FLOWS ON ORDERED BUNDLES

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In the investigation of the stability of an equilibrium in a smooth strongly monotone dynamical system (semiflow), an indispensable tool is the Frobenius–Perron theorem (in the finite-dimensional case), and the Kreĭn–Rutman theorem (in the infinite-dimensional case) applied to the linearization of the system at the equilibrium. (For the theory of strongly monotone semiflows the reader is referred in the continuous-time case to e. g. [**Hi**], or [**Sm–Th**], in the discrete-time case to [**Ta**], and for the definition of C^1 strongly monotone semiflows—to [**Mi**].) However, in many cases restricting oneself to equilibria does not suffice and one needs to consider the behavior of the linearization of the dynamical system on a more complex invariant set (for a recent analysis where it is necessary to consider *any* compact invariant set see [**Po–Te**]). To the author's knowledge, D. Ruelle was the first (1979) to generalize the Frobenius–Perron theorem to the case (in our terminology) of strongly monotone linear discrete-time semiflows on a finite-dimensional strongly ordered vector bundle (see [**R1**], Proposition 3.2).

The main result of the present paper (Theorem 1) is a generalization of Ruelle's result to a fairly broad class of strongly monotone semiflows on strongly ordered Banach bundles. The paper is organized as follows. Part 1 (Sections 1, 2 and 3) deals with abstract theorems about monotone linear semiflows on ordered Banach bundles. In Part 2 we show how the abstract results from Part 1 can be applied to linearizations of sufficiently smooth parabolic partial differential equations of second order.

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Part 1. The Abstract Results

1. Monotone semiflows on Banach bundles. Let V be a real Banach space with norm $\|\cdot\|$. For $v \in V \setminus \{0\}$, $\mathbb{P}v$ stands for its direction, and, for a subset $S \subset V$, $\mathbb{P}S := \{\mathbb{P}v : v \in S,\ v \neq 0\}$. For V, W Banach spaces, by $\mathcal{L}(V, W)$ we denote the Banach space of all bounded linear operators $A : V \to W$ endowed with the norm topology. We write $\mathcal{L}(V) := \mathcal{L}(V, V)$. $V^\star$ denotes the dual space to V, with the standard duality pairing $\langle v^\star, v\rangle$. $V^\star$ is endowed with the norm $\|v^\star\|_\star := \sup\{\langle v^\star, v\rangle : \|v\| = 1\}$.

A convex closed subset V_+ of V such that $\alpha V_+ \subset V_+$ for each $\alpha \geq 0$ is called a *cone* if $\{0\}$ is the only subspace contained in V_+. We will always assume that V_+ is *reproducing*, that is, $V_+ + (-V_+) = V$. A cone V_+ is called *solid* if its interior $\overset{\circ}{V}_+$ is nonempty. A pair (V, V_+) is called an *ordered Banach space*, and for V_+ solid a *strongly ordered Banach space*. For an ordered Banach space, we write $v \leq w$ if $w - v \in V_+$, and $v < w$ if $w - v \in V_+ \setminus \{0\}$, where v, $w \in V$. For V_+ solid, we write $v \ll w$ if $w - v \in \overset{\circ}{V}_+$. For $v, w \in V$, the *closed order interval* is defined as $[v, w] := \{u \in V : v \leq u \leq w\}$, and the *open order interval* as $[[v, w]] := \{u \in V : v \ll u \ll w\}$.

A cone V_+ is called *normal* if there exists $k > 0$ such that for each v, $w \in V_+$, $v \leq w$ one has $\|v\| \leq k\|w\|$. A pair (V, V_+) where V_+ is normal is called a *normally ordered Banach space*. For a normally ordered Banach space (V, V_+), the norm $\|\cdot\|$ is said to be *monotone* if $\|v\| \leq \|w\|$ for v, $w \in V_+$, $v \leq w$. From now on, we will assume that any normally ordered Banach space is endowed with a monotone norm (see [**K**–**L**–**S**], Thm. 4.4).

In the dual space $V^\star$ the *dual cone* is defined as $V_+^\star := \{v^\star \in V^\star : \langle v^\star, v\rangle \geq 0$ for all $v \in V_+\}$. It is well known that $V_+^\star$ is reproducing and normal if and only if V_+ is reproducing and normal (see e. g. [**K**–**L**–**S**], Thm. 4.5 and Thm. 4.6).

For $f \in V_+ \setminus \{0\}$ fixed we define the *order-unit norm*

$$\|v\|_f := \inf\{\alpha \geq 0 : -\alpha f \leq v \leq \alpha f\}.$$

The norm $\|\cdot\|_f$ is monotone. Let $V_f := \{v \in V : \|v\|_f < \infty\}$. Whenever V_+ is normal, for any $f \in V_+ \setminus \{0\}$ the normed space $(V_f, \|\cdot\|_f)$ is complete. Moreover, the cone $(V_f)_+ := V_f \cap V_+$ is solid (in $(V_f, \|\cdot\|_f)$), and $f \in (\overset{\circ}{V_f})_+$.

A strongly ordered Banach space (V, V_+) will be always considered with fixed $f \in \overset{\circ}{V}_+$ and $f^\star \in V_+^\star$ such that $\|\cdot\| = \|\cdot\|_f$ and $\langle f^\star, f\rangle = 1$.

Now we introduce an equivalence relation on $V_+ \setminus \{0\}$:

$$v \simeq w \quad \text{if there exist } 0 < \alpha \leq \beta \text{ such that } \alpha v \leq w \leq \beta v.$$

For $h \in V_+ \setminus \{0\}$, K_h stands for the equivalence class of h. If V_+ is solid, then $\overset{\circ}{V}_+ = K_h$ for any $h \in \overset{\circ}{V}_+$. For v, $w \in K_h$ we define

$$\tilde{\rho}(v, w) := \log \frac{\inf\{\beta > 0 : v \leq \beta w\}}{\sup\{\alpha > 0 : \alpha w \leq v\}}.$$

$(K_h, \tilde{\rho})$ is a pseudometric space. Moreover, $\tilde{\rho}(v, w) = \tilde{\rho}(\alpha v, \beta w)$ for all α, $\beta > 0$. Setting $\rho(\mathbb{P}v, \mathbb{P}w) := \tilde{\rho}(v, w)$ we obtain a metric ρ on $\mathbb{P}K_h$ (called *Hilbert's projective metric*). If the cone V_+ is normal, then for any $h \in V_+ \setminus \{0\}$, $(\mathbb{P}K_h, \rho)$ is a complete metric space (for proofs of the above facts see [**K**–**L**–**S**], Section 4.6).

By the *projectivization* of a linear operator $A : V \to V$ we understand the mapping $\mathbb{P}A : \mathbb{P}V \to \mathbb{P}V$ defined by $\mathbb{P}A(\mathbb{P}v) := \mathbb{P}(Av)$ for each $v \in V \setminus \{0\}$. For an ordered Banach space (V, V_+) a linear operator $A : V \to V$ is called *monotone* if $AV_+ \subset V_+$, and, for (V, V_+) strongly ordered, *strongly monotone* if $A(V_+ \setminus \{0\}) \subset \overset{\circ}{V}_+$. A monotone operator A is called *focusing* if

$$\sup\{\rho(\mathbb{P}A(\mathbb{P}v), \mathbb{P}A(\mathbb{P}w)) : v, w \in V_+ \setminus \{0\}, Av \neq 0, Aw \neq 0\} < \infty.$$

It is well known that if a linear operator A is monotone then so is its adjoint $A^\star : V^\star \to V^\star$. Moreover, if A is focusing then so is $A^\star$ (see [**K**–**L**–**S**], Thm. 10.1).

Let $\mathcal{V} := (X \times V, X, \pi)$ be a product Banach bundle with total space $X \times V$, base space X and projection π, where X is a compact metric space. The structure group of a Banach bundle $\mathcal{V}$ will always be the group $\mathcal{GL}(V)$ of all bounded linear automorphisms of V endowed with the uniform operator topology. We will frequently identify the bundle $\mathcal{V}$ with its total space $X \times V$. Generic elements of the total space $X \times V$ will be denoted by boldface letters $\mathbf{v}$, $\mathbf{w}$ etc. When we want to emphasize that $\mathbf{v}$ belongs to the fiber over a point $x \in X$, we write $(x, v) = \mathbf{v} \in \pi^{-1}(x) := \{x\} \times V$. $\mathcal{Z}$ stands for the zero section of $\mathcal{V}$. By a *Finsler* on $\mathcal{V}$ we mean a continuous function $(x, v) \ni X \times V \mapsto p(\mathbf{v}) \in [0, \infty)$ such that for $x \in X$ fixed, $(x, v) \mapsto p(\mathbf{v})$ is a norm on V, equivalent with $\|\cdot\|$. The Finslers $p(\cdot)$ and $p'(\cdot)$ are *uniformly equivalent* if there are constants $0 < k_1 \leq k_2$ such that $k_1 p(\mathbf{v}) \leq p'(\mathbf{v}) \leq k_2 p(\mathbf{v})$ for all $\mathbf{v} \in X \times V$. The *product* Finsler on $\mathcal{V}$ induced by the norm $\|\cdot\|$ is defined as $\|\mathbf{v}\| := \|v\|$.

