BESSEL POTENTIALS, HITTING DISTRIBUTIONS AND GREEN FUNCTIONS

T. BYCZKOWSKI, J. MAŁECKI, AND M. RYZNAR

Abstract. The purpose of the paper is to find explicit formulas for basic objects pertaining to the potential theory of the operator \((I - \Delta)^{\alpha/2}\), which is based on Bessel potentials \(J_\alpha = (I - \Delta)^{-\alpha/2}\), \(0 < \alpha < 2\). We compute the harmonic measure of the half-space and obtain a concise form for the corresponding Green function of the operator \((I - \Delta)^{\alpha/2}\). As an application we provide sharp estimates for the Green function of the half-space for the relativistic process.

1. Introduction

In this paper we deal with the potential theory of \((I - \Delta)^{\alpha/2}\), \(0 < \alpha < 2\), where \(\Delta\) is the Laplace operator on \(\mathbb{R}^d\). The (formal) inverse \(J_\alpha = (I - \Delta)^{-\alpha/2}\) is called the Bessel potential operator, and it has an integral representation with the following (Bessel) convolution kernel:

\[
2^{1-(d+\alpha)/2} \frac{K_{(d-\alpha)/2}(|x|)}{\Gamma(\alpha/2)\pi^{d/2}|x|^{(d-\alpha)/2}},
\]

where \(K_\alpha\) denotes the modified Bessel function of the third kind (see Section 2, Preliminaries). The reader interested in properties of Bessel potentials is referred to an exhaustive treatise [A1], [A2]. The significance of Bessel potentials is that the Sobolev space \(L^p_\alpha(\mathbb{R}^d)\) can be defined in terms of \(J_\alpha\). To be more specific, we define \(L^p_\alpha(\mathbb{R}^d)\) as the subspace of \(L^p(\mathbb{R}^d)\), consisting of all \(f\) which can be written in the form \(f = J_\alpha g, \ g \in L^p(\mathbb{R}^d)\). The norm of \(f\) is written as \(||f||_{p,\alpha}\), and it is defined as equal to the \(L^p\) norm of \(g\) (see [S], ch. V). The spaces \(L^p_\alpha(\mathbb{R}^d)\) play an important role in Harmonic Analysis and Partial Differential Equations (see e.g. [S] and [H]).

Closely related to the operator \((I - \Delta)^{\alpha/2}\) is the fractional Laplacian \((-\Delta)^{\alpha/2}\). Its inverse \(I_\alpha = (-\Delta)^{-\alpha/2}\), called the Riesz potential operator, has an integral (Riesz) convolution kernel of the form

\[
\frac{\Gamma((d - \alpha)/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)} \frac{1}{|x|^{d-\alpha}}, \quad \text{if} \ \alpha < d, \ 0 < \alpha < 2.
\]

If \(\alpha > d = 1\) we consider instead the so-called compensated potentials (see e.g. [BGR]). For \(\alpha > d = 1\) they are of the same form as above. When \(\alpha = d = 1\)
the compensated \((\text{logarithmic})\) potential is \((1/\pi) \ln(1/|x|)\). The reader interested in potential theory based on Riesz kernels is referred to \([La]\).

The potential theories based on kernels \(K_\alpha \) and \(I_\alpha\) can be analyzed in terms of stochastic processes (for \(0 < \alpha < 2\)): the Riesz potentials can be expressed by means of the \(\alpha\)-stable rotation invariant \(\text{Lévy process}\), and the Bessel potentials are related to the so-called \(\text{relativistic} \ \alpha\)-stable process.

The homogeneity of Riesz kernels yields elegant and transparent formulas of the potential theory of \((-\Delta)^{\alpha/2}\), much like in the Newtonian case. In particular, explicit formulas for the Poisson kernel and the Green function for the ball and half-space in \(\mathbb{R}^d\) (see e.g. \([BGR]\)) are available.

In contrast to this situation, up to now, there were no explicit formulas known either for harmonic measure or the Green function for the operator \((I - \Delta)^{\alpha/2}\), for sets such as half-spaces or balls. The purpose of the present paper is to fill in this gap by providing explicit formulas for half-spaces.

To formulate our main results denote by \(\mathbb{H} = \{(x_1, \ldots, x_d); \ x_d > 0\}\) the upper half-space in \(\mathbb{R}^d\) and let \(P_{\mathbb{H}}(x, u)\) be the Poisson kernel for \(\mathbb{H}\), that is, the \(\alpha\)-density of the harmonic measure of \(\mathbb{H}\) for \((I - \Delta)^{\alpha/2}\). Our first main result is the following:

**Poisson kernel of \(\mathbb{H}\) for \((I - \Delta)^{\alpha/2}\).**

\[
P_{\mathbb{H}}(x, u) = \frac{2 \sin(\pi \alpha/2)}{\pi} \frac{x_d}{2^{d/2}} \frac{K_{d/2}(|x - u|)}{|x - u|^{d/2}}, \quad u_d < 0 < x_d.
\]

The second main result can be stated as follows:

**Green function of \(\mathbb{H}\) for \((I - \Delta)^{\alpha/2}\).**

\[
G_{\mathbb{H}}(x, y) = \frac{2^{1-\alpha} \sin(\pi \alpha/2)}{(2\pi)^{d/4} \Gamma(\alpha/2)^2} \int_0^{4\pi a^2} t^{d-1} (t+1)^{d/4} \frac{K_{d/2}(|x-y|(t+1)^{1/2})}{(t+1)^{d/4}} dt, \quad x, y \in \mathbb{H}.
\]

We note that, in spite of the fact that our kernels are not homogeneous, the resulting formulas are transparent and very similar to those for the \(\alpha\)-stable case.

Throughout the paper we employ a mixture of probabilistic and analytic methods, even though an analytic approach to the above results is fully feasible. We consider the operator \(H_\alpha = I - (I - \Delta)^{\alpha/2}\), which is the infinitesimal generator of the relativistic \(\alpha\)-stable process. We then identify Bessel kernels as kernels of a \(\text{1-resolvent}\) for the semigroup associated with the relativistic \(\alpha\)-stable process and compute the corresponding \(1\)-Poisson kernel and \(1\)-Green function for half-spaces.

Let us note that the relativistic \(\alpha\)-stable process, apart from its usefulness in analyzing \((I - \Delta)^{\alpha/2}\), is an interesting subject of study on its own, mainly because of its applications in relativistic quantum mechanics. To be more specific, let us point out that for \(\alpha = 1\) the generator of this process has the form

\[
H_1 = I - (I - \Delta)^{1/2}
\]

and represents the kinetic energy of a relativistic particle with the unit mass. Generators of this kind were investigated for example by E. Lieb \([L]\) in connection with the problem of stability of relativistic matter. An interested reader will find references on this subject in the papers \([C],[Ry]\).

The organization of the paper is as follows. We first compute the formulas for the harmonic measure and Green function for the one-dimensional case. To this end we apply complex-variable methods and some real-variable manipulations with definite integrals to obtain a satisfactory form of the Green function. The
d-dimensional case is then settled via an application of (d − 1)-dimensional Fourier transform. For technical reasons, we have to consider the Poisson kernel and Green function not only for the operator $(I - \Delta)^{\alpha/2}$ but also for $(m^2/\alpha I - \Delta)^{\alpha/2}$. Let us point out that we do not apply Kelvin’s transform (see [La], IV. 5), which was an indispensable tool in the multi-dimensional α-stable case so far. By a limiting procedure we obtain the well-known formulas for the α-stable case (see e.g. [R]). The last section is devoted to various estimates for the Green function of the half-space $\mathbb{H}$, computed for the relativistic α-stable process (that is, for the operator $H_\alpha = I - (I - \Delta)^{\alpha/2}$). To distinguish it from the corresponding object computed for the operator $(I - \Delta)^{\alpha/2}$ we call it a $\theta$-Green function. Our estimates are sharp for $x, y \in \mathbb{H}$ with $|x - y| < 1$.

2. Preliminaries

In this section we collect some preliminary material. For more detailed information regarding the α-stable relativistic process, see [KV] and [C]. For questions regarding Markov processes, semigroup properties and basic potential theory the reader is referred to [ChZ] and [BG].

2.1. Basic notation. By $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$, we denote the set of all positive integers, integers, real numbers, and complex numbers, respectively. $\mathbb{R}^d$ denotes the d-dimensional Euclidean space and $(x, y)$ denotes the standard inner product of $x, y \in \mathbb{R}^d$, $(x, y) = \sum_{i=1}^d x_i y_i$, $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d)$, and $|\cdot|$ is the norm induced by $(\cdot, \cdot)$. We denote $\mathbb{R}_+ = [0, \infty)$. For any subset $A$ of $\mathbb{R}^d$, we denote its complement by $A^c = \mathbb{R}^d \setminus A$, its closure by $\bar{A}$, and its boundary by $\partial A$. By $1_A$ we denote the indicator function of a subset $A$ of $\mathbb{R}^d$.

2.2. Bessel functions. Various potential-theoretic objects in the theory of the relativistic process are expressed in terms of modified Bessel functions $K_\theta$ of the third kind, also called Macdonald functions. For convenience here we collect basic information about these functions.

The modified Bessel function $I_\theta$ of the first kind is defined by (see e.g. [E], 7.2.2 (12))

$$I_\theta(z) = \left(\frac{z}{2}\right)^\theta \sum_{k=0}^\infty \left(\frac{z}{2}\right)^{2k} \frac{1}{k!\Gamma(k + \theta + 1)}, \quad z \in \mathbb{C} \setminus (-\mathbb{R}_+),$$

where $\theta \in \mathbb{R}$ is not an integer and $z^\theta$ is the branch that is analytic on $\mathbb{C} \setminus (-\mathbb{R}_+)$ and positive on $\mathbb{R}_+ \setminus \{0\}$. The modified Bessel function of the third kind is defined by (see [E], 7.2.2 (13) and (36))

$$K_\theta(z) = \frac{\pi}{2\sin(\theta\pi)} [I_{-\theta}(z) - I_\theta(z)], \quad \theta \notin \mathbb{Z},$$

$$K_n(z) = \lim_{\theta \to n} K_\theta(z) = (-1)^n \frac{1}{2} \left[ \frac{\partial I_{-\theta}}{\partial \theta} - \frac{\partial I_\theta}{\partial \theta} \right]_{\theta=n}, \quad n \in \mathbb{Z}.$$

The asymptotic expansion of $K_\theta(z)$ is given by (see [E], 7.4.1. (4))

$$K_\theta(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left[ \sum_{k=0}^{M-1} \frac{\Gamma(1/2 + \theta + k)}{k!\Gamma(1/2 + \theta - k)}(2z)^{-k} + O(|z|^{-M}) \right],$$

where $M = 1, 2, \ldots$, and $-3\pi/2 < \arg z < 3\pi/2$. 


Observe that the series in (2.1) defines an analytic function of the complex variable $z$. From (2.1), (2.2) and (2.3) we have that $K_{\vartheta}(z)$ has a branch cut along the negative real axis in the complex $z$-plane. Moreover, the following formula for any integer $k$ holds (see [E], 7.11. (45)):

$$K_{\vartheta}(ze^{ik\pi}) = e^{-ik\pi\vartheta} K_{\vartheta}(z) - i\pi \frac{\sin(k\pi\vartheta)}{\sin(\pi\vartheta)} I_{\vartheta}(z), \quad \vartheta \notin \mathbb{Z}.$$

When $\vartheta$ is an integer $n$ we put

$$\frac{\sin(k\pi\vartheta)}{\sin(\pi\vartheta)} := k(-1)^{n(k+1)} \left( = \lim_{\zeta \to \vartheta} \frac{\sin(k\pi\zeta)}{\sin(\pi\zeta)} \right).$$

We will also use the following integral representations of the function $K_{\vartheta}(z)$ ([E], 7.11 (23) or [GR], 8.432 (6)):

$$K_{\vartheta}(z) = 2^{\vartheta-1} z^\vartheta \int_0^\infty e^{-t} e^{-\frac{z^2}{4t}} t^{-\vartheta-1} dt,$$

where $\Re(z^2) > 0$, $|\arg z| < \frac{\pi}{2}$. Moreover (see [GR], 8.432 (3)),

$$K_{\vartheta}(z) = \left(\frac{z}{2}\right)^\vartheta \frac{\Gamma(1/2)}{\Gamma(\vartheta + 1/2)} \int_1^\infty e^{-zt} \left(\frac{t}{t^2 - 1}\right)^{1/2-\vartheta} dt,$$

where $\Re(\vartheta + 1/2) > 0$, $|\arg z| < \frac{\pi}{2}$. In the sequel we will use the asymptotic behaviour of $K_{\vartheta}$, $\vartheta > 0$, as a function of the real variable $r$:

$$K_{\vartheta}(r) \approx \frac{\Gamma(\vartheta)}{2} \left(\frac{r}{2}\right)^{-\vartheta} r \to 0^+,$$

(2.9) $$K_{\vartheta}(r) \approx \frac{\sqrt{\pi}}{\sqrt{2r}} e^{-r}, \quad r \to \infty,$$

where $g(r) \equiv f(r)$ means that the ratio of $g$ and $f$ tends to 1. For $\vartheta < 0$ we have $K_{\vartheta}(r) = K_{-\vartheta}(r)$, which determines the asymptotic behaviour for negative indices. We also state the following results concerning differentiability properties of the functions $K_{\vartheta}(r)$ with respect to the real variable $r > 0$ (see [E], 7.21, 7.22):

$$\frac{d}{dr}[r^{-\vartheta} K_{\vartheta}(r)] = -r^{-\vartheta} K_{\vartheta+1}(r), \quad \frac{d}{dr}[r^\vartheta K_{\vartheta}(r)] = -r^\vartheta K_{\vartheta-1}(r).$$

Consequently, for fixed $\vartheta > 0$, we obtain

$$r^{-\vartheta} K_{\vartheta}(r), r^\vartheta K_{\vartheta}(r)$$

are decreasing in $r > 0$.

