Particles" in 27-28 June 2018 at Polytechnic of Wrocław. The author expresses his gratitude to the organizers, Jacek Małecki and Piotr Graczyk. The author is on sabbatical leave from Chuo University, and this manuscript was prepared at Fakultät für Mathematik, Universität Wien, in which the present author thanks Christian Krattenthaler very much for his hospitality. This work was supported by the Grant-in-Aid for Scientific Research (C) (No.26400405), (B) (No.18H01124), and (S) (No.16H06338) of Japan Society for the Promotion of Science.

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