If (V, V_+) is a (strongly, normally) ordered Banach space, $\mathcal{V}$ is said to be *(strongly, normally) ordered*. We write $\mathcal{V}_+ := X \times V_+$, and $\overset{\circ}{\mathcal{V}}_+ := X \times \overset{\circ}{V}_+$. For $h \in V_+ \setminus \{0\}$, $(\mathcal{V}_h)_+ := X \times (V_h)_+$. Let $(\mathcal{V}, \mathcal{V}_+)$ be an ordered Banach bundle, and let $h : X \to V_+ \setminus \{0\}$ be a fixed continuous function. We define the *order-unit Finsler* $\|\cdot\|_h$ as $\|\mathbf{v}\|_h := \|v\|_{h(x)}$, $x \in X$, $v \in V$. In the sequel we will frequently need the following

Lemma 1.1. *Let h_1, $h_2 : X \to \overset{\circ}{V}_+$ be continuous. Then the order-unit Finslers $\|\cdot\|_{h_1}$ and $\|\cdot\|_{h_2}$ on the strongly ordered Banach bundle $(\mathcal{V}, \mathcal{V}_+)$ are uniformly equivalent.*

Proof. Notice that over each $x \in X$, the order-unit norms $\|\cdot\|_{h_1(x)}$ and $\|\cdot\|_{h_2(x)}$ are equivalent, and make use of the continuity of h_1 and h_2 and the compactness of X. $\square$

By a *flow* [*semiflow*] φ on a metric space X we mean a continuous mapping

$$\varphi : \mathbb{R} \times X \to X \quad [\varphi : [0, \infty) \times X \to X] \quad \text{(we denote } \varphi_t(\cdot) := \varphi(t, \cdot))$$

such that $\varphi_0 = \mathrm{id}_X$ and for each s, $t \in \mathbb{R}$ [s, $t \in [0, \infty)$], $\varphi_s \circ \varphi_t = \varphi_{s+t}$.

A *discrete-time flow* [*discrete-time semiflow*] φ on X is a continuous mapping

$$\varphi : \mathbb{Z} \times X \to X \quad [\varphi : (\mathbb{N} \cup \{0\}) \times X \to X]$$

such that $\varphi_0 = \mathrm{id}_X$ and for each n_1, $n_2 \in \mathbb{Z}$ $\quad [n_1,\, n_2 \in \mathbb{N} \cup \{0\}]$, $\quad \varphi_{n_1} \circ \varphi_{n_2} = \varphi_{n_1+n_2}$. All our results will be formulated for (semi)flows, however they remain valid for discrete-time (semi)flows too.

For a semiflow φ on X, a set $Y \subset X$ is called *invariant* if $\varphi_t Y = Y$ for every $t \ge 0$, and *totally invariant* if it is invariant and $\varphi_t^{-1} Y \subset Y$.

Assume that $\mathcal{V} := (X \times V, X, \pi)$ is a Banach bundle. A semiflow Φ on the total space $X \times V$ is a *linear semiflow covering* a flow φ on X, if for each $t \in [0, \infty)$, Φ_t is a Banach bundle endomorphism:

$$\Phi_t \mathbf{v} = \Phi(t, x, v) = (\varphi_t x, \phi_t(x) v) \quad \text{for } x \in X, v \in V,$$

where the assignment $(0, \infty) \times X \ni (t, x) \mapsto \phi_t(x) \in \mathcal{L}(V)$ is continuous.

A linear semiflow Φ is called *compact* if for each $t > 0$ the set $\{(\varphi_t x, \phi_t(x) v) : x \in X, \|v\| = 1\}$ is precompact.

For $\mathcal{V}$ ordered, a linear semiflow Φ on $\mathcal{V}$ is called *monotone* if $\phi_t(x)$ is monotone for each $t \ge 0$, $x \in X$. For $\mathcal{V}$ strongly ordered, Φ is called *strongly monotone* if $\phi_t(x)$ is strongly monotone for each $t > 0$, $x \in X$.

Theorem 1. *Assume that*

1) *Φ is a compact linear semiflow on a strongly and normally ordered Banach bundle $\mathcal{V}$, and*

2) *For each $T > 0$ there are constants $0 < m_T \le M_T$ with the following property: for any $\mathbf{v} \in \mathcal{V}_+ \setminus \mathcal{Z}$ there is $l(\mathbf{v}) > 0$ such that*

$$m_T l(\mathbf{v}) f \le \phi_T(x) v \le M_T l(\mathbf{v}) f \quad \textit{for all } t \in [T/2, 2T], x \in X.$$

Then

i) *There exists a one-dimensional invariant subbundle $\mathcal{S}$ such that $\mathcal{S} \setminus \mathcal{Z} \subset \overset{\circ}{\mathcal{V}}_+ \cup -\overset{\circ}{\mathcal{V}}_+$. Such a subbundle is unique.*

ii) *There exists a one-codimensional totally invariant subbundle* $\mathcal{T}$ *such that* $\mathcal{T} \cap \mathcal{V}_+ = \mathcal{Z}$*. Such a subbundle is unique.*
iii) *There are constants* $\nu > 0$ *and* $c > 0$ *such that*

$$\frac{\|\Phi_t \mathbf{v}\|}{\|\Phi_t \mathbf{w}\|} \geq ce^{\nu t} \quad \textit{for all } t \geq 0, x \in X, \mathbf{v} \in \mathcal{S}_x, \mathbf{w} \in \mathcal{T}_x, \|\mathbf{v}\| = \|\mathbf{w}\| = 1.$$

(The last property is called *exponential separation*).

2. Proof of Theorem 1. In the first part of the proof we show that the theorem holds for the discrete-time semiflows.

1) Existence of a one-dimensional invariant subbundle. Denote by $\mathbb{P}\mathring{\mathcal{V}}_+$ the product fiber bundle $(X \times \mathbb{P}\mathring{V}_+, X, \mathbb{P}\pi)$ where $\mathbb{P}\mathring{V}_+$ is endowed with the projective metric ρ.

Lemma 2.1. *Let* σ *be a (not necessarily continuous) section of the fiber bundle* $\mathbb{P}\mathring{\mathcal{V}}_+$*, and let* $\boldsymbol{\sigma} : X \to \mathring{V}_+$ *be the (unique) function such that* $\|\boldsymbol{\sigma}(x)\| = 1$ *and* $\sigma(x) = (x, \mathbb{P}\boldsymbol{\sigma}(x))$ *for all* $x \in X$*. Then* σ *is continuous if and only if* $\boldsymbol{\sigma}$ *is continuous.*

Proof. Assume that σ is continuous. Let $x_k \in X$ be a sequence converging to x. By the definition of the metric ρ, there are sequences $0 < \alpha_k \leq \beta_k$ such that $\alpha_k \boldsymbol{\sigma}(x_k) \leq \boldsymbol{\sigma}(x) \leq \beta_k \boldsymbol{\sigma}(x_k)$ and $\beta_k/\alpha_k \to 1$. We claim that $\alpha_k \to 1$ and $\beta_k \to 1$. Indeed, if, for example, $\beta_{k_l} \to a < 1$ for some subsequence k_l, then by the monotonicity of the norm we would have $\|\boldsymbol{\sigma}(x)\| \leq a < 1$, a contradiction. We have

$$\|\boldsymbol{\sigma}(x_k) - \boldsymbol{\sigma}(x)\| \leq \|\boldsymbol{\sigma}(x_k) - \alpha_k \boldsymbol{\sigma}(x_k)\| + \|\boldsymbol{\sigma}(x) - \alpha_k \boldsymbol{\sigma}(x_k)\| \leq |1 - \alpha_k| + (\beta_k - \alpha_k),$$

which converges to 0.