2.3. Relativistic processes. A stochastic process $Y_t$, $t \geq 0$, is called a Lévy process on $\mathbb{R}^d$ if it has stationary independent increments, it is stochastically continuous, that is, $\lim_{\epsilon \to 0} P(|Y_s - Y_t| > \epsilon) = 0$ for every $\epsilon > 0$, and it is right-continuous with left-hand limits (see e.g. [Sa]). Observe that we do not assume that $Y_0 = 0$, because our processes start from various points of $\mathbb{R}^d$. As usual, by $E^x$ we denote the expectation with respect to the distribution $P^x$ of the process starting from $x \in \mathbb{R}^d$. Every Lévy process $Y_t$ is Markov with transition probabilities given by

$$P_t(x, A) = P^x(Y_t \in A) = \mu_t(A - x),$$

where $\mu_t$ is the one-dimensional distribution of $Y_t$ with respect to $P^0$. When $P_t(x, A) = \int_A p_t(x, y) \, dy$ then $p_t(x, y) = p_t(x - y, 0)$, and, with a certain abuse of notation, we denote this last object by $p_t(x - y)$ and call $p_t(x)$ the transition density function of the process $Y_t$.

A subordinator $W_t$, $t \geq 0$, is a one-dimensional Lévy process with increasing sample paths and such that $W_0 = 0$. According to general theory (see again [Sa], Ch. 6), if $Y_t$ is a Lévy process on $\mathbb{R}^d$ and $W_t$ is a subordinator and the processes $Y_t$
and \( W_t \) are independent, then \( X_t = Y_{W_t} \) is another Lévy process. We now adapt this procedure to our needs, specifying processes \( Y_t \) and \( W_t \).

We begin by recalling the definition of the standard \( \alpha/2 \)-stable subordinator \( S^\alpha_t \) with the Laplace transform \( E^0 e^{-\lambda S^\alpha_t} = e^{-\lambda^\alpha/2} \). Throughout the entire paper \( \alpha \) denotes the stability index of the process and we always assume \( 0 < \alpha < 2 \). The transition density function of \( S^\alpha_t \) will be denoted by \( \theta^\alpha_t(u) \). Here \( u, t > 0 \).

For \( m > 0 \) we define another subordinating process \( T^\alpha_m t \) by modifying \( \theta^\alpha_t(u) \) as follows:

\[
(2.11) \quad \theta^\alpha_{m,t}(u) = e^{mt} \theta^\alpha_t(u) e^{-m^2/nu}, \quad u > 0.
\]

The Laplace transform of \( T^\alpha_m t \) is

\[
E^0 e^{-\lambda T^\alpha_m t} = e^{mt} e^{-t(\lambda + m^2/\alpha)^{\alpha/2}}.
\]

Let \( B_t \) be the Brownian motion in \( \mathbb{R}^d \) with the characteristic function \( E^0 e^{i \langle \xi, B_t \rangle} = e^{-t|\xi|^2} \). The transition density function of \( B_t \) is denoted by \( g_t \) and is of the form

\[
g_t(u) = \frac{1}{(4\pi t)^{d/2}} e^{-|u|^2/4t}.
\]

Assume that the processes \( T^\alpha_m t \) and \( B_t \) are stochastically independent. Then the process \( X^\alpha_m t = B_{T^\alpha_m t} \) is called the \( \alpha \)-stable relativistic process (with parameter \( m \)). In the sequel we will use the generic notation \( X^m_t \) instead of \( X^\alpha_m t \). If \( m = 1 \), we then write \( T^\alpha_t \) instead of \( T^\alpha_1 t \) and \( X_t \) instead of \( X^1_t \).

When \( m = 0 \) we obtain the \( \alpha \)-stable rotation invariant Lévy process which is denoted by \( Z_t \).

We obtain

\[
P^x(X^m_t \in A) = E^x[1_A(B_{T^\alpha_m t})]
\]

\[
= \int_0^\infty \left[ \int_A g_u(x - y) dy \right] \theta^\alpha_{m,t}(u) du = \int_A \left[ \int_0^\infty g_u(x - y) \theta^\alpha_{m,t}(u) du \right] dy.
\]

This provides the formula for the transition density function of the process \( X^m_t \):

\[
(2.12) \quad p^m_t(x) = \int_0^\infty \theta^\alpha_{m,t}(u) g_u(x) du,
\]

where \( p^m_t(x), t > 0 \), is a semigroup under convolution. A particular case when \( \alpha = 1 \) yields the relativistic Cauchy semigroup on \( \mathbb{R}^d \) with parameter \( m \) and is denoted by \( p^m_t \).

The formula below exhibits the explicit form of this transition density function.

Lemma 2.1 (Relativistic Cauchy semigroup).

\[
(2.13) \quad p^m_t(x) = 2(m/2\pi)^{(d+1)/2} t e^{mt} \frac{K_{(d+1)/2}(m|x|^2 + t^2)^{1/2}}{(|x|^2 + t^2)^{d+1/2}}.
\]

Proof. Observe that \( \theta^1_t(u) \), the transition density function of the 1/2-stable subordinator, is of the form

\[
\theta^1_t(u) = \frac{t}{\sqrt{4\pi}} u^{-3/2} e^{-t^2/4u},
\]
so, taking into account (2.6), we obtain
\[ p_t^m(x) = e^{mt} \int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{-|x|^2/4u} e^{-m^2 u t} \frac{t}{\sqrt{4\pi}} u^{-3/2} e^{-t^2/4u} du \]
\[ = \frac{te^{mt}}{(4\pi)^{d/2}} \int_0^\infty e^{-m^2 u e^{-(|x|^2+2t^2)/4u}} \frac{du}{u^{d/2+1}} \]
\[ = 2(m/2\pi)^{(d+1)/2} te^{mt} K(d+1)/2(m(|x|^2 + t^2)^{1/2}) \]
\[ \frac{1}{(|x|^2 + t^2)^{d/4}}. \]

In the next lemma we compute the Fourier transform of the transition density function (2.12).

**Lemma 2.2** (Fourier transform of \( p_t^m \)). The Fourier transform of the \( \alpha \)-stable relativistic transition density function \( p_t^m \) is of the form
\[ \hat{p}_t^m(z) = e^{mt} e^{-t(|z|^2 + m^{2/\alpha})^{\alpha/2}}. \]

**Proof.**
\[ \hat{p}_t^m(z) = \int_{\mathbb{R}^d} p_t^m(x)e^{iz(x)} dx = \int_{\mathbb{R}^d} \int_0^\infty e^{mt} g_u(x)e^{-m^{2/\alpha} u} \theta^\alpha_t(u) du e^{i(z,x)} dx \]
\[ = e^{mt} \int_0^\infty e^{-u|z|^2} e^{-m^{2/\alpha} u} \theta^\alpha_t(u) du = e^{mt} \int_0^\infty e^{-u(|z|^2 + m^{2/\alpha})} \theta^\alpha_t(u) du \]
\[ = e^{mt} e^{-t(|z|^2 + m^{2/\alpha})^{\alpha/2}}. \]

Specializing this to the case \( \alpha = 1 \) we obtain
\[ (2.14) \quad \hat{p}_t^m(z) = e^{mt} e^{-t(|z|^2 + m)^{1/2}}. \]

From the Fourier transform we obtain the following scaling property:
\[ (2.15) \quad p_t^m(x) = m^{d/\alpha} p_{mt}(m^{1/\alpha} x). \]

In terms of one-dimensional distributions of the relativistic process (starting from the point 0) (2.15) reads as
\[ X_t^m \sim m^{-1/\alpha} X_{mt}^1, \]
where \( X_t^m \) denotes the relativistic \( \alpha \)-stable process with parameter \( m \) and \( "\sim" \) denotes the equality of distributions. Because of this scaling property, we often restrict our attention to the case when \( m = 1 \), if not specified otherwise. When \( m = 1 \) we omit the superscript "1", i.e. we write \( p_t(x) \) instead of \( p_t^1(x) \).

In what follows we will work within the framework of the so-called \( \lambda \)-potential theory, for \( \lambda > 0 \).

The kernel of the \( \lambda \)-resolvent of the semigroup generated by \( X_t^m \) will be denoted by \( U_\lambda^m(x) \) and will be called the \( \lambda \)-potential of the process \( X_t^m \). We have
\[ U_\lambda^m(x) = \int_0^\infty e^{-\lambda t} p_t^m(x) dt. \]

The function has a particularly simple expression when \( \lambda = m \), and we state it for further reference.
Lemma 2.3 (m-potential for the relativistic process with parameter m).

\[
U_m^m(x) = \frac{2^{1-(d+\alpha)/2} m^{d+\alpha} K_{(d-\alpha)/2}(m^{1/\alpha}|x|)}{\Gamma((\alpha/2)\pi^{d/2}) |x|^{(d-\alpha)/2}}.
\]

**Proof.** We provide calculations for \( m = 1 \); the general case follows from (2.15).

Observe first that the potential kernel of the \( \alpha/2 \)-stable subordinator is well known (and easy to obtain via the Laplace transform). Namely, we have

\[
\int_0^\infty \theta_i^\alpha(u) dt = \frac{u^{\alpha/2-1}}{\Gamma(\alpha/2)}.
\]

This and (2.16) yield

\[
U_1(x) = \int_0^\infty e^{-t} p_t(x) dt = \int_0^\infty \int_0^\infty g_u(x) e^{-u} \theta_i^\alpha(u) du dt
\]

\[
= \int_0^\infty \frac{1}{(4\pi u)^{d/2}} e^{-|x|^2/4u} \left( \int_0^\infty \theta_i^\alpha(u) dt \right) du
\]

\[
= \frac{1}{\Gamma(\alpha/2)(4\pi)^{d/2}} \int_0^\infty e^{-|x|^2/4u} \frac{du}{u^{\frac{d}{2}+1}}
\]

\[
= \frac{2^{1-(d+\alpha)/2} K_{(d-\alpha)/2}(|x|)}{\Gamma((\alpha/2)\pi^{d/2}) |x|^{(d-\alpha)/2}}.
\]

In what follows we denote by \( U_1 \) the \( \lambda \)-potential for \( \lambda = m = 1 \).

We also recall the formula for the density function \( \nu^m(x) \) of the Lévy measure of the relativistic \( \alpha \)-stable process (see [Ry]).

**Lemma 2.4** (Lévy measure of relativistic process with parameter m).

\[
\nu^m(x) = \frac{\alpha 2^{\frac{d-\alpha}{2}} m^{\frac{d+\alpha}{2}} K_{\frac{(d-\alpha)}{2}}(m^{1/\alpha}|x|)}{\pi^{d/2}(1-\frac{\alpha}{2}) |x|^{\frac{d+\alpha}{2}}}.
\]

When \( m = 1 \) we write \( \nu \) instead of \( \nu^1 \).

**Remark.** Observe that the density function of the Lévy measure of the \( \alpha \)-stable rotation invariant Lévy process is of the form

\[
\nu^\#(x) = \frac{2^\alpha \Gamma(\frac{d+\alpha}{2})}{\pi^{d/2} |\Gamma(-\alpha/2)| |x|^{d+\alpha}}.
\]

The formulas (2.15) and (2.17) exemplify a correspondence between the potential theories of \( (-\Delta)^{\alpha/2} \) and \( (I - \Delta)^{\alpha/2} \), which can be stated as follows: if an object in the potential theory of \( (-\Delta)^{\alpha/2} \) is expressed in terms of the kernel \( 1/|x|^\theta \), then the corresponding object in the potential theory of \( (I - \Delta)^{\alpha/2} \) has the kernel \( K_{\theta/2}(|x|)/|x|^\theta/2 \). We observe that this principle holds true for the Lévy measures with \( \theta = d + \alpha \) and for the potential kernels (1.2) and (1.1) with \( \theta = d - \alpha \). The objects of the Riesz potential theory are denoted with the superscript "\#".

The first exit time from an (open) set \( D \subset \mathbb{R}^d \) by the process \( X_t^m \) is defined by the formula

\[
\tau_D = \inf\{ t \geq 0; X_t^m \notin D \}.
\]

To facilitate the discussion we will assume (in what follows) that \( D \) is regular, e.g., with smooth boundary. In fact below \( D \) will be either a ball or a half-space.
The $\lambda$-harmonic measure of the set $D$ represents the basic object in the potential theory of $X_t^m$. In probabilistic terms it is defined as follows:

\[(2.19) \quad P_D^{\lambda,m}(x, A) = E^x[\tau_D < \infty; e^{-\lambda\tau_D} 1_A(X_{\tau_D}^m)].\]

Observe that for $x \in D^c$ we obtain $P_D^{\lambda,m}(x, \cdot) = \delta_x(\cdot)$, the point mass at $x$. Obviously, $P_D^{\lambda,m}(x, \mathbb{R}^d) \leq 1$. On the other hand, if $x \in D$, then, by the right-continuity of sample paths, $P_D^{\lambda,m}(x, \cdot)$ is concentrated on $D^c$. The kernel of the measure $P_D^{\lambda,m}(x, A)$, for $x \in D$ (if it exists), is called the $\lambda$-Poisson kernel of the set $D$. When $\lambda = m$ we denote the corresponding $\lambda$-harmonic measure by $P_D^{\lambda}(x, A)$ and its kernel by $P_D^{m}(x, y)$. The function $P_D^{m}$ is the primary object of our investigations. When $\lambda = m = 1$ we use the notation $P_D(x, y)$.

In Sections 3 and 4 we prove the existence of $P_D^{m}(x, y)$, while providing at the same time an explicit formula for it. The existence of a $\lambda$-Poisson kernel for general $\lambda$ can be deduced from papers [W] and [SH], but analogous explicit formulas are not available.

The process $X_t^{m,D}$ killed when exiting the set $D$ is described in terms of sample paths up to time $\tau_D$. Its transition probability, $P_t^{m,D}$, is defined by $P_t^{m,D}(x, A) = P^x(t < \tau_D; X_t^m \in A)$, $t > 0$. Correspondingly, $P_t^{m,D}$, the transition density function of $X_t^{m,D}$, can be expressed as

\[p_t^{m,D}(x, y) = p_t^{m}(x, y) - E^x[t \geq \tau_D; p_{t-\tau_D}^{m}(X_{\tau_D}^m - y)], \quad x, y \in \mathbb{R}^d.\]

Obviously, we obtain $p_t^{m,D}(x, y) \leq p_t^{m}(x, y)$, $x, y \in \mathbb{R}^d$ and $p_t^{m,D}(x, y) = 0$ if $x$ or $y$ does not belong to $D$.

$P_t^{m,D}$ is a strongly contractive semigroup (under composition), and it shares many properties of the semigroup $P_t^{m}$. In particular, it is strongly Feller and its transition density function is symmetric: $p_t^{m,D}(x, y) = p_t^{m,D}(y, x)$. When $m = 1$, we write, as before, $p_t^{D}$ instead of $p_t^{1,D}$.

The $\lambda$-potential kernel of the process $X_t^{m,D}$ is called the $\lambda$-Green function and is denoted by $G_D^{\lambda,m}$. Thus, we have

\[G_D^{\lambda,m}(x, y) = \int_0^\infty e^{-\lambda t} p_t^{m,D}(x, y) \, dt.\]

The “first passage time relation” (see [BGR]) provides another important formula for the $\lambda$-Green function of the set $D$, which is expressed in terms of the $\lambda$-harmonic measure:

\[(2.20) \quad G_D^{\lambda,m}(x, y) = U_D^m(x - y) - \int_{\mathbb{R}^d} U_D^m(z - y) P_D^{\lambda,m}(x, dz), \quad x \neq y, \quad x, y \in \mathbb{R}^d.\]

Observe that if $x \neq y$ and $x$ or $y$ belongs to $D^c$, then we obtain $G_D^{\lambda,m}(x, y) = 0$. This yields the following:

\[(2.21) \quad \int_{D^c} U_D^m(z - y) P_D^{\lambda,m}(x, dz) = U_D^m(x - y), \quad x \in D, \quad y \in D^c.\]

We are mainly interested when $\lambda = m$, and we then write $G_D^m$. If $\lambda = m = 1$ we write $G_D$. The formula (2.21) is a particular case of “balayage” or “sweeping out” (see [La], V.5) and, together with the following uniqueness lemma, is crucial for obtaining the explicit form of $P_D^{m}(x, y)$ as well as $G_D^{m}(x, y)$.
Lemma 2.5 (Uniqueness). Suppose that $\mu$ is a finite signed measure concentrated on a closed set $F \subseteq \mathbb{R}^d$ with the (finite energy) property

$$
(2.22) \int_F \int_F \frac{K_{(d-\alpha)/2}(m^{1/\alpha}|z-y|)}{|z-y|^{(d-\alpha)/2}} |\mu|(dz) |\mu|(dy) < \infty.
$$

If for every $z \in F$ we have

$$
(2.23) \int_F \frac{K_{(d-\alpha)/2}(m^{1/\alpha}|z-y|)}{|z-y|^{(d-\alpha)/2}} \mu(dy) = 0,
$$

then $\mu = 0$.

Proof. The proof of the above lemma is standard (see e.g. [BGR]); we include it for convenience of the reader. We provide the details for $m = 1$ only; the general case follows by scaling. Observe that condition (2.22) enables us to integrate equation (2.23) over the set $F$ and apply Fubini’s theorem in the calculations below:

$$
0 = \frac{2^{1-(d+\alpha)/2}}{\Gamma(\alpha/2)\pi^{d/2}} \int_F \int_F \int_0^\infty \frac{K_{(d-\alpha)/2}(z-y)}{|z-y|^{d-\alpha}} \mu(dz) \mu(dy)
$$

$$
= \int_F \int_0^\infty e^{-t} p_t(z-y) dt \mu(dy)
$$

$$
= (2\pi)^{-d} \int_F \int_0^\infty \int_{\mathbb{R}^d} e^{-t} e^{-i(|\xi|^2+1)^{\alpha/2}} d\xi dt \mu(dy)
$$

$$
= (2\pi)^{-d} \int_0^\infty \int_{\mathbb{R}^d} e^{-t(|\xi|^2+1)^{\alpha/2}} \left| \int_F e^{-i\xi z} \mu(dz) \right|^2 d\xi dt.
$$

This shows that the Fourier transform of the measure $\mu$ is zero, so the measure itself vanishes, which concludes the proof. \qed

We now state some scaling properties for $P^m_D(x,y)$, $G^m_D(x,y)$. The proof employs the scaling property (2.15) and consists of elementary but tedious calculation; hence it is omitted.

Lemma 2.6 (Scaling property). We have

$$
P^m_D(x,u) = m^{d/\alpha} P^m_m(x,m^{1/\alpha}u), \quad x \in D, u \in D^c,
$$

$$
G^m_D(x,y) = m^{-(d-\alpha)/\alpha} G^m_m(x,m^{1/\alpha}y), \quad x \in D, y \in D.
$$

In particular, if $D = \mathbb{H}$ we obtain

$$
P^m_\mathbb{H}(x,u) = m^{d/\alpha} P^m_\mathbb{H}(x,m^{1/\alpha}u), \quad x \in \mathbb{H}, u \in \mathbb{H}^c,
$$

$$
G^m_\mathbb{H}(x,y) = m^{-(d-\alpha)/\alpha} G^m_\mathbb{H}(x,m^{1/\alpha}y), \quad x \in \mathbb{H}, y \in \mathbb{H}.
$$

3. One-dimensional case

This section is basic for the whole paper. Here we establish the formulas for the Poisson kernel and Green function of $(0,\infty)$ for the operator $(m^{2/\alpha} I - \frac{d^2}{dx^2})^{\alpha/2}$. The presentation is divided into three parts. The first one relies on complex integration.
3.1. Complex-variable method. The following lemma is crucial for further purposes.

**Lemma 3.1.** For $x > 0 \geq y$ we have

\[
\sin\left(\frac{\pi\alpha}{2}\right) \int_{-\infty}^{0} \left(\frac{x}{-u}\right)^{\frac{\alpha}{2}} e^{-|x-u|} \frac{K_{\frac{1-\alpha}{2}}(|u-y|)}{|u-y|^{\frac{1-\alpha}{2}}} du = K_{\frac{1-\alpha}{2}}(|x-y|) \left|\frac{x}{|x-y|}\right|^{\frac{1-\alpha}{2}}.
\]

**Proof.** Let $x > 0 > y$ and consider the following function of the complex variable $z$:

\[
f(z) = \frac{1}{z^\frac{\alpha}{2}} \frac{e^{z-x}}{z-x} \frac{K_{\frac{1-\alpha}{2}}(z-y)}{(z-y)^{\frac{1-\alpha}{2}}}.
\]

Due to the properties of the function $K_\alpha$ given in the Preliminaries, it is easy to see that $f$ is a holomorphic function in $\mathbb{C}\setminus(-\infty,0)\setminus\{x\}$ with a branch cut along the negative real axis and has a simple pole at $z = x$. We are going to integrate the above function over the contour described below. We make the branch cut along the axis $(-\infty,0]$ and make the contour of integration wrap around this line (see the picture below). We choose the intervals $\gamma_1$ and $\gamma_2$ in such a way that their points are at the distance $0 < \epsilon < 1$ from the negative axis.

By the Cauchy theorem, we get

\[
\frac{1}{2\pi i} \int_\Gamma f(z) \, dz + \frac{1}{2\pi i} \left( \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \right) f(z) \, dz = \text{Res}_{z=x} f(z).
\]
The asymptotic expansion for the Macdonald function (2.4), which is valid for \(-\frac{3}{2}\pi < \arg z < \frac{3}{2}\pi\), gives

\[
\begin{align*}
    f(z) &= 1 + \frac{e^{z-x}}{z} \frac{e^{y-z}}{(z-y)^{1-\frac{1}{2}}} R_0(z-y) \\
    & \quad + \frac{1}{e^{y-x}} \frac{1}{z} (z-x) (z-y)^{1-\frac{1}{2}} R_0(z-y),
\end{align*}
\]

where \(R_0(z) = O(1)\) and \(z\) is large enough. Using (3.3) it is easy to show that the expression \(|f(z)||z|^2\) is bounded for large \(z\), and this implies that the integral over \(\Gamma\) of \(f(z)\) vanishes when \(r \to \infty\).

The function \(f(z)\) behaves like \(|z|^{-\alpha/2}\) near the origin (notice that \(y < 0\), and consequently the integral over \(\gamma_3\) vanishes when \(\epsilon \searrow 0\).

To calculate the limits of the integrals over \(\gamma_1\) and \(\gamma_2\) we examine the behaviour of the function \(f\) near the branch cut. For every \(u < y\), putting \(k = 1\) and \(k = -1\) in (2.4), we get

\[
\begin{align*}
    \lim_{\epsilon \to 0^+} K_{1/2}(u-y+i\epsilon) &= e^{\frac{\pi \alpha (1-\alpha)}{2}} K_{1/2}(|u-y|) - i\pi I_{1/2}(|u-y|), \\
    \lim_{\epsilon \to 0^+} K_{1/2}(u-y-i\epsilon) &= e^{\frac{\pi \alpha (1-\alpha)}{2}} K_{1/2}(|u-y|) + i\pi I_{1/2}(|u-y|).
\end{align*}
\]

We also have for \(u < y\)

\[
\begin{align*}
    \lim_{\epsilon \to 0^+} (u-y+i\epsilon)^{-\frac{1}{2}} &= e^{\frac{\pi \alpha (1-\alpha)}{2}} |u-y|^{-\frac{1}{2}}, \\
    \lim_{\epsilon \to 0^+} (u-y-i\epsilon)^{-\frac{1}{2}} &= e^{\frac{\pi \alpha (1-\alpha)}{2}} |u-y|^{-\frac{1}{2}}.
\end{align*}
\]

Next, for \(u < 0\) we have

\[
\begin{align*}
    \lim_{\epsilon \to 0^+} (u+i\epsilon)^{-\frac{1}{2}} &= e^{\frac{\pi \alpha u}{2}} |u|^{-\frac{1}{2}}, \\
    \lim_{\epsilon \to 0^+} (u-i\epsilon)^{-\frac{1}{2}} &= e^{\frac{\pi \alpha u}{2}} |u|^{-\frac{1}{2}}.
\end{align*}
\]

Using all the relations given above we obtain that for \(u < y\)

\[
\begin{align*}
    f_1^+(u) &= \lim_{\epsilon \to 0^+} f(u+i\epsilon) = -\frac{1}{|u|^2} e^{u-x} \left[ e^{\frac{\pi \alpha u}{2}} K_{1/2}(|u-y|) + i\pi I_{1/2}(|u-y|) \right], \\
    f_1^-(u) &= \lim_{\epsilon \to 0^+} f(u-i\epsilon) = -\frac{1}{|u|^2} e^{u-x} \left[ e^{-\frac{\pi \alpha u}{2}} K_{1/2}(|u-y|) + i\pi I_{1/2}(|u-y|) \right].
\end{align*}
\]

Similarly for \(y < u < 0\) we get

\[
\begin{align*}
    f_2^+(u) &= \lim_{\epsilon \to 0^+} f(u+i\epsilon) = \frac{1}{|u|^2} e^{u-x} e^{\frac{\pi \alpha u}{2}} K_{1/2}(|u-y|), \\
    f_2^-(u) &= \lim_{\epsilon \to 0^+} f(u-i\epsilon) = \frac{1}{|u|^2} e^{u-x} e^{-\frac{\pi \alpha u}{2}} K_{1/2}(|u-y|).
\end{align*}
\]

Observe that the family of functions \(g_\epsilon(u) := f(u+i\epsilon) - f(u-i\epsilon), u < 0\), indexed by positive \(\epsilon < 1\), has the following properties:

\[
\begin{align*}
    |g_\epsilon(u)| &= O(|u|^{-2}), & u \to \infty, \\
    |g_\epsilon(u)| &= O(|u|^{-\alpha/2}), & u \to 0,
\end{align*}
\]

uniformly with respect to \(\epsilon\).
For \( u \to y \), using (2.1) and (2.2), we get that

\[
|g_\varepsilon(u)| = O(|u - y|^{\alpha - 1}), \quad \alpha < 1,
\]

\[
|g_\varepsilon(u)| = O(1), \quad \alpha > 1,
\]

uniformly with respect to \( \varepsilon \).