Assume that $\boldsymbol{\sigma}$ is continuous, and let $x_k \in X$ be a sequence converging to x. The norms $\|\cdot\|$ and $\|\cdot\|_{\boldsymbol{\sigma}(x)}$ are equivalent, so the family $\{[(1-1/n)\boldsymbol{\sigma}(x), (1+1/n)\boldsymbol{\sigma}(x)] : n = 1, 2, \ldots\}$ forms a neighborhood basis at $\boldsymbol{\sigma}(x)$ in V. Therefore there are sequences $\alpha_k \to 1$, $\beta_k \to 1$ such that $\alpha_k \boldsymbol{\sigma}(x) \leq \boldsymbol{\sigma}(x_k) \leq \beta_k \boldsymbol{\sigma}(x)$. From this it follows that $\tilde{\rho}(\boldsymbol{\sigma}(x_k), \boldsymbol{\sigma}(x)) \leq \log(\beta_k/\alpha_k) \to 0$ as $k \to \infty$. $\square$

Let Σ denote the complete metric space of all continuous sections of $\mathbb{P}\mathring{\mathcal{V}}_+$ with the metric $d(\sigma', \sigma'') := \sup\{\rho(\sigma'(x), \sigma''(x)) : x \in X\}$. For notational convenience, denote $\varphi := \varphi_1$, $\Phi := \Phi_1$, $\phi := \phi_1$, $m := m_1$, $M := M_1$. The mapping Φ induces in a natural way the fiber bundle endomorphism $\mathbb{P}\Phi : \mathbb{P}\mathring{\mathcal{V}}_+ \to \mathbb{P}\mathring{\mathcal{V}}_+$. Consider the *graph transform* $\mathfrak{F}$ from Σ into the set of all sections of $\mathbb{P}\mathring{\mathcal{V}}_+$ defined as

$$\mathfrak{F}\sigma(x) := \mathbb{P}\Phi(\sigma(\varphi^{-1}x)), x \in X.$$

Lemma 2.2. *$\mathfrak{F}$ maps Σ into Σ.*

Proof. Denote by $\mathbf{\Sigma}$ the Banach space of all continuous sections of $\mathcal{V}$, endowed with the supremum norm. Fix $\sigma \in \Sigma$ and let $\boldsymbol{\sigma} \in \mathbf{\Sigma}$ be as in Lemma 2.1.

We have

$$\mathfrak{F}\sigma(x) = \mathbb{P}\phi(\varphi^{-1}x)\boldsymbol{\sigma}(\varphi^{-1}x), x \in X.$$

By the continuity of $\phi(x)$ in x we get that the function $\Gamma : X \to V$ defined as $\Gamma(x) := \phi(\varphi^{-1}x)\boldsymbol{\sigma}(\varphi^{-1}x)$, $x \in X$, belongs to $\mathbf{\Sigma}$. $\phi(x)$ is strongly monotone, so $\Gamma(x) \in \overset{\circ}{V}_+$ for all $x \in X$, therefore the section γ defined as $\gamma(x) := \Gamma(x)/\|\Gamma(x)\|$ is in $\mathbf{\Sigma}$. By Lemma 2.1, $\mathfrak{F}\sigma \in \Sigma$. $\square$

From hypothesis 2) it follows that $\phi(x)v \gg 0$ for all $v \in V_+ \setminus \{0\}$, $x \in X$, and that the projective diameter of the image $\mathbb{P}\phi(x)(\mathbb{P}V_+)$ is not bigger than $\log(M/m)$. By [**K–L–S**], Lemma 10.3, $\mathbb{P}\phi(x)$ contracts projective distances by factor not bigger than $(\sqrt{M/m} - 1)/(\sqrt{M/m} + 1) < 1$. Therefore $\mathfrak{F}$ is a contraction on Σ, so there exists a unique fixed point τ_1 of $\mathfrak{F}$. The section τ_1 of $\mathbb{P}\overset{\circ}{\mathcal{V}}_+$ corresponds to a one-dimensional continuous Φ_1-invariant subbundle $\mathcal{S}_1$ of $\mathcal{V}$. The uniqueness of $\mathcal{S}_1$ is equivalent to the uniqueness of a fixed point of the graph transform $\mathfrak{F}$.

2) Existence of a family of complementary subspaces. The normal cone V_+ is solid, hence reproducing, so the dual cone $V_+^\star$ is normal but, in general, only reproducing. For any $x \in X$, $v^\star \in V_+^\star \setminus \{0\}$, we have $\phi^\star(x)v^\star \in V_+^\star \setminus \{0\}$. Moreover, by [**K–L–S**], Thm. 10.1, the operators $\phi^\star(x) : V_+^\star \to V_+^\star$ are focusing, with the projective diameters of $\mathbb{P}\phi^\star(\varphi x)\mathbb{P}V_+^\star$ not bigger than $\log(M/m)$.

For $x \in X$, let D_x denote the closure of $\mathbb{P}\phi^\star(\varphi x)\mathbb{P}V_+^\star$ in the topology corresponding to the projective metric ρ. (D_x, ρ) is a complete metric space. $\mathcal{D}$ will denote the product fiber bundle with base X_{d}(=X with the discrete topology) and fiber over $x \in X$ equal to $\{x\} \times D_x$. Let $\Sigma^\star$ denote the set of all sections $\sigma^\star$ of $\mathcal{D}$ endowed with the metric $d(\sigma^{\star\prime}, \sigma^{\star\prime\prime}) := \sup\{\rho(\sigma^{\star\prime}(x), \sigma^{\star\prime\prime}(x)) : xinX\}$. $(\Sigma^\star, d)$ is a complete metric space. Making use of the appropriate graph transform, we obtain the existence of a section $\tau_1^\star$ of the bundle $\mathcal{D}$. The mapping $g^\star : X \to V_+^\star$ is defined so that $\|g^\star(x)\| = \langle g^\star(x), f\rangle = 1$ and $\tau_1^\star(x) = (x, \mathbb{P}g^\star(x))$ for all $x \in X$. By $\mathcal{T}_1$ we denote the Φ_1-invariant family of the nullspaces of the functionals $g^\star(x)$.

3) Exponential separation. Let, for each $x \in X$, $g(x)$ denote the unique unit vector contained in $(\overset{\circ}{\mathcal{V}}_+ \cap \mathcal{S}_1)_x$. By Lemma 1.1, the Finslers $\|\cdot\|_g$ and $\|\cdot\|$ are uniformly equivalent. Define the function osc : $\mathcal{T}_1 \to [0, \infty)$ as

$$\operatorname{osc}(\mathbf{w}) := \beta(\mathbf{w}) - \alpha(\mathbf{w})$$

where $\beta(\mathbf{w}) := \min\{\beta : \beta g(x) \geq w\}$, $\alpha(\mathbf{w}) := \max\{\alpha : \alpha g(x) \leq w\}$. By [**K–L–S**], Section 12.2, $\operatorname{osc}|(\mathcal{T}_1)_x$ is a norm. Moreover, since for $\mathbf{w} \in \mathcal{T}_1 \setminus \mathcal{Z}$, $\beta(\mathbf{w}) > 0$, $\alpha(\mathbf{w}) < 0$, we have

$$\|\mathbf{w}\|_g \leq \operatorname{osc}(\mathbf{w}) \leq 2\|\mathbf{w}\|_g \tag{2.1}$$

(recall that $\|\mathbf{w}\|_g = \min\{\alpha \geq 0 : -\alpha g(x) \leq w \leq \alpha g(x)\}$).

From [**K–L–S**], Lemma 12.1, it follows that for $\kappa := (\sqrt{M/m}-1)/(\sqrt{M/m}+1) < 1$ and for any $\mathbf{w} \in \mathcal{T}_1$, $\operatorname{osc}(\mathbf{w}) = 1$ one has

$$\frac{\operatorname{osc}(\Phi_1 \mathbf{w})}{\|\phi(x)g(x)\|_{g(\varphi x)}} \leq \kappa.$$

Therefore for any $\mathbf{v} \in \mathcal{S}_1$, $\|\mathbf{v}\| = \|\mathbf{v}\|_g = 1$, we have

$$\frac{\|\Phi_1 \mathbf{v}\|_g}{\operatorname{osc}(\Phi_1 \mathbf{w})} \geq \theta > 1$$

with $\theta := \kappa^{-1} > 1$. By induction we prove that

$$\frac{\|\Phi_n \mathbf{v}\|_g}{\operatorname{osc}(\Phi_n \mathbf{w})} \geq \theta^n, n \in \mathbb{N}.$$

By (2.1) the assertion follows.

In the sequel we will need the following result.