Moreover, \( |g_\varepsilon(u)| = O(-\log(|u - y|)) \) uniformly with respect to \( \varepsilon \), when \( \alpha = 1 \) (see [E], 7.2.6 (39)).

Combining the above and using the dominated convergence theorem we find that

\[
\left( \int_{\gamma_1} + \int_{\gamma_2} \right) f(z) \, dz \to \int_{-\infty}^{y} (f_1^+(u) - f_1^-(u)) \, du + \int_{y}^{0} (f_2^+(u) - f_2^-(u)) \, du
\]

\[
= (2\pi i) \frac{\sin(\pi \alpha)}{\pi} \int_{-\infty}^{0} \frac{1}{(u - x)^{\frac{\alpha}{2}}} e^{-|u-x|K_{1-\alpha}(|u-y|)} \, du,
\]

as \( r \to \infty \) and \( \varepsilon \to 0 \). We also have

\[
\text{Res}_{z=x} f(z) = \frac{1}{x^{\frac{\alpha}{2}}} K_{\frac{\alpha}{2}}(|x-y|).
\]

Using (2.2) and the relations given above we obtain the desired formula. To end the proof we observe that both sides of (3.1) are right-continuous at \( y = 0 \), as functions of \( y \), \( y \leq 0 < x \), so the formula (3.1) also holds for \( y = 0 < x \). \( \square \)

### 3.2. Poisson kernel of \((0, \infty)\)

For \( x > 0 \) we denote

\[
Q^m_{(0, \infty)}(x, u) = \begin{cases} 
\frac{\sin(\pi \alpha/2)}{\pi} \left( \frac{x}{u} \right)^{\alpha/2} e^{-m^{1/\alpha}(x-u)}, & u < 0, \\
0, & u \geq 0.
\end{cases}
\]

For \( m = 1 \) we write \( Q_{(0, \infty)}(x, u) \) instead of \( Q^1_{(0, \infty)}(x, u) \).

**Theorem 3.2** (Poisson kernel). \( Q^m_{(0, \infty)}(x, u) \) is the Poisson kernel of \((0, \infty)\) for the operator \((m^2/\alpha I - d^2/dx^2)^{\alpha/2}\).

Let us observe that in probabilistic terms we thus identify the density function of the \( m \)-harmonic measure \( P^m_{(0, \infty)}(x, \cdot) \) (see (2.19)) for the \( \alpha \)-stable relativistic process \( X^m_t \); that is \( E^x[e^{-m^\tau_{(0, \infty)}}; X^m_{\tau_{(0, \infty)}} \in \cdot] \).

**Proof.** By the scaling property we may assume that \( m = 1 \). To prove the theorem it is enough to check that condition (2.22) of Lemma 2.5 is satisfied for the measure \( Q_{(0, \infty)}(x, u) \) concentrated on the set \( F = (-\infty, 0) \), where \( x > 0 \) is fixed. Observe that the above measure is finite:

\[
\int_{-\infty}^{0} Q_{(0, \infty)}(x, u) \, du = \int_{-\infty}^{0} \left( \frac{x}{u} \right)^{\alpha/2} e^{-x-u} \, du
\]

\[
= e^{-x} x^{\alpha/2} \int_{0}^{\infty} w^{-\alpha/2} e^{-w} \, dw \leq e^{-x} x^{\alpha/2} \int_{0}^{\infty} e^{-w} w^{(1-\alpha)/2-1} \, dw
\]

\[
= e^{-x} x^{\alpha/2-1} \Gamma(1 - \alpha/2).
\]
This property, Lemma 3.1, and the first property in (2.10) yield for fixed \( x > 0 \):
\[
\int_{-\infty}^{0} \int_{-\infty}^{0} K_{(1-\alpha)/2}(|z-y|/|x-y|^{(1-\alpha)/2}) Q_{(0,\infty)}(x, z) \, dz \, Q_{(0,\infty)}(x, y) \, dy
\]
\[
= \int_{-\infty}^{0} K_{(1-\alpha)/2}(|z-x|) Q_{(0,\infty)}(x, z) \, dz
\]
\[
\leq \frac{K_{(1-\alpha)/2}(|x|)}{|x|^{(1-\alpha)/2}} \int_{0}^{1} Q_{(0,\infty)}(x, z) \, dz < \infty.
\]

Observe that the measure \( P_{1}^{0}(\infty, x) \) is finite. From (2.21) and the last part of the calculations above, carried out for \( P_{1}^{0}(\infty) \) instead of \( Q_{(0,\infty)} \), it follows that \( P_{1}^{0}(\infty, x) \) satisfies condition (2.22) as well. Thus, the measure \( \mu_{x}(dz) = P_{1}^{0}(\infty, x, dz) - Q_{(0,\infty)}(x, z) \, dz \) satisfies all the conditions of Lemma 2.4, for \( x \in (0, \infty) \). Thus, \( \mu_{x}(dz) = 0 \), so \( Q_{(0,\infty)}(x, z) \) is the desired Poisson kernel. The proof is complete.

3.3. Green function of \((0,\infty)\).

**Theorem 3.3** (Green function). The Green function of \((0,\infty)\) for the operator \((m^{2/\alpha}I - d^{2}/dx^{2})^{\alpha/2}\) is of the form
\[
G_{\mu}(x, y) = \frac{|x-y|^{\alpha-1}}{2^{\alpha} \Gamma(\alpha/2)^{2}} \int_{0}^{(x-y)^{2}} e^{-m^{1/\alpha}|x-y|(t+1)^{1/2}} t^{\alpha/2-1} (t+1)^{-1/2} \, dt,
\]
where \( x, y > 0 \).

**Proof.** By the scaling property we may assume that \( m = 1 \). Due to the symmetry of the Green function, it is enough to determine \( G_{(0,\infty)}(x, y) \) for \( 0 < x < y \). We compute the compensator of the 1-Green function for the one-dimensional \( \alpha \)-stable relativistic process, that is, the integral in the formula (2.20). We have

\[
H(x, y) = E^{x}[e^{-\tau_{(0,\infty)}} U_{1}(X_{\tau_{(0,\infty)}}) - y]
\]

(3.4)
\[
= C(\alpha, 1) \frac{\sin(\pi \alpha/2)}{\pi} \int_{-\infty}^{0} \frac{x-\alpha/2}{-u} e^{-(x-u)} \frac{K_{1-\alpha}(y-u)}{y-u} \frac{1}{(y-u)^{1-\alpha/2}} \, du,
\]

where \( C(\alpha, 1) = \frac{\alpha^{1-\alpha/2}}{\Gamma(\alpha/2)^{2}} \). Substituting \((u)^{1-\alpha/2} = v\) and taking into account the two following well-known identities (see (2.7) for the second one)

\[
\frac{1}{x + v^{2}(2-\alpha)} = \int_{0}^{\infty} e^{-w(x+w^{2}(2-\alpha))} \, dw,
\]

\[
U_{1}(z) = C(\alpha, 1) \frac{K_{1-\alpha}(z)}{z^{1-\alpha/2}}
\]

\[
= \int_{1}^{\infty} e^{-zt} \frac{1}{(t^{2}-1)^{\alpha/2}} \, dt, \quad z > 0,
\]

we obtain that the right-hand side of (3.4) is of the form

\[
C(x) \int_{0}^{\infty} \left\{ \int_{0}^{\infty} e^{-wz} e^{-w^{2}(2-\alpha)} \, dw \right\} e^{-z^{2}/(2-\alpha)} \int_{1}^{\infty} \frac{e^{-(y+z^{2}/(2-\alpha))}}{(t^{2}-1)^{\alpha/2}} \, dt \, dv
\]

\[
= C(x) \int_{1}^{\infty} \int_{0}^{\infty} e^{-wx^{2}/(2-\alpha)} e^{-(w^{2}+t^{2}/(2-\alpha))} \, dt \, dv \frac{e^{-yt}}{(t^{2}-1)^{\alpha/2}} \, dt \, dv.
\]
where \( C(x) = \frac{2\sin(\pi\alpha/2)x^{\alpha/2}e^{-x}}{(2-\alpha)\Gamma(1/2)\Gamma(1-\alpha/2)} \). The interior integral can be expressed as

\[
\int_0^\infty e^{-(w+t+1)x^{\alpha/2}} \, dw = (w + t + 1)^{(\alpha-2)/2} \int_0^\infty e^{-u^{2/(2-\alpha)}} \, du
\]

\[
= \frac{2 - \alpha}{2} \Gamma(1 - \alpha/2)(w + t + 1)^{(\alpha-2)/2}.
\]

Consequently, we get

\[
H(x, y) = \frac{\sin(\pi\alpha/2) x^{\alpha/2} e^{-x}}{\pi \Gamma(\alpha/2)} \int_1^\infty \int_0^\infty e^{-w^2} (w + t + 1)^{(\alpha-2)/2} \, dw \, \frac{e^{-yt}}{(t^2 - 1)^{\alpha/2}} \, dt
\]

\[
= \frac{\sin(\pi\alpha/2)}{\pi \Gamma(\alpha/2)} \int_1^\infty \left[ x^{\alpha/2} \int_0^\infty e^{-x(w+t+1)} (w + t + 1)^{(\alpha-2)/2} \, dw \right] \frac{e^{-(y-x)t}}{(t^2 - 1)^{\alpha/2}} \, dt.
\]

Observe that the expression in brackets can be computed as follows:

\[
x^{\alpha/2} \int_0^\infty e^{-x(w+t+1)} (w + t + 1)^{(\alpha-2)/2} \, dw = \int_{x(t+1)}^\infty e^{-s} s^{\alpha/2 - 1} \, ds
\]

\[
= \Gamma(\alpha/2) - \int_0^{x(t+1)} e^{-s} s^{\alpha/2 - 1} \, ds
\]

\[
= \Gamma(\alpha/2) - \frac{2}{\alpha} e^{\alpha/2}(t+1)^{\alpha/2} \int_0^1 e^{-w^{2/\alpha}x(t+1)} \, dw.
\]

Thus we get

\[
H(x, y) = \frac{\sin(\pi\alpha/2)}{\pi} \int_1^\infty e^{-(y-x)t} \left( \frac{1}{(t^2 - 1)^{\alpha/2}} \right) \, dt
\]

\[
- \frac{2\sin(\pi\alpha/2)x^{\alpha/2}}{\alpha \pi \Gamma(\alpha/2)} \int_1^\infty 1 \int_0^1 e^{-w^{2/\alpha}x(t+1)} \, dw \, \frac{e^{-(y-x)t}}{(t - 1)^{\alpha/2}} \, dt
\]

\[
= U_1(x - y) - \frac{2\sin(\pi\alpha/2)x^{\alpha/2}}{\alpha \pi \Gamma(\alpha/2)} \int_1^\infty 1 \int_0^1 e^{-w^{2/\alpha}x(t+1)} \, dw \, \frac{e^{-(y-x)t}}{(t - 1)^{\alpha/2}} \, dt.
\]

Since

\[
G_{(0, \infty)}(x, y) = U_1(x - y) - H(x, y),
\]

we have

\[
G_{(0, \infty)}(x, y) = \frac{2x^{\alpha/2}}{\alpha \Gamma(\alpha/2)^2 \Gamma(1 - \alpha/2)} \int_0^\infty 1 \int_0^1 e^{-xw^{2/\alpha}(t+1)} \, dw \, \frac{e^{-(y-x)t}}{(t - 1)^{\alpha/2}} \, dt
\]

\[
= \frac{2x^{\alpha/2}}{\alpha \Gamma(\alpha/2)^2 \Gamma(1 - \alpha/2)} \int_0^1 1 \int_1^\infty e^{-xw^{2/\alpha}} \, dw \, \frac{e^{-(y-x)t}}{(t - 1)^{\alpha/2}} \, dt \, dw
\]

\[
= \frac{2x^{\alpha/2}e^{x-y}}{\alpha \Gamma(\alpha/2)^2 \Gamma(1 - \alpha/2)} \int_0^1 1 \int_0^\infty e^{-2xw^{2/\alpha}} \, dw \, \frac{e^{-(y-x)t}}{w^{\alpha/2}} \, dt \, dw
\]

\[
= \frac{2\alpha^{\alpha/2}e^{x-y}}{\alpha \Gamma(\alpha/2)^2} \int_0^1 e^{-2xw^{2/\alpha}} \int_0^\infty e^{-u(xw^{2/\alpha} + y - x)^{\alpha/2 - 1}} \, du \, dw
\]

\[
= \frac{e^{x-y}}{\Gamma(\alpha/2)^2} \int_0^\infty e^{-2v^{\alpha/2 - 1}} \, dv.
\]
Substituting $4v(v + y - x) = u$ in the last integral we finally obtain

$$
G_{(0, \infty)}(x, y) = \frac{1}{2\pi \Gamma(\alpha/2)^2} \int_0^{4xy} e^{-(u+(y-x)^2)^{1/2}} u^{\alpha/2-1} (u + (y - x)^2)^{-1/2} du
$$

$$
= \frac{(y - x)^{\alpha-1}}{2\pi \Gamma(\alpha/2)^2} \int_0^{4xy} e^{-(y-x)(t+1)^{1/2} t^{\alpha/2-1}(t + 1)^{-1/2}} dt. \quad \Box
$$

4. Multi-dimensional case

In this section we rely on the computation of the $(d - 1)$-dimensional Fourier transform of the $d$-dimensional Poisson kernel as well as the corresponding $d$-dimensional Green function. We show that this Fourier transform can be expressed in terms of the corresponding one-dimensional object, which we can explicitly invert.

4.1. Notation. In the proofs below one-dimensional quantities play an important rôle. Hence, to distinguish one-dimensional and $d$-dimensional objects, we introduce the following notation:

$$
q^m_t(x) \text{ denotes the one-dimensional } \alpha\text{-stable relativistic density with parameter } m,
$$

$$
V^m_\lambda(x) \text{ denotes the corresponding one-dimensional } \lambda\text{-potential kernel.}
$$

Recall that by $Q^m_{(0, \infty)}(x, u)$ we denote the one-dimensional $m$-Poisson kernel of $(0, \infty)$.

For $x \in \mathbb{R}^d$ we write $x = (x, x_d) \in \mathbb{R}^d$, where $x \in \mathbb{R}^{d-1}$. We begin with computation of the $(d - 1)$-dimensional Fourier transform of $U_\lambda((x, x_d), (y, y_d)) = U_\lambda((x - y), (x_d - y_d))$, that is, the one obtained by integrating with respect to the variable $y \in \mathbb{R}^{d-1}$. We denote it by $U_\lambda(x, y_d, \cdot)(z)$. We employ this notation throughout the entire section.

4.2. Poisson kernel of $\mathbb{H}$. We have

**Lemma 4.1.**

$$
U_\lambda(x, y_d, \cdot)(z) = e^{i(z, x)} V^\kappa_\lambda (x_d - y_d),
$$

where $\kappa = (|z|^2 + 1)^{1/2}$ and $\lambda = \kappa^\alpha + \lambda - 1$. Specifying to the case $\lambda = 1$:

$$
U_1(x, y_d, \cdot)(z) = e^{i(z, x)} V^\kappa (x_d - y_d)
$$

$$
= \frac{2^{1-\alpha}}{\sqrt{\pi}} \frac{e^{i(z, x)}}{\kappa^{1/2}} \frac{\Gamma(1-\alpha)}{\Gamma(\alpha/2)} \frac{K_{1-\alpha}(\kappa|x_d - y_d|)}{|x_d - y_d|^{\frac{\alpha}{2}}}.\quad \Box
$$

**Proof.** We begin with computation of the $(d - 1)$-dimensional Fourier transform of the transition density function $g_u(x - y)$ of the normal distribution:

$$
g_u(x, y_d, \cdot)(z) = \int_{\mathbb{R}^{d-1}} g_u(x - y) e^{i(z, y)} dy = \frac{e^{i(z, x)}}{(4\pi u)^{1/2}} e^{-|z|^2/4u} e^{-\frac{(x_d - y_d)^2}{4u}}.
$$
In the next step we use this to find the \((d - 1)\)-dimensional Fourier transform of \(p_\gamma(x - y)\):

\[
p_\gamma(x, y, \cdot)(z) = \int_{\mathbb{R}^{d-1}} p_\gamma(x - y) e^{i(x, y)} dy
\]

\[
= e^t \int_0^\infty g_\gamma(x, y, \cdot)(z) e^{-u} \theta^\alpha_t(u) du
\]

\[
= e^{i(x, z)} e^t \int_0^\infty \frac{1}{(4\pi u)^{1/2}} e^{-(|z|^2 + 1)u} e^{-\frac{(x - y)^2}{4u}} \theta^\alpha_t(u) du
\]

\[
= e^{i(x, z)} e^{t - (|z|^2 + 1)t} \int_0^\infty e^{\kappa^\alpha t} e^{-\kappa u} \frac{1}{(4\pi u)^{1/2}} e^{-\frac{(x - y)^2}{4u}} \theta^\alpha_t(u) du.
\]

Note that the integral expression in the last line is the one-dimensional \(\alpha\)-stable relativistic transition density function with parameter \(\kappa^\alpha\) which we denoted by \(q_t^\kappa\). Hence we obtain

\[
p_\gamma(x, y, \cdot)(z) = e^{i(x, z)} e^{(1 - \kappa^\alpha)t} q_t^\kappa(x_d - y_d).
\]

As a consequence we obtain the \((d - 1)\)-dimensional Fourier transform of the \(\lambda\)-potential:

\[
U_\lambda(x, y, \cdot)(z) = e^{i(x, z)} \int_0^\infty e^{(1 - \kappa^\alpha - \lambda)t} q_t^\kappa(x_d - y_d) dt
\]

\[
= e^{i(x, z)} V_\lambda^{\kappa^\alpha}(x_d - y_d) = e^{i(x, z)} V_\lambda^{\kappa^\alpha}(x_d - y_d),
\]

where \(\lambda = (\kappa^\alpha + \lambda - 1)\) and \(V_\lambda^{\kappa^\alpha}\) is the \(\lambda\) potential for the one-dimensional relativistic \(\alpha\)-stable semigroup \(q_t^\kappa\) with parameter \(\kappa^\alpha\). Now, if we take \(\lambda = 1\), then \(\lambda = (\kappa^\alpha + \lambda - 1) = (|z|^2 + 1)^{\alpha/2} = \kappa^\alpha\). By \((2.10)\) we have

\[
V_\lambda^{\kappa^\alpha}(x_d - y_d) = V_\kappa^\alpha(x_d - y_d) = \frac{2^{1 - \alpha}}{\sqrt{\pi^{1/2} \Gamma(\alpha/2)}} \frac{K_{1 - \alpha}(\kappa |x_d - y_d|)}{|x_d - y_d|^{1 - \alpha}}.
\]

For \(x \in \mathbb{H}\) we denote

\[
f(x, u) = \begin{cases} 2\sin(\pi\alpha/2) \left( \frac{x_d}{-u_d} \right)^{\alpha/2} \frac{K_{1 - \alpha}(\kappa |x - y|)}{|x - y|^{1 - \alpha}}, & u \in \mathbb{H}, \\
0, & \text{otherwise}. \end{cases}
\]

**Lemma 4.2.** The \((d - 1)\)-dimensional Fourier transform of \(f(x, u)\) with respect to \(u\) is given by

\[
f(x, u, \cdot)(z) = e^{i(x, z)} \sin(\pi\alpha/2) \left( \frac{x_d}{-u_d} \right)^{\alpha/2} \frac{e^{-\kappa(x_d - u_d)}}{x_d - u_d}, \quad u_d < 0 < x_d.
\]

For any \(x \in \mathbb{H}\) the function \(f(x, u)\) is a density of a finite measure supported on \(\mathbb{H}_c\).

**Proof.** Taking into account \((2.14)\) and \((2.13)\) with \((d - 1)\) instead of \(d\) and \(x_d - y_d\) instead of \(t\), we obtain the proof for the first part of the lemma.
The second part follows from the observation that
\[ \int_{\mathbb{H}} f(x, u) du = \int_{-\infty}^{0} \hat{f}(u, \cdot)(0) du_d < \infty. \]
The finiteness of the last integral was shown in the proof of Theorem 3.2. □

**Theorem 4.3 (Poisson kernel).** The Poisson kernel of \( \mathbb{H} \) for \( (m^{2/\alpha} I - \Delta)^{\alpha/2} \) exists and is given by
\[ P^m_{\mathbb{H}}(x, u) = 2 \frac{\sin(\pi \alpha/2) m^{d/2\alpha}}{\pi(2\pi)^{d/2}} \left( \frac{x_d}{-u_d} \right)^{\alpha/2} K_{d/2}(m^{1/\alpha}|x - u|) \frac{1}{|x - u|^{d/2}}, \quad u_d < 0 < x_d. \]

**Proof.** We will consider only the case \( m = 1 \), since the general case follows from the scaling property.

We claim that
\[ U_1(x, u) = \int_{\mathbb{H}} f(x, u) U_1(u, y) du, \quad x \in \mathbb{H}, y \in \mathbb{H}^c. \]

If we show that the measure with the \( u \)-density \( f(x, u) \) has the finite energy \( (2.22) \), then the application of Lemma 2.5 will complete the argument. Define
\[ R(x, y) = \int_{u_d < 0} f(x, u) U_1(u, y) du. \]

From Theorem 3.2 (with \( m = \kappa^\alpha \)) and Lemma 4.2 it follows that
\[ f(\hat{x}, y, \cdot)(\Omega) = e^{i(x, z) Q^\kappa_{\alpha}(x, u, y)} \frac{f(x, u) V_{\kappa^\alpha}(u_d - y_d)}{Q^\kappa_{\alpha}(x, u, y)} \]

Since \( U_1(u, y, \cdot)(\Omega) = e^{i(x, z) V_{\kappa^\alpha}(u_d - y_d)} \) we obtain
\[ R(x, y, \cdot)(\Omega) = \int_{u_d < 0} f(x, u) U_1(u, y, \cdot)(\Omega) du = \int_{u_d < 0} e^{i(x, z) f(x, u) V_{\kappa^\alpha}(u_d - y_d)} du = \int_{-\infty}^{0} f(x, u_d, \cdot) V_{\kappa^\alpha}(u_d - y_d) du_d = e^{i(x, z)} \int_{-\infty}^{0} Q_{\alpha}(x, u_d) V_{\kappa^\alpha}(u_d - y_d) du_d. \]

Next, Theorem 3.2 applied for \( m = \kappa^\alpha \) together with the “sweeping out” formula \( (2.21) \) yield
\[ V_{\kappa^\alpha}(x_d - y_d) = \int_{-\infty}^{0} Q_{\alpha}(x_d, u_d) V_{\kappa^\alpha}(u_d - y_d) du_d, \quad y_d \leq 0 < x_d. \]

Hence, by the uniqueness of Fourier transforms, we verify \( (4.1) \). It remains only to show the finiteness of energy. By \( (4.1) \),
\[ \int_{\mathbb{H}} \int_{\mathbb{H}} U_1(x - y) f(x, u) f(x, y) du dy = \int_{\mathbb{H}} U_1(x - y) f(x, y) dy \leq \sup_{y \in \mathbb{H}} U_1(x - y) \int_{\mathbb{H}} f(x, y) dy < \infty, \]
since \( \sup_{y \in \mathbb{H}} U_1(x - y) \) is finite, as \( x_d > 0 \). □
4.3. Green function of $\mathbb{H}$.

**Theorem 4.4** (Green function). The Green function of $\mathbb{H}$ for $(m^{2/\alpha} I - \Delta)^{\alpha/2}$ is given by the formula

\[
G^m_{\mathbb{H}}(x, y) = \frac{2^{1-\alpha} m^{d/2\alpha} |x - y|^{\alpha-d/2}}{(2\pi)^{d/2} \Gamma(\alpha/2)^2} \int_0^{\frac{t^2}{t^2 - 1}} \left( \frac{t^2 - 1}{(t + 1)^{d/4}} \right) K_{d/2}(m^{1/\alpha} |x - y|(t + 1)^{1/2}) \, dt,
\]

where $x, y \in \mathbb{H}$.