Lemma 2.3. *For any $\mathbf{w} \notin \mathcal{T}_1$, there exists $N \in \mathbb{N}$ such that $\Phi_n \mathbf{w} \in (\mathring{\mathcal{V}}_+ \cup -\mathring{\mathcal{V}}_+)$ for all $n \geq N$.*

Proof. For $h^\star \in V_+^\star$ such that $\langle h^\star, f\rangle = 1$, let $P(h^\star)$ and $P'(h^\star)$ denote the projections corresponding to the direct sum decomposition $V = \operatorname{span}\{f\} \oplus \operatorname{nullspace}\{h^\star\}$. Notice that these projections are defined as:

$$P(h^\star)v = \langle h^\star, v\rangle f \ , \ P'(h^\star)v = v - \langle h^\star, v\rangle f \quad \text{for } v \in V.$$

Since $h^\star$ is nonnegative, we have $\langle h^\star, v\rangle \geq m$, therefore $\|P(h^\star)v\| \geq m$, provided that $v \in [mf, Mf]$. On the other hand,

$$\begin{aligned}\|P'(h^\star)v\| = \|(v-f) - (\langle h^\star, v\rangle f - f)\| &\leq \|v - f\| + \|\langle h^\star, v\rangle f - f\| \\ \|v - f\| + \|\langle h^\star, v - f\rangle f\| &\leq 2\max\{|m-1|, |M-1|\}.\end{aligned}$$

for $v \in [mf, Mf]$. So we have

$$\frac{\|P(h^\star)v\|}{\|P'(h^\star)v\|} \geq \frac{m}{2\max\{|m-1|, |M-1|\}}, \tag{2.2}$$

The inequality (2.2) holds also for all $v \in \overset{\circ}{V}_+$ such that $v = au$ for some $a > 0$ and $u \in [mf, Mf]$.

By the exponential separation for the discrete-time semiflows we have that for $\mathbf{w} \notin \mathcal{T}_1$ there is $c(\mathbf{w}) > 0$ such that

$$\frac{\|\Phi_n \mathbf{w}^{(1)}\|}{\|\Phi_n \mathbf{w}^{(2)}\|} \geq c(\mathbf{w}) e^{\nu n} \quad \text{for all } n \in \mathbb{N},$$

where $\mathbf{w} = \mathbf{w}^{(1)} + \mathbf{w}^{(2)}$, $\mathbf{w}^{(1)} \in \mathcal{S}_1$, $\mathbf{w}^{(2)} \in \mathcal{T}_1$. Let $x_n := \varphi_n x$ and w_n be such that $\Phi_n \mathbf{w} = (x_n, w_n)$. Making use of Lemma 1.1 we obtain that

$$\frac{\|P(g^\star(x_n))w_n\|}{\|P'(g^\star(x_n))w_n\|} \geq c_1(\mathbf{w}) e^{\nu n}$$

for some $c_1(\mathbf{w}) > 0$ and all $n \in \mathbb{N}$. The desired result follows easily. $\square$

4) Invariance. We get the invariance of $\mathcal{S}_1$ and $\mathcal{T}_1$ in the proof of their existence. In order to prove that $\mathcal{T}_1$ is totally invariant, suppose to the contrary that there are $\mathbf{v} \notin \mathcal{T}_1$ and $n \in \mathbb{N}$ such that $\Phi_n \mathbf{v} \in \mathcal{T}_1$. Then, by Lemma 2.3, $\Phi_{n_1} \mathbf{v} \in \overset{\circ}{\mathcal{V}}_+ \cup -\overset{\circ}{\mathcal{V}}_+$ for some $n_1 \in \mathbb{N}$, $n_1 \geq n$, which contradicts the (forward) invariance of $\mathcal{T}_1$.

5) Continuity of the family $\mathcal{T}_1$. We will prove that, in fact, $\mathcal{T}_1$ is a trivial Banach bundle over X. This is the content of the following

Proposition 2.1. *Let V_1 be the subspace spanned by f and let V_2 be the nullspace of $f^\star$. Then there exists a continuous mapping $X \ni x \mapsto F(x) \in \mathcal{L}(V_2, V_1)$ such that $\mathcal{T}_1 = \{(x, v + F(x)v) : x \in X, v \in V_2\}$.*

Proof. Recall that $g^\star(x)$, for any $x \in X$, denotes the functional defining $(\mathcal{T}_1)_x$, normalized so that $\langle g^\star(x), f\rangle = 1$. The operator $F(x)$ is defined as:

$$F(x)v := -\langle g^\star(x), v\rangle f.$$

We have also the equality

$$\langle g^\star(x), v\rangle = -\langle f^\star, F(x)v\rangle.$$

First, we wish to prove that the *set* $\mathcal{T}_1$ is closed. Assume that $\mathbf{v} \notin \mathcal{T}_1$. By Lemma 2.3, there exists $n \in \mathbb{N}$ such that $\Phi_n \mathbf{v} \in \overset{\circ}{\mathcal{V}}_+ \cup -\overset{\circ}{\mathcal{V}}_+$. The mapping Φ_n is continuous, so $\Phi_n \mathbf{w} \in \overset{\circ}{\mathcal{V}}_+ \cup -\overset{\circ}{\mathcal{V}}_+$ for all $\mathbf{w}$ in some neighborhood of $\mathbf{v}$. By the invariance of the set $\mathcal{T}_1$, no such $\mathbf{w}$ can belong to $\mathcal{T}_1$.

Now, assume that $x_k \to x \in X$, $v_k \to v$. From Lemma 2.3 we deduce that the set $\{\|F(x)v\| : x \in X, \|v\| = 1\}$ is bounded, so, since V_1 is one-dimensional, the set $\{F(x_k)v_k : k \in \mathbb{N}\}$ is precompact. Choosing a subsequence, if necessary, we can assume that $F(x_k)v_k \to w$. So we have that $x_k \to x, v_k \to v$ and $F(x_k)v_k \to w$. By the closedness of $\mathcal{T}_1$ we get $w = F(x)v$. It follows that the mapping $x \mapsto F(x)$ is continuous *in the strong operator topology*. The desired continuity in the uniform topology follows in a standard way from the compactness of Φ_1 (compare e. g. [**Pa**], Thm. 3.2 on p. 48). □

6) $\mathcal{S}_t = \mathcal{S}_1$ *for all* $t > 0$. Precisely as in part *1)* we consider for any $t > 0$ the graph transform $\mathfrak{F}_t$. From assumption 2) we deduce that, for $T > 0$ fixed, $\mathfrak{F}_t$'s contract uniformly in $t \in [T/2, 2T]$. From the well-known facts (see e. g. [**D–G**], Problem 6.4(b) on p. 17) it follows that we prove the continuous dependence of the fixed point τ_t on $t \in [T/2, 2T]$ if we prove that, for $\sigma \in \Sigma$ fixed, the assignment $[T/2, 2T] \ni t \mapsto \mathfrak{F}_t\sigma$ is continuous.

Lemma 2.4. *Let* $t_k \to t > 0$. *Then for* $\sigma \in \Sigma$ *fixed one has* $\lim_{k\to\infty} d(\mathbb{P}\Phi_{t_k}\sigma, \mathbb{P}\Phi_t\sigma) = 0$.

Proof. As in the proof of Lemma 2.2, denote by $\mathbf{\Sigma}$ the Banach space of all continuous functions from X into V, endowed with the supremum norm, and let $\boldsymbol{\sigma} : X \to \overset{\circ}{V}_+$ stand for the (unique) function such that $\|\boldsymbol{\sigma}(x)\| = 1$ and $(x, \mathbb{P}\boldsymbol{\sigma}(x)) = \sigma(x)$ for all $x \in X$. Denote

$$\Gamma(x) := \phi(\varphi_{-t}x)\boldsymbol{\sigma}(\varphi_{-t}x), \Gamma_k(x) := \phi(\varphi_{-t_k}x)\boldsymbol{\sigma}(\varphi_{-t_k}x),$$

for $x \in X$. Since $\boldsymbol{\sigma}$ is uniformly continuous and ϕ depends continuously on x in the uniform topology, we get that $\sup_{x\in X} \|\Gamma_k(x) - \Gamma(x)\| \to 0$ as $k \to \infty$. Further, $\sup_{x\in X} \|\gamma_k(x) - \gamma(x)\| \to 0$, where $\gamma(x) := \Gamma(x)/\|\Gamma(x)\|$ and $\gamma_k(x) := \Gamma_k(x)/\|\Gamma_k(x)\|$. By Lemma 1.1, the Finslers $\|\cdot\|$ and $\|\cdot\|_\gamma$ are uniformly equivalent. Notice that the family $\{\mathbf{R}_l : l \in \mathbb{N}\}$, where $\mathbf{R}_l := \{\mathbf{r} \in \mathbf{\Sigma} : \mathbf{r}(x) \in [(1-1/l)\gamma(x), (1+1/l)\gamma(x)]\}$, forms a neighborhood basis of γ in $\mathbf{\Sigma}$. Now the desired statement follows easily (compare Lemma 2.1). □

By uniqueness, $\tau_1 = \tau_t$ for each $t \in \mathbb{Q}$, $t > 0$. The set $\{t \in \mathbb{Q}, t > 0\}$ is dense in $(0, \infty)$, so all τ_t, $t > 0$, are the same. We put $\mathcal{S} = \mathcal{S}_1$.

7) $\mathcal{T}_t = \mathcal{T}_1$ *for all* $t > 0$. As in *2)* we prove that for each $t > 0$ there exists a one-codimensional subbundle $\mathcal{T}_t$, complementary to $\mathcal{S}_t = \mathcal{S}$. Suppose to the contrary that there exists $\mathbf{v} \in \mathcal{T}_t \setminus \mathcal{T}_1$ for some $t \neq 1$. By Lemma 2.3, $\Phi_n\mathbf{v} \in \mathring{\mathcal{V}}_+ \cup -\mathring{\mathcal{V}}_+$ for some $n \in \mathbb{N}$, hence $\Phi_n\mathbf{v} \notin \mathcal{T}_t$, which contradicts the invariance of $\mathcal{T}_t$.

8) Invariance for the continuous time. The proof goes precisely as for the discrete time (see *4)*.

9) Exponential separation for the continuous time. Now we introduce some concepts form the spectral theory of linear flows on vector bundles. By the *dynamical spectrum* of a linear flow Φ defined on a (finite-dimensional) vector bundle $\mathcal{B}$ we understand the complement in $\mathbb{R}$ of the set of those λ for which there are a continuous invariant decomposition $\mathcal{B} = \mathcal{B}_{(1)} \oplus \mathcal{B}_{(2)}$ and constants $c_{(1)} > 0$, $c_{(2)} > 0$ such that

$$\|\Phi_t\mathbf{v}_{(\mathbf{1})}\| \le c_{(1)}e^{\lambda t}\|\mathbf{v}_{(\mathbf{1})}\| \quad \text{for all } \mathbf{v}_{(\mathbf{1})} \in \mathcal{B}_{(1)}, t \ge 0$$

and

$$\|\Phi_t\mathbf{v}_{(\mathbf{2})}\| \ge c_{(2)}e^{\lambda t}\|\mathbf{v}_{(\mathbf{2})}\| \quad \text{for all } \mathbf{v}_{(\mathbf{2})} \in \mathcal{B}_{(2)}, t \ge 0.$$

(The case $\mathcal{B}_{(i)} = \mathcal{Z}$ is not excluded.)

Lemma 2.5. *The semiflow $\Phi|\mathcal{S}$ extends naturally to a flow on $\mathcal{S}$. If we denote the infimum and the supremum of its dynamical spectrum by $\underline{\lambda}_1$ and $\overline{\lambda}_1$ respectively, then for each λ', λ'', $\lambda' < \underline{\lambda}_1 \le \overline{\lambda}_1 < \lambda''$, there are constants $k(\lambda') > 0$ and $K(\lambda'') > 0$ such that*

$$\|\Phi_t\mathbf{v}\| \ge k(\lambda')e^{\lambda' t}\|\mathbf{v}\| \tag{2.3}$$

and

$$\|\Phi_t\mathbf{v}\| \le K(\lambda'')e^{\lambda'' t}\|\mathbf{v}\| \tag{2.4}$$

for all $\mathbf{v} \in \mathcal{S}$ and all $t \ge 0$.

Proof. The semiflow $\Phi|\mathcal{S}$ is defined on a one-dimensional subbundle and for all $t \ge 0$ the mappings $\Phi_t|\mathcal{S}$ are invertible, so $\Phi|\mathcal{S}$ extends naturally to a flow. The remaining statements follow from the results contained in [**Sa–Se**], Section 3. □

Now, consider the semiflow $\Phi|\mathcal{T}$. Making use of the Uniform Boundedness Theorem we deduce that there are constants $d > 0$ and λ_2 such that

$$\|\Phi_t\mathbf{w}\| \le de^{\lambda_2 t}\|\mathbf{w}\| \quad \text{for all } \mathbf{w} \in \mathcal{T}, t \ge 0 \tag{2.5}$$

(the proof goes precisely in the same way as in the case of semigroups, compare [**Pa**], Thm. 2.2 on p. 4). Combining the exponential separation for the discrete time with (2.3) and (2.5) we get the desired result.

3. Generalizations. Let (V, V_+) be a strongly ordered Banach space. Following [**Hi**], we can define the *order topology* on V as the topology generated by the order-unit norm $\|\cdot\|_h$ for some $h \in \overset{\circ}{V}_+$ (such norms corresponding to different choices of h are equivalent, cf. Lemma 1.1). Let $\widehat{V}$ be the completion of V in any of these norms, and let $j : V \to \widehat{V}$ be the natural embedding. The Banach space $\widehat{V}$ is strongly and normally ordered. Assume that Φ is a strongly monotone semiflow on the bundle $\mathcal{V} = (X \times V, X, \pi)$. For all $x \in X$, $t \geq 0$, the mapping $\phi_t(x)$ induces in a natural way a mapping $\widehat{\phi}_t(x) \in \mathcal{L}(\widehat{V}, \widehat{V})$ (see [**Hi**], Proposition 1.10). If we define $\widehat{\Phi}_t(\mathbf{v}) := (\varphi_t x, \widehat{\phi}_t(x)v)$ for $x \in X$, $v \in \widehat{V}$ and $t \geq 0$, then $\widehat{\Phi}$ satisfies all the properties of a strongly monotone linear semiflow on the bundle $\widehat{\mathcal{V}} := (X \times \widehat{V}, \widehat{V}, \widehat{\pi})$ except perhaps continuity in $x \in X$. We say that Φ *factorizes through* $\widehat{\Phi}$ if the following conditions are satisfied:

F1) $\widehat{\Phi}$ is a (strongly monotone) linear semiflow on $\widehat{\mathcal{V}}$, and

F2) For $t > 0$, $x \in X$, $\widehat{\phi}_t(x)$ factorizes as $\widehat{\phi}_t(x) = j \circ \psi_t(x)$, where the assignment $(0, \infty) \times X \ni (t, x) \mapsto \psi_t(x) \in \mathcal{L}(\widehat{V}, V)$ is continuous.

Theorem 2. *Assume that a strongly monotone semiflow Φ factorizes through $\widehat{\Phi}$, and that $\widehat{\Phi}$ satisfies all the hypotheses of Theorem 1. Then all the assertions of Theorem 1 hold for the original semiflow Φ.*

Indication of proof. The sets $\mathcal{S}$ and $\mathcal{T}$ constructed for the semiflow $\widehat{\Phi}$ are obviously contained in $X \times V$. The rest goes along the lines of the corresponding parts of the proof of Theorem 1. $\square$

Now we proceed to another generalization of Theorem 1. The flow φ on X is called *uniquely ergodic* if there exists precisely one probability Borel measure μ on X invariant under φ.

Theorem 3. *Assume that a semiflow Φ covering a uniquely ergodic flow φ satisfies all the hypotheses of Theorem 1 (or Theorem 2). Then there are constants $-\infty \leq \lambda_2 < \lambda_1 < \infty$ such that for any λ, $\lambda_2 < \lambda < \lambda_1$, one can find $c(\lambda)$ such that*

$$\|\Phi_t \mathbf{v}\| \geq c(\lambda) e^{\lambda t} \|\mathbf{v}\| \quad \text{for } \mathbf{v} \in \mathcal{S}, t \geq 0$$

and

$$\|\Phi_t \mathbf{w}\| \leq c(\lambda) e^{\lambda t} \|\mathbf{w}\| \quad \text{for } \mathbf{v} \in \mathcal{T}, t \geq 0.$$

(this property is called *spectral separation*).