**Proof.** We will consider only the case $m = 1$, since the general case follows from the scaling property. Also it is enough to consider $x = (0, x_d)$. We will find the $(d - 1)$-dimensional Fourier transform of the Green function. If we write

\[
G_{\mathbb{H}}(x, y) = U_1(x - y) - \int_{\mathbb{H}} P_{\mathbb{E}}(x, u) U_1(u - y) \, du = U_1(x - y) - R(x, y),
\]

then by (4.2) and by Theorem 3.3

\[
G_{\mathbb{H}}(x, y_d, \cdot)(z) = U_1(x, y_d, \cdot)(z) - R(x, y_d, \cdot)(z)
\]

\[
= V_{\nu}^\alpha(x - y) - \int_{-\infty}^\infty Q_{(0, \infty)}^\alpha(x_d, u_d) V_{\nu}^\alpha(u_d - y_d) \, du_d
\]

\[
= G_{(0, \infty)}^\alpha(x, y_d)
\]

\[
= \frac{1}{2\pi \Gamma(\alpha/2)^2} \int_0^{4\pi y_d} \frac{s^{\alpha-1}}{(s + (x_d - y_d)^2)^{\alpha/2}} e^{-(s + (x_d - y_d)^2)^{1/2}/(s + (x_d - y_d)^2)^{1/2}} \, ds.
\]

Taking into account (2.11) and (2.13), with $(d - 1)$ instead of $d$, and with $(s + (x_d - y_d)^2)^{1/2}$ instead of $t$, we obtain for $d > 1$

\[
G_{\mathbb{H}}(x, y) = \frac{2^{1-\alpha} m^{d/2\alpha} |x - y|^{\alpha-d/2}}{(2\pi)^{d/2} \Gamma(\alpha/2)^2} \int_0^{\frac{t^2}{t^2 - 1}} \frac{K_{d/2}(|y|^{2} + (x_d - y_d)^2 + s)^{1/2}}{(|y|^{2} + (x_d - y_d)^2 + s)^{d/4}} \, ds
\]

\[
= \frac{2^{1-\alpha} m^{d/2\alpha} |x - y|^{\alpha-d/2}}{(2\pi)^{d/2} \Gamma(\alpha/2)^2} \int_0^{\frac{t^2}{t^2 - 1}} \frac{K_{d/2}(|x - y|^{2} + s)^{1/2}}{(|x - y|^{2} + s)^{d/4}} \, ds
\]

\[
= \frac{2^{1-\alpha} m^{d/2\alpha} |x - y|^{\alpha-d/2}}{(2\pi)^{d/2} \Gamma(\alpha/2)^2} \int_0^{\frac{t^2}{t^2 - 1}} \frac{t^{\alpha-1}}{(t + 1)^{d/4}} \, K_{d/2}(t - (t + 1)^{1/2}) \, dt.
\]

We now compute $E_x e^{-m z}$, where $\tau_{\mathbb{H}}$ is the first exit time from $\mathbb{H}$ of the relativistic $\alpha$-stable process with parameter $m > 0$. Its importance, among other things, is due to the fact that it is harmonic on $\mathbb{H}$ for the operator $(m^{2/\alpha} I - \Delta)^{\alpha/2}$. Indeed, according to the theory of Schrödinger operators (see [ChZ], 4.5), the function $\phi(x) = E_x[\exp(\int_0^{\tau_{\mathbb{H}}} q(X_t^m) \, dt)]$ is harmonic on $\mathbb{H}$ with respect to the Schrödinger operator $m I - (m^{2/\alpha} I - \Delta)^{\alpha/2} + q I$ based on the generator of our relativistic process (with parameter $m$) with the potential $q$. Taking $q = -m$ we obtain that $E_x[\exp(\int_0^{\tau_{\mathbb{H}}} 1 \, dt)] = E_x e^{-m \tau_{\mathbb{H}}}$ is harmonic on $\mathbb{H}$ for $(m^{2/\alpha} I - \Delta)^{\alpha/2}$.

**Corollary 4.5.**

\[
E_x e^{-m z} = \frac{1}{\Gamma(\alpha/2)} \int_0^{\infty} t^{\alpha/2 - 1} e^{-t} \, dt, \quad z \in \mathbb{H}.
\]

**Proof.** The proof consists of computing the mass of the $m$-Poisson kernel and will be carried out for $m = 1$, since the general case follows from the scaling property. It is obvious that we may assume that $d = 1$. 


Substituting \((-u) = \frac{v^2}{2\pi}\) and taking into account the following identity:

\[
\frac{1}{x + \frac{v^2}{2\pi}} = \int_0^{\infty} e^{-w(x + \frac{v^2}{2\pi})} \, dw,
\]

we obtain, after changing order of integration

\[
E^z e^{-\tau_z} = \frac{\sin(\alpha \pi/2)}{\pi} \int_{-\infty}^{0} \left( \frac{z}{u} \right)^{\alpha/2} e^{-u} \left( \frac{u}{z} - 1 \right) du
\]

\[
= \frac{2 \sin(\pi \alpha/2)}{(2 - \alpha)\pi} z^{\alpha/2} e^{-z} \int_0^{\infty} \left\{ \int_0^{\infty} e^{-wz} e^{-\frac{v^2}{2\pi}w} \, dw \right\} e^{-\frac{v^2}{2\pi}v} \, dv
\]

\[
= \frac{2 z^{\alpha/2} e^{-z}}{(2 - \alpha)\Gamma(\alpha/2)\Gamma(1 - \alpha/2)} \int_0^{\infty} e^{-wz} \left\{ \int_0^{\infty} e^{-\frac{v^2}{2\pi}(w+1)} \, dv \right\} \, dw
\]

\[
= \frac{1}{\Gamma(\alpha/2)} \int_0^{\infty} (z(w + 1))^{\alpha/2 - 1} e^{-(w+1)z} \, z \, dw
\]

\[
= \frac{1}{\Gamma(\alpha/2)} \int_0^{\infty} z^{\alpha/2 - 1} e^{-t \, dt}.
\]

(4.4) **Riesz potential theory.** We provide an alternative proof for the formulas of the Poisson kernel and the Green function of \(\mathbb{H}\) for the \(\alpha\)-stable rotation invariant process \(Z_t\). As far as we know, it is the first proof in the multi-dimensional case which does not use Kelvin’s transform. We define

\[
P^m_{\mathbb{H}}(x, u) = \lim_{m \to 0^+} P_m(x, u) = \frac{\sin(\pi \alpha/2)\Gamma(d/2)}{\pi^{1+d/2}} \left( \frac{x-u}{u} \right)^{\alpha/2} \frac{1}{|x-u|^{d}},
\]

where \(u_d < 0 < x_d\) and

\[
G^m_{\mathbb{H}}(x, y) = \lim_{m \to 0^+} G_m^m(x, y) = \frac{\Gamma(d/2)}{\pi^{d/2}\alpha\Gamma(\alpha/2)z} |x-y|^{\alpha-d} \int_0^{t} \frac{t^{\alpha-1}}{(t+1)^{d/2}} \, dt,
\]

where \(x_d, y_d > 0\). Note that the above limits are obtained using (2.15). Also observe that \(P^m_{\mathbb{H}}(x, u), G^m_{\mathbb{H}}(x, y)\) and \(U^m_{\mathbb{H}}(x, y)\) are decreasing functions of the parameter \(m\). This easily follows from the fact that \(r^\theta K_\theta(r)\) is a decreasing function of \(r > 0\); see (2.10). This shows in particular that

(4.4) \[G^m_{\mathbb{H}}(x, y) \leq G^m_{\mathbb{H}}(x, y)\]

and

\[P^m_{\mathbb{H}}(x, y) \leq P^m_{\mathbb{H}}(x, y).
\]

**Corollary 4.6.** \(P^m_{\mathbb{H}}, G^m_{\mathbb{H}}\) are the 0-Poisson kernel of \(\mathbb{H}\) or 0-Green function of \(\mathbb{H}\), respectively, for the \(d\)-dimensional \(\alpha\)-stable rotation invariant process.

**Remark.** Comparing the formulas for \(P^m_{\mathbb{H}}, P_{\mathbb{H}}\) and for \(G^m_{\mathbb{H}}, G_{\mathbb{H}}\), we observe that the correspondence between the potential theories of \((-\Delta)^{\alpha/2}\) and \((I - \Delta)^{\alpha/2}\) we mentioned in the Remark below Lemma (2.4) extends onto the Poisson kernel and the Green function of \(\mathbb{H}\). More specifically, the expression \(\frac{1}{|x-u|^{d}}\left( \frac{1}{|x-u|(t+1)^{d/2}} \right)\) in the formula for \(P^m_{\mathbb{H}}\) \((G^m_{\mathbb{H}})\), respectively, is replaced by \(\frac{K_{d/2}(|x-u|)}{|x-u|^{d}}\left( \frac{K_{d/2}(|x-u|(t+1)^{d/2})}{|x-u|(t+1)^{d/2}} \right)\) in the one for \(P_{\mathbb{H}}\) \((G_{\mathbb{H}})\), respectively. No such correspondence is known so far for the corresponding Poisson kernels or the Green functions for balls.
Proof. We provide the arguments only in the case $\alpha < d$. The case $d = 1 \leq \alpha$ can be handled in a similar way but requires the compensated potential kernels. We omit the details.

To deal with the Poisson kernel we write the sweeping out formula (2.21) for the $m$-Poisson kernel:

$$U^m_m(x, y) = \int_{\mathbb{H}} P^m_m(x, u)U^m_m(u, y)du, \quad x \in \mathbb{H}, \ y \in \mathbb{H}. \tag{4.5}$$

Next, note that for $\alpha < d$

$$\lim_{m \to 0^+} U^m_m(u, y) = C^\#(\alpha, d)|u - y|^{\alpha - d},$$

where $C^\#(\alpha, d) = 2^{-\alpha}\pi^{-d/2}\Gamma((d - \alpha)/2)/\Gamma(\alpha/2)$ and $C^\#(\alpha, d)|u - y|^{\alpha - d}$ is the Riesz kernel. Observe that

$$P^m_m(x, u)U^m_m(u, y) \succ C^\#(\alpha, d)P^\#(x, u)|u - y|^{\alpha - d}, \text{ as } m \searrow 0.$$

Then, after taking the limit in (4.5) and using the monotone convergence theorem, we obtain

$$|x - y|^{\alpha - d} = \int_{\mathbb{H}} P^\#(x, u)|u - y|^{\alpha - d}du, \quad x \in \mathbb{H}, \ y \in \mathbb{H},$$

which gives the formula (4.5) rewritten in terms of Riesz kernels and $P^\#_m$.

Applying the uniqueness theorem for Riesz kernels (see [BGR]) we obtain that $P^\#_m(x, y)$ is the Poisson kernel of $\mathbb{H}$ for the rotation invariant $d$-dimensional $\alpha$-stable process.

To deal with the Green function we use the identity (2.20), written for the $m$-Poisson kernel:

$$G^m_m(x, y) = U^m_m(x, y) - \int_{\mathbb{H}} U^m_m(u, y)P^m_m(x, u)du, \quad x, y \in \mathbb{H}. \tag{4.6}$$

After taking the limit and using the monotone convergence theorem we obtain

$$G^\#_m(x, y) = C^\#(\alpha, d)|x - y|^{\alpha - d} - \int_{\mathbb{H}} C^\#(\alpha, d)|u - y|^{\alpha - d}P^\#_m(x, u)du, \quad x, y \in \mathbb{H},$$

which shows that $G^\#_m(x, y)$ is the Green function of $\mathbb{H}$ for the rotation invariant $d$-dimensional $\alpha$-stable process.

5. Appendix

Throughout this section by $c, C, C_1, \ldots$ we denote positive constants. The notation $p(u) \approx q(u), \ u \in A$, means that the ratio $p(u)/q(u), \ u \in A$, is bounded from below and above by positive constants. For any $y = (y, y_d) \in \mathbb{R}^d$ we denote $y^* = (y, -yd)$.

In this section we apply results obtained so far in the paper to obtain estimates of the Green function of $\mathbb{H}$ for the relativistic $\alpha$-stable process $X^m_t$, under the assumption that $m = 1$. Equivalently, it is the Green function of $\mathbb{H}$ for the operator $I - (I - \Delta)^{\alpha/2}$. To distinguish it from the previously considered Green function $G^0_\mathbb{H}$ for the operator $(I - \Delta)^{\alpha/2}$, we call it the $0$-Green function and denote it as $G^0_\mathbb{H}$. From the point of view of the potential theory of $X^m_t$ the Green function $G^0_\mathbb{H}$ is of prime importance. In particular, questions of boundary behaviour of harmonic functions or of potential theory of the Schrödinger operator based on $I - (I - \Delta)^{\alpha/2}$ require more detailed information about the Green function (cf. [OZ]). For $m = 0$...
we obtain the $\alpha$-stable rotation invariant Lévy process $Z_t$, whose potential theory is considerably more advanced. The process $X_t^{\mu}$ is another important example of a Lévy process whose potential theory is still in its early stages.

In principle we can use the resolvent equation to represent $G^0_H$ by a series involving powers of $G^0_H$. That is, we have the following identity:

$$G^0_H(x, y) - G^0_H(x, y) = \int_H G^0_H(x, u) G^0_H(u, y) \, du, \quad x, y \in H.$$ 

The iteration of the above identity yields

$$G^0_H(x, y) = \sum_{n=1}^{\infty} G^{(n)}_H(x, y), \quad x, y \in H,$$

where $G^{(n)}_H$ is the $n$-th iteration of the kernel $G_H$. The validity of the representation (5.1) requires further justification, which is omitted here. Besides, the consecutive terms of the series (5.1) are not decaying sufficiently fast, as $n \to \infty$, in order to obtain an appropriate estimate for $G^0_H(x, y)$. The present, alternative approach, exploits the following formula for the Green function:

$$G^0_H(x, y) = \int_0^\infty p_t^H(x, y) \, dt$$

and relies on estimation of the transition density function $p_t^H(x, y)$.

**Comparison results.** One of our motivations to explore bounds for $G^0_H$ stems from comparison results for the relativistic 0-Green function and its stable counterpart. We recall that the potential theoretic objects corresponding to $(-\Delta)^{\alpha/2}$, that is, in probabilistic terms, to the potential theory of the $\alpha$-stable rotation invariant process $Z_t$, are denoted with the superscript “#”. In several papers (Ry, CS, GRy, KL) it was shown, under various assumptions on an open bounded set $D$, that there is a constant $C$ depending on $D$ such that

$$C^{-1} G_H^#(x, y) \leq G^0_H(x, y) \leq C G^#_D(x, y), \quad x, y \in D,$$

where $G^#_D$ is the 0-Green function for $Z_t$. One of the important questions which may be raised is: to what extent does the above comparison hold if an unbounded set is considered? We show that for $D = H$ such comparison holds if $d \geq 3$, and we restrict $x, y$ to $H$ with $|x - y| \leq 1$. On the other hand (7.2) does not hold for $d \leq 2$ even if $|x - y| \leq 1$. In the forthcoming paper GRy 2 the optimal estimates for $G^0_H$ are provided, and they extend the results obtained in this section.