Proof. The invariant measure μ on X is unique, so the metric space X is connected. Since the bundle $\mathcal{S}$ is one-dimensional, from [**J–P–Se**], Thm. 2.3, we deduce that the dynamical spectrum of the flow $\Phi|\mathcal{S}$ equals $\{\lambda_1\}$ for some $\lambda_1 \in \mathbb{R}$. Therefore, by Lemma 2.5, for any $\lambda' < \lambda_1 < \lambda''$ there exist constants $k(\lambda') > 0$ and $K(\lambda'') > 0$ such that

$$\|\Phi_t \mathbf{v}\| \geq k(\lambda') e^{\lambda' t} \|\mathbf{v}\| \quad \text{for } \mathbf{v} \in \mathcal{S}, t \geq 0 \tag{3.1}$$

and

$$\|\Phi_t \mathbf{v}\| \leq K(\lambda'') e^{\lambda'' t} \|\mathbf{v}\| \quad \text{for } \mathbf{v} \in \mathcal{S}, t \geq 0. \tag{3.2}$$

Making use of the exponential separation we obtain from (3.2) that there exists $\lambda_2 := \lambda_1 - \nu$ such that for each $\lambda > \lambda_2$ there is a constant $c(\lambda)$ such that

$$\|\Phi_t \mathbf{w}\| \leq c(\lambda) e^{\lambda t} \|\mathbf{w}\| \quad \text{for } \mathbf{v} \in \mathcal{T}, t \geq 0.$$

□

Part 2. Applications

4. Parabolic partial differential equations of second order. As mentioned in the introduction, now we show how our abstract results in Part 1 can be applied in the investigation of parabolic partial differential equations (PDE's) of second order.

In this section the general setting will be as follows. Assume that $\mathfrak{A}$ is a compact metric space. We consider a family of second order linear nonautonomous partial differential equations of parabolic type, parameterized by $\mathfrak{a} \in \mathfrak{A}$:

$$\frac{\partial u}{\partial t}(\mathfrak{a}; t, x) = \mathcal{A}(\mathfrak{a}; t, x) u(\mathfrak{a}; t, x), \quad t \in (0, \infty), x \in \Omega, \mathfrak{a} \in \mathfrak{A}, \tag{4.1a}$$

where

$$\begin{aligned} \mathcal{A}(\mathfrak{a}; t, x) u := & \sum_{i,j=1}^{n} a_{ij}(\mathfrak{a}; t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} \\ & + \sum_{i=1}^{n} b_i(\mathfrak{a}; t, x) \frac{\partial u}{\partial x_i} + c(\mathfrak{a}; t, x) u, \quad t \in [0, \infty), x \in \overline{\Omega}, \end{aligned}$$

is, for $T > 0$ fixed, a uniformly elliptic operator on $[0, T] \times \overline{\Omega}$, where $\overline{\Omega}$ is the closure of a bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$ of class $C^{2+\alpha}$, $\alpha > 0$.

The equation (4.1a) is complemented with Neumann-type homogeneous boundary conditions:

$$\frac{\partial u}{\partial \nu}(t,x) = 0 \quad \text{for } t \in (0,\infty), x \in \partial\Omega, \tag{4.1b}$$

where $\nu : \partial\Omega \to \mathbb{R}^n$ is a C^1 vector field pointing out of Ω,

or with Dirichlet homogeneous boundary conditions:

$$u(t,x) = 0 \quad \text{for } t \in (0,\infty), x \in \partial\Omega. \tag{4.1b$'$}$$

From now on, our standing assumptions are:

A1) For $T > 0$ and $\mathfrak{a} \in \mathfrak{A}$ fixed the functions $a_{ij}(\mathfrak{a};\cdot,\cdot)$, $b_i(\mathfrak{a};\cdot,\cdot)$, $c(\mathfrak{a};\cdot,\cdot)$ belong to the Banach spaces $C^{\alpha/2,2+\alpha}([0,T]\times\overline{\Omega})$, $C^{\alpha/2,1+\alpha}([0,T]\times\overline{\Omega})$, $C^{\alpha/2,\alpha}([0,T]\times\overline{\Omega})$ respectively.

A2) There exists a Banach space W such that $C^2(\overline{\Omega}) \subset W \subset C^0(\overline{\Omega})$ with both inclusions continuous having the property: if $U(\mathfrak{a};\tau;t)w(x)$ denotes the unique solution to the problem (4.1a)+(4.1b) (or (4.1a)+(4.1b$'$)) for the parameter value $\mathfrak{a}$ with the initial condition:

$$u(\tau,x) = w(x) \quad \text{for } w \in W, x \in \overline{\Omega}, \tag{4.1c}$$

then the mapping

$$(\mathfrak{a};\tau,t,w) \mapsto U(\mathfrak{a};\tau;t)w$$

defined for $\mathfrak{a} \in \mathfrak{A}$, $\tau \in [0,\infty)$, $t \in [0,\infty)$, $\tau \le t$, $w \in W$, is continuous.

Remark. The reason why we choose to formulate assumption A2) in the above way is that, since there are several approaches to the problem of the generation of a semiflow by a parabolic PDE (compare e. g. [**He**], or [**Am1**]–[**Am3**], or else [**An**]), we do not want to be too specific. For a very broad class of conditions implying the fulfillment of assumption A2) the reader is referred to [**Am2**], Sections 3 and 7.

We will have two sources of examples in mind. In both cases, we assume that the bounded domain $\Omega \subset \mathbb{R}^n$ is C^∞ and that all the functions are also C^∞ on their domains.

A) Linearization of a quasilinear PDE on an invariant set. Consider a quasilinear time-independent parabolic partial differential equation of second order

$$\frac{\partial \tilde{u}}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\tilde{a}_{ij}(x,\tilde{u})\frac{\partial \tilde{u}}{\partial x_j}\right) + \tilde{F}(x,\tilde{u},\nabla\tilde{u}), \quad t \in (0,\infty), x \in \Omega. \tag{4.2}$$

Equation (4.2) together with appropriate boundary conditions generates a local semiflow $\tilde{\varphi}$ on a subset Z of a suitable function space W (see e. g. [**Am2**]). Under natural conditions (see e. g. [**He**], Section 7.3) the mappings $\tilde{\varphi}_t$ for $t > 0$ are injective. If, in addition, we put restrictions on the dependence of $\tilde{a}_{ij}$ on $\nabla \tilde{u}$, then bounded trajectories are precompact (see e. g. [**Am2**], Introduction).

As X we can take any compact totally invariant set for $\tilde{\varphi}$, for instance, the ω-limit set of a bounded trajectory, or the whole global attractor (if it exists). The semiflow $\tilde{\varphi}$ restricted to X extends naturally to a flow φ. Linearizing (4.2) along the trajectory of each point in X (for an explicit formula see e. g. [**Am2**], Thm. 10.4), we obtain a linear semiflow Φ on the tangent bundle $X \times W$, covering φ. Note that Φ is generated by an equation of the form (4.1), where the parameter set $\mathfrak{A}$ equals X and $\mathcal{A}(\mathfrak{a}; t, \cdot)$ for $(\mathfrak{a}, t) \in \mathfrak{A} \times [0, \infty)$ is the linearization of (4.2) at $\varphi_t \mathfrak{a}$. The solutions of (4.2) corresponding to initial conditions in X are classical ones for all $t \in (-\infty, \infty)$ (see [**A2**], Thm. 9.2), so it is easy to verify that assumptions A1) and A2) are fulfilled in this situation.

The situation is quite similar in the case where the right-hand side of (4.2) is T-periodic in time, $T > 0$. Then the local semiflow is defined on $Z \times [0, T]$ rather than on Z. The Poincaré map restricted to a compact totally invariant set is a homeomorphism whose (integer) iterates form a discrete-time flow. Now, taking the Poincaré map for the linearization of the equation along any solution contained in the invariant set, we obtain a discrete-time linear semiflow on the corresponding product bundle.

It should be noted that many of the above conditions (smoothness, for instance) can be considerably relaxed. Also, equation (4.2) can be in a more general form.

B) Partial differential equations almost periodic in time. Consider a linear homogeneous nonautonomous parabolic PDE of second order

(4.3)
$$\frac{\partial \tilde{u}}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\tilde{a}_{ij}(x, \tilde{u}) \frac{\partial \tilde{u}}{\partial x_j} \right) + \sum_{i=1}^{n} \tilde{b}_i(t, x) \frac{\partial \tilde{u}}{\partial x_i} + \tilde{c}(t, x) \tilde{u}, \quad t \in (-\infty, \infty), x \in \Omega,$$

complemented with (say) homogeneous Dirichlet boundary conditions. Assume that the mapping $(-\infty, \infty) \ni t \mapsto (\tilde{a}_{ij}(t, \cdot), \tilde{b}_i(t, \cdot), \tilde{c}(t, \cdot)) \in C^2(\overline{\Omega}, \mathbb{R}^{n^2}) \times C^2(\overline{\Omega}, \mathbb{R}^n) \times C^1(\overline{\Omega}, \mathbb{R})$ is (Bohr) almost periodic.

As X we take the *hull* of the right-hand side of (4.3), that is, the closure in the Banach space $C(\mathbb{R}, C^2(\overline{\Omega}, \mathbb{R}^{n^2}) \times C^2(\overline{\Omega}, \mathbb{R}^n) \times C^1(\overline{\Omega}, \mathbb{R}))$ (with the uniform norm) of the set of all time translates of it. X is a compact connected metric space. Moreover, the flow φ on X defined as the time translation has precisely one invariant probability measure (see [**Ne–St**], Thm. 9.34 on p. 510). The linear semiflow Φ is generated by the natural extension of (4.3) to the hull of its right-hand side. It is easy now to check that assumptions A1) and A2) are fulfilled, with $\mathfrak{A} = X$.

The linear mapping $U(\mathfrak{a};\tau;t) : W \to W$ can be represented as an integral operator with kernel G defined on $\{(\mathfrak{a},t,x,\xi) \in \mathfrak{A} \times [0,\infty) \times (0,\infty) \times \overline{\Omega} \times \overline{\Omega} : \tau < t\}$:

$$U(\mathfrak{a};\tau;t)w(x) = \int_{\Omega} G(\mathfrak{a};t,x;\tau,\xi)w(\xi)d\xi$$

for $\mathfrak{a} \in \mathfrak{A}$, $\tau \in [0,\infty)$, $t \in (0,\infty)$, $\tau < t$, $w \in W$, x, $\xi \in \overline{\Omega}$ (see [**F**], Section 3.7 for the Dirichlet case, and Problem 5.5 on p. 155 for the Neumann-type case).

Lemma 4.1. *For $\tau \in [0,\infty)$ fixed, the mapping $\mathfrak{a} \times (\tau,\infty) \times \overline{\Omega} \ni (\mathfrak{a},t,\xi) \mapsto G(\mathfrak{a};t,\cdot;\tau,\xi) \in C^0(\overline{\Omega})$ is continuous.*

Proof. Fix, for a moment, $\tau < T_1 \le T_2 < \infty$. Since $\mathcal{A}$ is uniformly elliptic in $[T_1,T_2]$, [**È–I**], Thm. 1.1 implies that G and its derivatives $\partial G/\partial x_i$ are bounded uniformly in $(\mathfrak{a},t,\xi) \in \mathfrak{A} \times [T_1,T_2] \times \overline{\Omega}$. Therefore the family $\{G(\mathfrak{a};t,\cdot;\tau,\xi) : (\mathfrak{a},t,\xi) \in \mathfrak{A} \times [T_1,T_2] \times \overline{\Omega}\} \subset C^0(\overline{\Omega})$ is bounded and equicontinuous, thus precompact.

Let $(\mathfrak{a}_k,t_k,\xi_k) \in \mathfrak{A} \times (\tau,\infty) \times \overline{\Omega}$ be a sequence converging to $(\mathfrak{a}_0,t_0,\xi_0)$, $t_0 > \tau$. Take $T_1 := (t_0+\tau)/2$, $T_2 := 2t_0$. By choosing a subsequence, if necessary, we can assume that $G(\mathfrak{a}_k,t_k,\cdot;\tau,\xi_k)$ converges in $C^0(\overline{\Omega})$ to some g. Suppose to the contrary that there exists $\tilde{x} \in \overline{\Omega}$ such that $g(\tilde{x}) \ne G(\mathfrak{a}_0,t_0,\tilde{x};\tau,\xi_0)$. Recall that for $\mathfrak{a}$, t and x fixed, the function G is the kernel for the adjoint equation (see [**F**], Thm. 17 on p. 84). By A1), the adjoint equation satisfies the assumptions of [**È–I**], Thm. 1.1, so the derivatives $(\partial G/\partial\xi)(\mathfrak{a}_k,t_k,\cdot;\tau,\xi)$ are bounded uniformly in $(k,\xi) \in \mathbb{N} \times \overline{\Omega}$. Therefore there exists a relatively open neighborhood Ω_0 of ξ_0 in $\overline{\Omega}$ such that $\lim_{k\to\infty}(\inf\{|G(\mathfrak{a}_k,t_k,\tilde{x};\tau,\xi) - G(\mathfrak{a}_0,t_0,\tilde{x};\tau,\xi)| : \xi \in \Omega_0\}) > 0$. Now take a $\tilde{w} \in W$ with support contained in Ω_0. We have $U(\mathfrak{a}_0;\tau;t_0)\tilde{w}(\tilde{x}) \ne \lim_{k\to\infty} U(\mathfrak{a}_k;\tau;t_k)\tilde{w}(\tilde{x})$, which contradicts the continuous dependence of the solution on $(\mathfrak{a},t) \in \mathfrak{A} \times (0,\infty)$. □

We will consider two cases: first the case of Neumann-type boundary conditions, then the case of Dirichlet boundary conditions.

1) Neumann-type case. The Banach space $V := C^0(\overline{\Omega})$ is strongly and normally ordered via the cone V_+ of nonnegative functions. By f we will denote the function equal to 1.

For $\mathfrak{a} \in \mathfrak{A}$ and $\xi \in \overline{\Omega}$ fixed, $G(\mathfrak{a};\cdot,\cdot;0,\xi)$ is a solution to (4.1a)+(4.1b). By [**Hi**], Thm. 4.1, $G(\mathfrak{a};t,\cdot;0,\xi) \gg 0$ for all $t > 0$. From our Lemma 4.1 it follows that the assignment $\mathfrak{A} \times (0,\infty) \times \overline{\Omega} \ni (\mathfrak{a},t,\xi) \mapsto G(\mathfrak{a};t,\cdot;0,\xi) \in V$ is continuous. Therefore for any $T > 0$ there are constants $0 < m_T \le M_T$ such that for all $\mathfrak{a} \in \mathfrak{A}$, $t \in [T/2, 2T]$, $x \in \overline{\Omega}$ and $\xi \in \overline{\Omega}$ one has $m_T \le G(\mathfrak{a}; t, x; 0, \xi) \le M_T$. Thus we have

$$m_T \int_\Omega v(\xi) d\xi \le \int_\Omega G(\mathfrak{a}, t, x; 0, \xi) v(\xi) d\xi \le M_T \int_\Omega v(\xi) d\xi,$$

that is,

$$m_T \int_\Omega v(\xi) d\xi \le U(\mathfrak{a}; 0; t) v \le M_T \int_\Omega v(\xi) d\xi$$

for $\mathfrak{a} \in \mathfrak{A}$, $t \in [T/2, 2T]$, $v \in V_+ \setminus \{0\}$.

2) Dirichlet case. In this case, let $f : \overline{\Omega} \to \mathbb{R}$ be the normalized (in the C^0-norm) nonnegative eigenfunction pertaining to the principal eigenvalue of the elliptic problem:

$$\Delta u = 0 \quad \text{on } \Omega, \quad u|\partial\Omega = 0.$$

By the theory of elliptic equations, f is of class C^1 on $\overline{\Omega}$, $f(x) > 0$ for $x \in \Omega$, $f(x) = 0$ for $x \in \partial\Omega$, and $(\partial f/\partial \nu)(x) < 0$ for $x \in \partial\Omega$, where ν is the unit normal vector field on $\partial\Omega$, pointing out of Ω.

The Banach space V is defined to be the set of those $v \in C^0(\overline{\Omega})$ for which the order-unit norm $\|v\|_f := \sup\{|v(x)|/f(x) : x \in \Omega\}$ is finite. V is strongly and normally ordered via the cone V_+ of nonnegative functions.

Lemma 4.2. *The derivatives* $(\partial^2 G/\partial x_i \partial \xi_j)(\mathfrak{a}; t, x; 0, \xi)$, i, $j = 1, \dots, n$, *are continuous in* $(\mathfrak{a}, t, x, \xi) \in \mathfrak{A} \times [0, \infty) \times \overline{\Omega} \times \overline{\Omega}$.