We start with several lemmas leading to estimates of the semigroup $p_t^H(x, y)$.

5.1. **Estimates of transition densities $p_t^H(x, y)$**. We begin with the following estimate [Ry]:

**Lemma 5.1.** There exists a constant $c = c(\alpha, d)$ such that

$$\max_{x \in \mathbb{R}^d} p_t(x) \leq c(t^{-d/2} + t^{-d/\alpha}).$$

The next lemma will be very useful in the sequel.

**Lemma 5.2.** There is $C$ such that

$$p_t(x - y) - p_t(x - y*) \leq p_t^H(x, y) \leq C(t^{-d/2} + t^{-d/\alpha}) P^x(\tau_H \geq t/3) P^y(\tau_H \geq t/3)$$

for $x, y \in \mathbb{H}$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Proof. We start with the upper bound. Since \( p^\mathbb{H}_t(x,y) \) is a transition density function, the semigroup property and the inequality \( p^\mathbb{H}_t(x,y) \leq \max_{z \in \mathbb{R}^d} p_t(z) \) yield

\[
p^\mathbb{H}_t(x,y) = \int_{\mathbb{R}^d} p^\mathbb{H}_t(x,z) p^\mathbb{H}_t(z,y) \, dz \leq \max_{z \in \mathbb{R}^d} p_t(z) \int_{\mathbb{R}^d} p^\mathbb{H}_t(x,z) \, dz
\]

\[
= \max_{z \in \mathbb{R}^d} p_t(z) P^x(\tau_{\mathbb{H}} \geq t).
\]

We repeat this argument to obtain

\[
p^\mathbb{H}_t(x,y) = \int_{\mathbb{R}^d} p^\mathbb{H}_t(x,z) p^\mathbb{H}_t(z,y) \, dz \leq \max_{z \in \mathbb{R}^d} p_t(z) P^x(\tau_{\mathbb{H}} \geq t) \int_{\mathbb{R}^d} p^\mathbb{H}_t(z,y) \, dz
\]

\[
= \max_{z \in \mathbb{R}^d} p_t(z) P^x(\tau_{\mathbb{H}} \geq t) P^y(\tau_{\mathbb{H}} \geq t).
\]

By Lemma 5.1, \( \max_{z \in \mathbb{R}^d} p_t(z) \leq C(t^{-d/2} + t^{-d/\alpha}) \); hence we obtain the upper bound.

To get the lower bound we use the subordination of the process to the Brownian motion: \( X_t = B_{T^\alpha} \). Let \( [y, y + \epsilon) = \bigotimes_{k=1}^d [y_k, y_k + \epsilon) \subseteq \mathbb{H}, \epsilon > 0 \). Then

\[
P^x(X_t \in [y, y + \epsilon), t < \tau_{\mathbb{H}}) = P^x(B_{T^\alpha} \in [y, y + \epsilon), B_{T^\alpha} \in \mathbb{H}, 0 \leq s < t)
\]

\[
\geq P^x(B_{T^\alpha} \in [y, y + \epsilon), B_s \in \mathbb{H}, 0 \leq s < T^\alpha_t).
\]

As a consequence of the reflection principle we have that the transition densities for the the Brownian motion killed on exiting \( \mathbb{H} \) are equal to \( g_t(x-y) - g_t(x-y^*) \) (see e.g. [ChZ]). Using the independence of \( T^\alpha \) and \( B \) we obtain

\[
P^x(B_{T^\alpha} \in [y, y + \epsilon), B_s \in \mathbb{H}, 0 \leq s < T^\alpha_t | T^\alpha_t)
\]

\[
= \int_{[y, y + \epsilon]} (g_{T^\alpha_t}(x-v) - g_{T^\alpha_t}(x-v^*)) \, dv.
\]

Taking the expectation, passing \( \epsilon \searrow 0 \) and using the fact that \( E^0 g_{T^\alpha_t}(z) = p_t(z) \), we obtain the lower bound. \( \square \)

The next lemma is taken from [Ch], but for the reader’s convenience we provide its proof.

Lemma 5.3. For \( t \geq 2, x_d > 0 \) we have

\[
P^x(\tau_{\mathbb{H}} \geq t) \leq C \frac{x_d + \ln t}{t^{1/2}}.
\]

Proof. Let \( Y_t = X_t^{(d)} \), where \( X_t = (X_t^{(1)}, \ldots, X_t^{(d)}) \) and let \( \nu^Y \) be the density of the Lévy measure of \( Y_t \). By the symmetry of the process \( Y_t \) we obtain

\[
P^x(\tau_{\mathbb{H}} \geq t) = P^x(\inf_{s \leq t} Y_s \geq 0) = P^0(\inf_{s \leq t} (-Y_s + x_d) \geq 0) = P^0(\sup_{s \leq t} Y_s \leq x_d).
\]

Using a version of the Lévy inequality ([Io], Ch.7, 37.9) we have for any \( \epsilon, y > 0 \) that

\[
2P^0(Y_t \geq y + 2\epsilon) - 2 \sum_{k=1}^n P^0(Y_{t+1/n} - Y_{t+1/n} \geq \epsilon) \leq P^0(\sup_{k \leq n} Y_{t+1/n} \geq y), \quad n \in \mathbb{N}.
\]
Note that \( \sum_{k=1}^{n} P^0(Y_{\frac{k}{n}} - Y_{\frac{k-1}{n}} \geq \varepsilon) = nP^0(Y_i \geq \varepsilon) \rightarrow t \int_{\varepsilon}^{\infty} \nu^Y(x) \, dx \); hence, by the symmetry again

\[
P^0(|Y_t| \geq y + 2\varepsilon) - 2t \int_{\varepsilon}^{\infty} \nu^Y(x) \, dx \leq P^0(\sup_{s \leq t} Y_s \geq y).
\]

This implies that

\[ P^x(\tau_D \geq t) = P^0(\sup_{s \leq t} Y_s \leq x_d) \leq P^0(|Y_t| \geq x_d + 2\varepsilon) + 2t \int_{\varepsilon}^{\infty} \nu^Y(x) \, dx. \quad (5.4) \]

From (2.17) and (2.9) we obtain

\[
\int_{\varepsilon}^{\infty} \nu^Y(x) \, dx \leq Ce^{-\varepsilon \varepsilon^{-\alpha/2}} \varepsilon \geq 1.
\]

Because of Lemma 5.1 we have the fact that the density of \( Y(t) \) is bounded by \( Ct^{-1/2}, t \geq 2 \). Hence taking \( \varepsilon = \frac{1}{2} \ln t \), we obtain by (5.4)

\[ P^x(\tau_D \geq t) \leq C(x_d + \ln t) t^{-1/2}. \quad \square \]

In order to improve the above estimate for \( x \) close to the boundary, we use the following result proved recently in \([GRy1]\).

**Lemma 5.4.** Assume that \( d = 1 \). Let \( D = (0, 1) \) and \( x \in D \). Then for \( y > 1 \)

\[ P^0_D(x, y) \approx \frac{(x(1-x))^{\alpha/2}}{(y-1)^{\alpha/2}(y-x)} e^{-y}. \]

We also have

\[ E^x[X_{\tau_D} > 1; X_{\tau_D}] \approx P^x(X_{\tau_D} > 1) \approx x^{\alpha/2} \]

and

\[ E^x\tau_D \approx (x(1-x))^{\alpha/2}. \]

We note that in \([GRy1]\) it was shown that the 0-Green function of \( D \) is comparable with the 0-Green function of the corresponding stable process (see \([5.2]\)). Then we can use the result of Ikeda-Watanabe \([IW]\) which describes the 0-Poisson kernel in terms of the 0-Green function and the \( \text{Lévy} \) measure:

\[ P^0_D(x, z) = \int_D G^0_D(x, y) \nu(y - z) \, dy, \quad x \in D, z \in D^c. \]

The conclusion of the lemma follows from arguments used in \([Ry]\) (see Theorem 6).

We further need the following strengthening of Lemma 5.3.

**Lemma 5.5.** For \( 0 < x_d < 1/2 \) we have

\[ P^x(\tau_D \geq t) \leq Cx_d^{\alpha/2} \ln t/t^{1/2}, \quad t \geq 4. \quad (5.5) \]

**Proof.** It is enough to prove the claim for \( d = 1 \). Let \( D = (0, 1) \) and assume that \( 0 < x < 1/2 \).
By the strong Markov property and then by Lemma 5.3 we obtain for \( t \geq 4:\)

\[
P^x(\tau_H \geq t) = P^x(\tau_D \geq t/2, \tau_H \geq t) + P^x(\tau_D < t/2, \tau_H \geq t)
\]

\[
\leq P^x(\tau_D \geq t/2) + P^x(\tau_H - \tau_D \geq t/2)
\]

\[
= P^x(\tau_D \geq t/2) + E^x[\tau_D < \tau_H; P^{X,0}(\tau_H \geq t/2)]
\]

\[
\leq P^x(\tau_D \geq t/2) + CE^x[\tau_D < \tau_H; X_{\tau_D} + \ln t]/t^{1/2}
\]

\[
\leq 2E^x[\tau_D/t + CE^x[X_{\tau_D} > 1; X_{\tau_D}]/t^{1/2} + C\ln t P^x(X_{\tau_D} > 1)/t^{1/2}
\]

\[
\leq Cx^{\alpha/2}\ln t/t^{1/2}.
\]

The last inequality follows from Lemma 5.4. The proof is complete. \( \square \)

5.2. **Bounds for 0-Green function.** We start with a lower estimate of \( G^0_H(x, y) \) by the 0-Green function of the Brownian motion for \( H \) which we denote by \( G^0_H(x, y) \).

As a consequence of the reflection principle we have

\[
G^0_H(x, y) = \int_0^\infty (g_u(x - y) - g_u(x - y^*)) \, du, \quad x, y \in H,
\]

which yields the following formulas (see e.g. [ChZ]) depending on the dimension \( d \):

\[
G^0_H(x, y) = \begin{cases} 
  x \land y, & d = 1, \\
  \frac{1}{2} \ln \frac{|x^* - y|}{|x - y|}, & d = 2, \\
  \frac{1}{4\pi^{(d-2)/2}} \left( \frac{1}{|x - y|^2} - \frac{1}{|x^* - y|^2} \right), & d \geq 3.
\end{cases}
\]

**Lemma 5.6.** For any \( x, y \in H \)

\[
G^0_H(x, y) \geq G^0_H(x, y).
\]

**Proof.** Let \( V(x, y) = \int_0^\infty (p_t(x - y) - p_t(x - y^*)) \, dt \). By Lemma 5.2 it is enough to prove that \( V(x, y) \geq G^0_H(x, y) \). For this purpose, let us write the 0-potential of the subordinator \( T^\alpha_t \) (2.11):

\[
G(u) = e^{-u} \int_0^\infty e^t \theta^\alpha_t(u) \, dt.
\]

Using (2.12) we have

\[
V(x, y) = \int_0^\infty (p_t(x - y) - p_t(x - y^*)) \, dt
\]

\[
= \int_0^\infty e^t \int_0^\infty (g_u(x - y) - g_u(x - y^*)) e^{-u} \theta^\alpha_u(u) \, du \, dt
\]

\[
= \int_0^\infty (g_u(x - y) - g_u(x - y^*)) e^{-u} \int_0^\infty e^t \theta^\alpha_t(u) \, du \, dt
\]

\[
= \int_0^\infty (g_u(x - y) - g_u(x - y^*)) G(u) \, du.
\]

It was proved in [RSV] that \( G(u) \) is a completely monotone (hence decreasing) function and \( \inf_{u > 0} G(u) = \lim_{u \to \infty} G(u) = C \). We find the constant \( C = \lim_{u \to \infty} G(u) \)
by taking into account the asymptotics of the Laplace transform of \( G(u) \) at the origin:

\[
\int_0^\infty e^{-\lambda u} G(u) \, du = \int_0^\infty e^t \int_0^\infty e^{-u(1+\lambda)} \theta_t^\alpha(u) \, du \, dt = \int_0^\infty e^t e^{-(1+\lambda)t^\alpha/2} \, dt = \frac{1}{(1+\lambda)^{\alpha/2}} \sim \frac{2}{\lambda\alpha}.
\]

Applying the Karamata Tauberian Theorem (see [1], 1.7) we obtain that \( C = 2/\alpha > 1 \). Since \( g_u(x - y) - g_u(x - y^*) \geq 0 \) we finally obtain

\[
V(x, y) = \int_0^\infty (g_u(x - y) - g_u(x - y^*)) G(u) \, du \geq \int_0^\infty (g_u(x - y) - g_u(x - y^*)) \, du = G^\#_{\mathbb{H}}(x, y).
\]

**Lemma 5.7.** Assume that \( |x - y| \leq 1 \). Then there is \( C = C(\alpha, d) \) such that

\[
C(x_d y_d \wedge 1)^{\alpha/2} \leq G_{\mathbb{H}}(x, y) \leq G^\#_{\mathbb{H}}(x, y).
\]

Moreover for \( d \geq 2 \) there is a constant \( C_1 = C_1(\alpha, d) \) such that \( G^\#_{\mathbb{H}}(x, y) \leq C_1 G_{\mathbb{H}}(x, y) \).