Proof. Fix $0 < T_1 \le T_2 < \infty$. By Lemma 4.1 the mapping $\mathfrak{A} \times [T_1, T_2] \times \overline{\Omega} \ni (\mathfrak{a}, t, \xi) \mapsto (\partial G/\partial x_i)(\mathfrak{a}; t, \cdot; T_1/2, \xi) \in C^0(\overline{\Omega})$ is continuous. For $(\mathfrak{a}, t, x) \in \mathfrak{A} \times [T_1, T_2] \times \overline{\Omega}$ fixed, the kernel G satisfies the adjoint equation:

$$\begin{cases} \dfrac{\partial G(\mathfrak{a}; t, x; \tau, \xi)}{\partial \tau} = \mathcal{A}^\star G(\mathfrak{a}; t, x; \tau, \xi) \\ G(\mathfrak{a}; t, x; \tau, \xi) = 0 \quad \text{for } \xi \in \partial\Omega, \tau \in [0, T_1/2], \end{cases} \tag{4.4}$$

where $\mathcal{A}^\star$ is the adjoint operator to $\mathcal{A}$ (see [**F**], p. 26). Differentiating (4.4) in x_i, $i = 1, \dots, n$, we obtain

$$\begin{cases} \dfrac{\partial^2 G(\mathfrak{a}; t, x; \tau, \xi)}{\partial x_i \partial \tau} = \mathcal{A}^\star \dfrac{\partial G(\mathfrak{a}; t, x; \tau, \xi)}{\partial x_i} \\ \dfrac{\partial G(\mathfrak{a}; t, x; \tau, \xi)}{\partial x_i} = 0 \quad \text{for } \xi \in \partial\Omega, \tau \in [0, T_1/2]. \end{cases} \tag{4.5}$$

We will now consider (4.5) with the following family of initial conditions parameterized by $(\mathfrak{a}, t, x) \in \mathfrak{A} \times [T_1, T_2] \times \overline{\Omega}$:

$$\frac{\partial G}{\partial x_i}(\mathfrak{a}; t, x; T_1/2, \xi) \quad \text{for } \xi \in \overline{\Omega}. \tag{4.5a}$$

Applying Lemma 4.1 to the adjoint equation, we obtain that the mapping

$$\mathfrak{A} \times [T_1, T_2] \times \overline{\Omega} \ni (\mathfrak{a}, t, x) \mapsto \frac{\partial^2 G}{\partial x_i \partial \xi_j}(\mathfrak{a}; t, x; 0, \cdot) \in C^0(\overline{\Omega})$$

is continuous, hence the derivatives $\partial^2 G/\partial x_i \partial \xi_j$ for $\tau = 0$ fixed depend continuously on $(\mathfrak{a}, t, x, \xi) \in \mathfrak{A} \times [T_1, T_2] \times \overline{\Omega} \times \overline{\Omega}$. $\square$

Proposition 4.1. *If u is a classical solution of* (4.1a)+(4.1b′) *such that $u(t, x) = 0$ for all $t > 0$ and $x \in \partial\Omega$, then $(\partial u/\partial \nu)(t, x) < 0$ for all $t > 0$ and $x \in \partial\Omega$, where $\nu : \partial\Omega \to \mathbb{R}^n$ is any vector field pointing out of Ω.*

Proof. For $t > 0$ and $\mathfrak{a} \in \mathfrak{A}$ fixed, the set $\{c(\mathfrak{a}; s, x) : s \in [0, t], x \in \overline{\Omega}\}$ is bounded. Now we apply Thm. 3.3.7 together with Remark (ii) on p. 175 of [**Pr–We**] to the function $-u$. $\square$

Since $(t, x) \mapsto G(\mathfrak{a}; t, x; \tau, \xi)$ is a solution of (4.1a)+(4.1b′), by Proposition 4.1 we have that

$$\frac{\partial G}{\partial \nu_x}(\mathfrak{a}; t, x; \tau, \xi) < 0 \quad \text{for } \mathfrak{a} \in \mathfrak{A}, t \in [T_1, T_2], 0 \le \tau < t, x \in \partial\Omega, \xi \in \overline{\Omega}.$$

Now, $(\tau, \xi) \mapsto (\partial G/\partial \nu_x)(\mathfrak{a}; t, x; \tau, \xi)$ is a solution of the equation adjoint to (4.1a)+(4.1b′), thus, applying again Proposition 4.1 we get

$$\frac{\partial^2 G}{\partial \nu_x \partial \nu_\xi}(\mathfrak{a}; t, x; 0, \xi) > 0$$

for all $\mathfrak{a} \in \mathfrak{A}$, $t \in [T_1, T_2]$, $x \in \partial\Omega$, $\xi \in \partial\Omega$.

By Lemma 4.2, we can prove, using l'Hôpital's rule, that for $T > 0$ fixed there are constants $0 < m_T \le M_T$ such that

$$m_T \le \frac{G(\mathfrak{a}; t, x; 0, \xi)}{f(x) f(\xi)} \le M_T \quad \text{for all } \mathfrak{a} \in \mathfrak{A}, t \in [T/2, 2T], x \in \overline{\Omega}, \xi \in \overline{\Omega}.$$

Take an initial condition $v \in V_+ \setminus \{0\}$. For any $\mathfrak{a} \in \mathfrak{A}$, $t \in [T/2, 2T]$, $x \in \overline{\Omega}$ we have

$$\frac{U(\mathfrak{a}; 0; t) v(x)}{f(x)} = \int_{\Omega} \frac{G(\mathfrak{a}, t, x; 0, \xi)}{f(x) f(\xi)} v(\xi) f(\xi) d\xi,$$

hence

$$m_T \int_{\Omega} v(\xi) f(\xi) d\xi \le \frac{U(\mathfrak{a}; 0; t) v(x)}{f(x)} \le M_T \int_{\Omega} v(\xi) f(\xi) d\xi.$$

This means that for any $v \in V_+ \setminus \{0\}$, one has

$$l(v) m_T f \le U(\mathfrak{a}; 0; t) v \le l(v) M_T f \quad \text{for all } \mathfrak{a} \in \mathfrak{A}, t \in [T/2, 2T]$$

with $l(v) := \int_{\Omega} v(\xi) f(\xi) d\xi > 0$. The Dirichlet case is done.

In both cases above we have shown that Theorem 1 can be applied to the linear semiflow Φ considered on the space V with a fairly weak topology (especially in the Neumann-type case), whereas in most settings for the dynamical theory of parabolic PDE's the flow φ is defined on a Banach space W with much stronger topology (for example, in the theory presented in [**Mo**] W can be $C^k(\overline{\Omega})$). In those cases the cone W_+ of nonnegative functions is solid but usually not normal. However, for $k \in \mathbb{N}$ sufficiently big the Banach space $C^k(\overline{\Omega})$ embeds continuously into such a W. Therefore, in order to prove that the semiflow on the strongly and normally ordered space V as above factorizes through the semiflow on W it suffices to show that the assignment

$$\mathfrak{A} \times (0, \infty) \times \overline{\Omega} \ni (\mathfrak{a}, t, \xi) \mapsto G(\mathfrak{a}; t, \cdot; 0, \xi) \in C^k(\overline{\Omega})$$

is continuous. This can be proved in a similar way as in Lemma 4.1 (one must assume that the domain Ω and the coefficients of the operator $\mathcal{A}$ are sufficiently smooth).

5. Information about applications. In the present section we would like to address the relevance of our results in the theory of strongly monotone dynamical systems (semiflows). As mentioned in Section 4, such systems are generated by (semilinear, quasilinear, fully nonlinear) parabolic partial differential equations of second order, complemented with boundary conditions admitting the strong maximum principle. When all the functions appearing in the equation and the boundary condition are sufficiently smooth, the semiflow is of class C^1 (for a C^1 theory the reader is referred to [**He**], or the series [**Am1**]–[**Am3**], or else e. g. [**An**], and for strong monotonicity—to e. g. [**Hi**]). The derivative semiflow is generated by the linearization of the original PDE along a solution (see e. g. [**Am1**]–[**Am3**], or [**He**]). The linearization is a (usually nonautonomous, even nonperiodic) *linear* parabolic PDE of second order, complemented with linear boundary conditions. When we restrict ourselves to a compact invariant set, that linear PDE gives rise to a strongly monotone linear semiflow.

Theorems asserting the occurrence of exponential separation (like our Theorem 1) enable one to use such powerful tools as (forward) invariant foliations (laminations) for normally hyperbolic manifolds. Further, if the semiflow is of class $C^{1+\alpha}$, $\alpha > 0$, one can use the theory of measurable invariant families of (embedded) submanifolds (Pesin's theory), as developed in the infinite-dimensional case by D. Ruelle ([**R2**]) and R. Mañé ([**Ma**]). Those ideas have been extensively used in a paper [**Po–Te**] by P. Poláčik and I. Tereščák. They proved, among others, that, under appropriate hypotheses, the set of points converging to a cycle (a periodic trajectory) is open and dense. That result provides a refinement of a classification of ω-limit sets given by P. Takáč in [**Ta2**].

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After submitting this paper the author learned that P. Poláčik and I. Tereščák proved a result analogous to Theorem 1 assuming only that the linear semiflow is strongly monotone. Their paper, entitled *Exponential separation and invariant bundles for maps in ordered Banach spaces with applications to parabolic equations,* will appear in J. Dynamics Differential Equations. Also, making use of exponential separation P. Hess and P. Poláčik have proved that the set of periods of cycles which are not linearly unstable is bounded from above (*Boundedness of prime periods of stable cycles and convergence to fixed points in discrete monotone dynamical systems,*to appear in SIAM J. Math. Anal.).

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