**Proof.** The inequality

\[
G_{\mathbb{H}}(x, y) \leq G^\#_{\mathbb{H}}(x, y), \quad x, y \in \mathbb{H},
\]

is obviously identical with (4.4) written for \( m = 1 \).

Since \( \frac{x_d y_d}{|x - y|^d} \geq x_d y_d \wedge 1 \), applying the formulas (4.3) and (2.8) we obtain

\[
G_{\mathbb{H}}(x, y) = C|x - y|^\alpha - d/2 \int_{x_d y_d \wedge 1}^{\frac{x_d y_d}{|x - y|^d}} \frac{t^{\alpha - 1}}{(t + 1)^{d/4}} K_{d/2}(|x - y|(t + 1)^{1/2}) \, dt
\]

\[
\geq C|x - y|^\alpha - d/2 \int_{x_d y_d \wedge 1}^{\frac{x_d y_d}{|x - y|^d}} \frac{t^{\alpha - 1}}{(t + 1)^{d/4}} K_{d/2}(|x - y|(t + 1)^{1/2}) \, dt
\]

\[
\geq C|x - y|^\alpha - d \int_0^1 \frac{t^{\alpha - 1}}{(t + 1)^{d/2}} \, dt \geq C(x_d y_d \wedge 1)^{\alpha/2}.
\]

Assume that \( d \geq 2 \). If \( \frac{x_d y_d}{|x - y|^d} \leq 1 \) and \( |x - y| \leq 1 \), then using the formulas (4.3) and (2.8) we obtain \( G_{\mathbb{H}}(x, y) \geq CG^\#_{\mathbb{H}}(x, y) \). If \( \frac{x_d y_d}{|x - y|^d} \geq 1 \) and \( |x - y| \leq 1 \) we apply (2.8) to arrive at

\[
G_{\mathbb{H}}(x, y) = C|x - y|^\alpha - d/2 \int_{x_d y_d \wedge 1}^{\frac{x_d y_d}{|x - y|^d}} \frac{t^{\alpha - 1}}{(t + 1)^{d/4}} K_{d/2}(|x - y|(t + 1)^{1/2}) \, dt
\]

\[
\geq C|x - y|^\alpha - d \int_0^1 \frac{t^{\alpha - 1}}{(t + 1)^{d/2}} \, dt
\]

\[
\geq C_1|x - y|^\alpha - d \int_0^1 \frac{t^{\alpha - 1}}{(t + 1)^{d/2}} \, dt
\]

\[
= C_2 G^\#_{\mathbb{H}}(x, y).
\]
Note that the only place where we used the assumption \( d \geq 2 \) is the passage from the third to the fourth line in the above string of inequalities.

The next result provides a general bound for the Green function. Before proving it we introduce some notation. For \( d \in \mathbb{N} \) we define a function \( \psi_d \) in the following way:

\[
\psi_d(v) = \begin{cases} 
  v^{\alpha/2}, & 0 < v < 1, \\
  v^{1/2}, & v \geq 1, \\
  \ln^{1/2}(1+v), & v \geq 1, \\
  1, & d \geq 3.
\end{cases}
\]

**Theorem 5.8.** There is a constant \( C \) such that

\[
\max\{G_{\mathbb{H}}(x, y), G_{\mathbb{H}}^0(x, y)\} \leq G_{\mathbb{H}}^0(x, y) \leq C[\psi_d(x_d)\psi_d(y_d) + G_{\mathbb{H}}(x, y)], \quad x, y \in \mathbb{H}.
\]

**Proof.** The lower bound is an obvious consequence of Lemma 5.6. The proof of the upper bound will rely on the estimates of \( P^x(\tau_H \geq t) \) and the application of Lemma 5.2.

We proceed to estimate the Green function from above. We split the integration,

\[
G_{\mathbb{H}}^0(x, y) = \int_0^\infty p_t^H(x, y) \, dt = \int_0^6 p_t^H(x, y) \, dt + \int_6^\infty p_t^H(x, y) \, dt.
\]

The first integral is estimated as follows:

\[
\int_0^6 p_t^H(x, y) \, dt \leq e^6 \int_0^6 e^{-t} p_t^H(x, y) \, dt \leq e^6 G_{\mathbb{H}}(x, y);
\]

in the second one we apply Lemma 5.2 to obtain

\[
\int_6^\infty p_t^H(x, y) \, dt \leq C \int_2^\infty P^x(\tau_H \geq t) P^y(\tau_H \geq t) \frac{dt}{t^{d/2}} = C R(x, y).
\]

For \( d \geq 3 \) we have

\[
R(x, y) \leq C P^x(\tau_H \geq 2) P^y(\tau_H \geq 2) \int_2^\infty \frac{dt}{t^{d/2}} \leq C \langle x_d \rangle^{\alpha/2} \langle y_d \rangle^{\alpha/2},
\]

because of (5.3), which completes the proof in this case.

To deal with \( d = 1 \) and \( 2 \) we observe that by the Schwarz inequality

\[
R^2(x, y) \leq R(x, x) R(y, y),
\]

so we need to estimate \( R(x, x) \). For the case \( x_d \leq 2 \), using (5.3) and (5.5), we have

\[
R(x, x) = \int_2^\infty (P^x(\tau_H \geq t))^2 \frac{dt}{t^{d/2}} \leq C x_d^2 \int_2^\infty \frac{(\ln t)^2}{t} \, dt.
\]

If \( x_d > 2 \), then applying (5.3) we estimate

\[
R(x, x) = \int_2^\infty (P^x(\tau_H \geq t))^2 \frac{dt}{t^{d/2}} \leq \int_2^{x_d^2} (P^x(\tau_H \geq t))^2 \frac{dt}{t^{d/2}} + \int_{x_d^2}^\infty (P^x(\tau_H \geq t))^2 \frac{dt}{t^{d/2}} \leq C \int_2^{x_d^2} \frac{dt}{t^{d/2}} + C \int_{x_d^2}^{\infty} \frac{x_d^2 + \ln t}{t^{d/2}} \, dt.
\]
Thus for \( d = 1 \) we have \( R(x, x) \leq Cx \), while for \( d = 2 \) we arrive at \( R(x, x) \leq C \ln x_2 \).

Taking into account all cases we get

\[
\int_6^\infty \psi_1^R(x, y) \, dt \leq C \psi_d(x) \psi_d(yd).
\]

The proof of the theorem is complete. \(\Box\)

Our final result provides optimal bounds for \( G_0^0(x, y) \) when the points \( x, y \) are close to each other.

**Theorem 5.9.** For \( d = 1 \) and \( |x - y| < 1 \) we have that

\[
(G_0^0(x, y) = G(x, y) + x \land y). \tag{5.7}
\]

For \( d = 2 \) and \( |x - y| < 1 \) we have that

\[
G_0^0(x, y) = G(x, y) + \ln(1 \lor (x_2 \land y_2)). \tag{5.8}
\]

For \( d \geq 3 \) and \( |x - y| < 1 \) we have that

\[
G_0^0(x, y) \approx G(x, y). \tag{5.9}
\]

**Proof.** Step 1. We first show that if \( |x - y| < 1 \), then

\[
G(x, y) \approx G_0^0(x, y) \quad \text{if} \quad x_d \land y_d \leq 2 \quad \text{or} \quad d \geq 3.
\]

Recall that by Lemma 5.7 we have

\[
(x_d y_d \land 1)^{\alpha/2} \leq CG(x, y). \tag{5.8}
\]

Let \( x_d \land y_d \leq 2 \). The condition \( |x - y| < 1 \) implies that \( x_d \lor y_d \leq 3 \), hence

\[
x_d y_d \leq 9(1 \land x_d y_d).
\]

Then due to Theorem 5.8 and (5.8) we obtain

\[
G(x, y) \leq G_0^0(x, y) \leq C_1(G(x, y) + (x_d y_d)^{\alpha/2}) \leq C_2G(x, y). \tag{5.9}
\]

We now consider the case when \( d \geq 3 \). It is elementary that

\[
(x_d \land 1)(y_d \land 1) \leq x_d y_d \lor 1.
\]

Then the inequality (5.9) can be rewritten in the following way regardless of the assumption on \( x_d \land y_d \):

\[
G(x, y) \leq G_0^0(x, y) \leq C(G(x, y) + ((x_d \land 1)(y_d \land 1))^{\alpha/2})
\]

\[
\leq C(G(x, y) + (x_d y_d \land 1)^{\alpha/2})
\]

\[
\leq C_1G(x, y),
\]

where in the last line we again applied (5.8). This completes the proof of Step 1 and proves the theorem for \( d \geq 3 \).

Step 2. In this step we complete the proof of the case \( d = 1 \).

The lower bound follows from the estimate proved in Lemma 5.6 and (5.6): \( G_0^0(x, y) \geq G_{(0, \infty)}(x, y) = x \land y \). In the case \( x \land y \leq 2 \) we also get the upper bound since \( G_0^0(x, y) \approx G_{(0, \infty)}(x, y) \) by Step 1.

If \( x \land y \geq 2 \) we obtain by Theorem 5.8

\[
G_0^0(x, y) \leq C_2(G_{(0, \infty)}(x, y) + (xy)^{1/2}).
\]

Since \( |x - y| < 1 \) and \( x \land y \geq 2 \) we have \( (xy)^{1/2} \leq 2(x \land y) \), which completes the proof of (5.7).
Step 3. Now we deal with the case $d = 2$. We claim that

$$G_\mathbb{H}^g(x, y) \geq C \ln(1 \vee (x_2 \wedge y_2)).$$

It is enough to show it for $x_2 \wedge y_2 \geq 1$. If $|x - y| \leq 1$, then using (5.6) we arrive at

$$G_\mathbb{H}^g(x, y) = \frac{1}{2\pi} \ln \left| \frac{x^* - y}{x - y} \right| \geq \frac{1}{2\pi} \ln |x^* - y| \geq \frac{1}{2\pi} \ln (x_2 + y_2) \geq \frac{1}{2\pi} \ln (1 \vee (x_2 \wedge y_2)).$$

As a consequence of the above inequality and Lemma [5.6] we obtain

$$\frac{1}{2\pi} \ln (1 \vee (x_2 \wedge y_2)) \leq G_\mathbb{H}^g(x, y) \leq G_\mathbb{H}^0(x, y),$$

which proves the lower bound.

If $x_2 \wedge y_2 \leq 2$, then by Step 1, $G_\mathbb{H}(x, y) \approx G_\mathbb{H}^0(x, y)$, which yields the upper bound in this case.

It remains to consider the case $x_2 \wedge y_2 \geq 2$. Note that $1 + x_2 \vee y_2 \leq (1 \vee (x_2 \wedge y_2))^2$,

$$\ln^{1/2}(1 + y_2) \ln^{1/2}(1 + x_2) \leq \ln(1 + x_2 \vee y_2) = 2 \ln(1 \vee (x_2 \wedge y_2)).$$

Thus, by Theorem [5.8]

$$G_\mathbb{H}^0(x, y) \leq C(G_\mathbb{H}(x, y) + \ln^{1/2}(1 + y_2) \ln^{1/2}(1 + x_2)) \leq C(G_\mathbb{H}(x, y) + \ln(1 \vee (x_2 \wedge y_2))),$$

which completes the proof of Step 3 and of the theorem.

We conclude with a partial answer to the question posed at the beginning of this section.

**Corollary 5.10.** For $d \leq 2$ we obtain

$$\sup_{|x - y| \leq 1} \frac{G_\mathbb{H}^0(x, y)}{G_\mathbb{H}^g(x, y)} = \infty.$$ 

For $d \geq 3$

$$G_\mathbb{H}^0(x, y) \approx G_\mathbb{H}^g(x, y), \quad |x - y| \leq 1.$$ 

**Proof.** For $d \geq 3$ the proof easily follows from Lemma [5.7] and the preceding theorem.

For $d \leq 2$ from the formula (4.4) we easily obtain the following upper bounds for $G_\mathbb{H}^g(x, y)$:

$$G_\mathbb{H}^g(x, y) \leq \left\{ \begin{array}{ll}
C \frac{(x \wedge y)^{n/2}}{|x - y|^{n-1}}, & x, y \in \mathbb{H}, \quad d = 1, \\
C \frac{1}{|x - y|^d}, & x, y \in \mathbb{H}, \quad d = 2.
\end{array} \right.$$ 

The application of the above inequalities together with Theorem [5.9] completes the proof.

**Remark.** It would be very interesting to obtain optimal estimates for the 0-Green function for balls. For balls of moderate radii one may show that (5.2) holds with a universal constant, so optimal estimates are easily derived from the well-known estimates in the stable case. However for large balls the comparison (5.2) is not satisfactory, as the constant grows to $\infty$ with the radius of the ball.
ACKNOWLEDGEMENT

The authors are very grateful to the referee for critical remarks and comments which enabled them to greatly improve the presentation of the paper.

REFERENCES


Institute of Mathematics and Computer Sciences, Wrocław University of Technology, ul. Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

E-mail address: tomasz.byczkowski@pwr.wroc.pl

Institute of Mathematics and Computer Sciences, Wrocław University of Technology, ul. Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

E-mail address: jacek.malecki@pwr.wroc.pl

Institute of Mathematics and Computer Sciences, Wrocław University of Technology, ul. Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

E-mail address: michal.ryznar@pwr.wroc.pl